Size-Biased Poisson-New Linear-Exponential Distribution

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ABSTRACT
To construct this distribution, the size-biased Poisson distribution has been mixed with size-biased Linear-exponential distribution (LED) of Sah (2022). It can also be obtained by size biasing Poisson-New Linear-exponential distribution (PNLED) of Sah and Sahani (2024). