On finding integer solutions to fifth degree Diophantine Equations with five unknowns

$$6(x^3+y^3)=13(z^2-w^2)p^3$$

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ABSTRACT
Varieties of solution patterns to not uniform fifth degree polynomial equation having five variables given by $6(x^3+y^3)=13(z^2-w^2)p^3$ are presented in this paper.

Keywords: Quintic with five variable, Non-homogeneous quintic, Integer solutions

Launch
The subject of polynomial equations offers a plenty of motivating problems. Particularly uniform or non uniform fifth degree equations with multiple variables have aroused the interest of numerous mathematicians for a long time [1-3]. In particular, [4-6] refers to integer solutions for fifth degree equation having three variables and [7-14] refers to fifth degree degree equation having five variables. The above problems motivated us to obtain choices of integer solutions to the non uniform fifth degree equation having five variables presented by $6(x^3+y^3)=13(z^2-w^2)p^3$.

Technical procedure
Non-homogeneous fifth degree Diophantine equation with five variables for solving is

$$6(x^3+y^3)=13(z^2-w^2)p^3 \qquad (1)$$

The procedure to obtain varies patterns of solution for (1) through different ways is presented as follows:
Way 1
In (1) choosing

$$x=u+v,\ y=u-v,\ z=3u+1,\ w=3u+1 \qquad (2)$$

gives

$$u^2+3v^2=13p^3 \qquad (3)$$

Take

$$p=a^2+3b^2 \qquad (4)$$

where $a \neq b$
Take

$$13=\left(1+i2\sqrt{3}\right)(1-i2\sqrt{3}) \qquad (5)$$

Utilize (4) and (5) in (3)and applying the resolving activity, we have

$$u+i\sqrt{3}\,v=\left(1+i2\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^3$$

After specific algebraic computations and clarification, we obtain

$$u=a^3-9ab^2+18b^3-18a^2b$$

$$v=2\,(a^3-9ab^2)+(-3b^3+3a^2b)$$

This replacement of u and v in (2), the solutions of equation (1) are as follows

$$x = 3a^3 + 15b^3 - 27ab^2 - 15a^2b$$

$$y = -a^3 + 21b^3 + 9ab^2 - 21a^2b$$

$$z = 3(a^3 - 9ab^2 + 18b^3 - 18a^2b) + 1$$

$$w = 3(a^3 - 9ab^2 + 18b^3 - 18a^2b) - 1$$

along with (4)

Way 2

Take

$$13 = \frac{(5 + i3\sqrt{3})(5 - i3\sqrt{3})}{4} \quad (6)$$

Utilize (4) and (6) in (3) and applying the resolving activity, we have

$$u + i\sqrt{3}\,v = \left(\frac{5 + i3\sqrt{3}}{2}\right)(a + i\sqrt{3}b)^3$$

After specific algebraic computations and clarification, we obtain

$$u = \frac{1}{2}(5a^3 - 45ab^2 - 27a^2b + 27b^3)$$

$$v = \frac{1}{2}(3a^3 - 27ab^2 + 15a^2b - 15b^3)$$

Writing a by 2A & b by 2B, we have

$$u = 4(5A^3 - 45AB^2 - 27A^2B + 27B^3)$$

$$v = 4(3A^3 - 27AB^2 + 15A^2B - 15B^3)$$

This replacement of u and v in (2), the solutions of equation (1) are as follows

$$x = 32A^3 - 288AB^2 - 48A^2B + 48B^3$$

$$y = 8A^{3^3} - 72AB^2 - 168A^2B + 168B^3$$

$$z = 3(20A^3 - 180AB^2 - 108A^2B + 108B^3) + 1$$

$$w = 3(20A^3 - 180AB^2 - 108A^2B + 108B^3) - 1$$

$$P = 4(A^2 + 3B^2)$$

Way 3

Rewrite (3) as

$$u^2 + 3v^2 = 13p^3 * 1 \quad (7)$$

Take

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \quad (8)$$

Utilize (4),(6) and (8) in (7) and applying the resolving activity, we have

$$u + i\sqrt{3}\,v = (1 + i2\sqrt{3})\left(\frac{1 + i\sqrt{3}}{2}\right)(a + i\sqrt{3}b)^3$$

After specific algebraic computations and clarification, we obtain

$$u = \frac{1}{2}(-5a^3 + 45ab^2 - 27a^2b + 27b^3)$$

$$v = \frac{1}{2}(3a^3 - 27ab^2 - 15a^2b + 15b^3)$$

Writting a by 2A and b by 2B, we have

$$u = 4(-5A^3 + 45AB^2 - 27A^2B + 27B^3)$$
$$v = 4(3A^3 - 27AB^2 - 15A^2B + 15B^3)$$

This replacement of u and v in (2), the solutions of equation (1) are as follows

$$x = -8A^3 + 72AB^2 - 168A^2B + 168B^3$$
$$y = -32A^{3^3} + 288AB^2 - 48A^2B + 48B^3$$
$$z = 3(-20A^3 + 180AB^2 - 108A^2B + 108B^3) + 1$$
$$w = 3(-20A^3 + 180AB^2 - 108A^2B + 108B^3) - 1$$
$$P = 4(A^2 + 3B^2)$$

Way 4

Take

$$1 = \frac{\left(3p^2 - q^2 + i\sqrt{3}\,2pq\right)\left(3p^2 - q^2 + i\sqrt{3}\,2pq\right)}{\left(3p^2 + q^2\right)^2} \quad (9)$$

Utilize (4), (5) and (9) in (7) and applying the resolving activity, we have

$$u + i\sqrt{3}\,v = (1 + i2\sqrt{3})\left(\frac{3p^2 - q^2 + i\sqrt{3}\,2pq}{3p^2 + q^2}\right)\left(a + i\sqrt{3}b\right)^3$$

Consider

$$\left(a + i\sqrt{3}b\right)^3 = f(a,b) + i\sqrt{3}g(a,b)$$

Where, $f(a,b) = a^3 - 9ab^2$

$g(a,b) = (3a^2b - 3b^3)$

Therefore

$$u + i\sqrt{3}\,v = (1 + i2\sqrt{3})\left(\frac{3p^2 - q^2 + i\sqrt{3}\,2pq}{3p^2 + q^2}\right)(f(a,b) + i\sqrt{3}g(a,b))$$

$$u + i\sqrt{3}\,v = \frac{1}{(3p^2 + q^2)}\begin{bmatrix} \{(3p^2 - q^2)[f(a,b) - 6g(a.b)] - 6pq[g(a,b) + 2f(a,b)\} \\ + i\sqrt{3}\{(3p^2 - q^2)[2f(a,b) + g(a,b)] + 2pq[f(a,b) - 6g(a,b)]\} \end{bmatrix}$$

Replacing a by $(3p^2 + q^2)$ A, b by $(3p^2 + q^2)$ B and equating coefficients of corresponding terms, we have

$$u = (3p^2 + q^2)^2\{(3p^2 - q^2)[f(a,b) - 6g(a.b)] - 6pq[g(a,b) + 2f(a,b)\}$$
$$v = (3p^2 + q^2)^2\{(3p^2 - q^2)[2f(a,b) + g(a.b) + 2pq[f(a,b) - 6g(a,b)\}$$

This replacement of u and v in (2), the solutions of equation (1) are as follows

$$x = (3p^2 + q^2)^2\{(3p^2 - q^2)[3f(a,b) - 5g(a.b)] - 10pqf(a,b) - 18pqg(a,b)\}$$
$$y = (3p^2 + q^2)^2\{(3p^2 - q^2)[-f(a,b) - 7g(a.b)] - 14pqf(a,b) - 6g(a,b)\}$$
$$z = 3\{(3p^2 + q^2)^2\{(3p^2 - q^2)[f(a,b) - 6g(a.b)] - 6pq[g(a,b) + 2f(a,b)\} + 1$$
$$w = 3\{(3p^2 + q^2)^2\{(3p^2 - q^2)[f(a,b) - 6g(a.b)] - 6pq[g(a,b) + 2f(a,b)\} - 1$$
$$p = (3p^2 + q^2)^2(A^2 + 3B^2)$$

Way 5

Taking

$$v = 2p \quad (10)$$

in (3),it reduces to

$$u^2 = p^2(13p - 12) \quad (11)$$

Take

$$\alpha^2 = 13p - 12 \tag{12}$$

Whose smallest positive integer solutions are $p_0 = 1, \alpha_0 = 1$

Let the second solution of (12) be

$$\alpha_1 = h - \alpha_0, p_1 = h + p_0 \tag{13}$$

Subsituting (13) in (12)& on simplifying, we get

$$h = 2\alpha_0 + 13$$

From (13),it is seen that

$$\alpha_1 = \alpha_0 + 13, p_1 = p_0 + 2\alpha_0 + 13$$

Repeating the above process,the solution of (12) in general be

$$p_n = p_0 + 2n\alpha_0 + 13n^2 = 1 + 2n + 13n^2 \tag{14}$$

$$\alpha_n = \alpha_0 + 13n = 1 + 13n$$

In view of (10) and (11),one obtains

$$v_n = 2\left(1 + 2n + 13n^2\right)$$

$$u_n = p_n \alpha_n = \left(1 + 2n + 13n^2\right)\left(1 + 13n\right) \tag{15}$$

Using (15) in (2), we have

$$x_n = (1 + 2n + 13n^2)(3 + 13n)$$

$$y_n = (1 + 2n + 13n^2)(-1 + 13n)$$

$$z_n = 3(1 + 2n + 13n^2)(1 + 13n) + 1$$

$$w_n = 3(1 + 2n + 13n^2)(1 + 13n) - 1 \tag{16}$$

Thus (1) is satisfied by (16) and (17)

Way 6

Taking

$$v = 2kp \tag{17}$$

in (3),it reduces to

$$u^2 = p^2(13p - 12k^2) \tag{18}$$

Take

$$\alpha^2 = 13p - 12k^2 \tag{19}$$

Whose smallest positive integer solutions are $p_0 = k^2, \alpha_0 = k$

Let the second solution of (19) be

$$\alpha_1 = h - \alpha_0, p_1 = h + p_0 \tag{20}$$

Subsituting (20) in (19) & on simplifying, we get

$$h = 2\alpha_0 + 13$$

From (19),it is seen that

$$\alpha_1 = \alpha_0 + 13, p_1 = p_0 + 2\alpha_0 + 13$$

Repeating the above process,the general solution to (19) is given by

$$p_n = p_0 + 2n\alpha_0 + 13n^2 = k^2 + 2nk + 13n^2 \tag{21}$$

$$\alpha_n = \alpha_0 + 13n = k + 13n$$

In view of (17) and (18),one obtains

$$v_n = 2k\left(k^2 + 2nk + 13n^2\right)$$

$$u_n = p_n \alpha_n = \left(k^2 + 2nk + 13n^2\right)\left(k + 13n\right) \tag{22}$$

Using (22) in (2), we have

$$x_n = (k^2 + 2nk + 13n^2)(3k + 13n)$$
$$y_n = (k^2 + 2nk + 13n^2)(-k + 13n)$$
$$z_n = 3(k^2 + 2nk + 13n^2)(k + 13n) + 1$$
$$w_n = 3(k^2 + 2nk + 13n^2)(k + 13n) - 1$$
(23)

Thus (1) is satisfied by (23) and (21).
Way 7
Taking
$$v = 3p \quad (24)$$
in (3),it reduces to
$$u^2 = p^2(13p - 27) \quad (25)$$
Take the second solution of (26),
$$\alpha^2 = 13p - 27 \quad (26)$$
Whose smallest positive integer solutions are $p_0 = 4, \alpha_0 = 5$
Let
$$\alpha_1 = h - \alpha_0, p_1 = h + p_0 \quad (27)$$
Substituting (27) in (26) & on simplification, we get
$$h = 2\alpha_0 + 13$$
From (27),it is seen that
$$\alpha_1 = \alpha_0 + 13, p_1 = p_0 + 2\alpha_0 + 13$$
Repeating the above process, the general solution to (26) is given by
$$p_n = p_0 + 2n\alpha_0 + 13n^2 = 4 + 10n + 13n^2 \quad (28)$$
$$\alpha_n = \alpha_0 + 13n = 5 + 13n$$
In view of (24) and (25),one obtains
$$v_n = 3(4 + 10n + 13n^2)$$
$$u_n = p_n \alpha_n = (4 + 10n + 13n^2)(5 + 13n) \quad (29)$$
Using (29) in (2), we have
$$x_n = (4 + 10n + 13n^2)(8 + 13n)$$
$$y_n = (4 + 10n + 13n^2)(2 + 13n)$$
$$z_n = 3(4 + 10n + 13n^2)(5 + 13n) + 1$$
$$w_n = 3(4 + 10n + 13n^2)(5 + 13n) - 1$$
(30)

Thus (1) is satisfied by (30) and (28).

Inference

Many choices of solution patterns to not uniform fifth degree polynomial equation having five variables given by $6(x^3 + y^3) = 13(z^2 - w^2)p^3$ are presented in this paper . One may search for integer solutions to other forms of quintic Diophantine equations with three or more variables

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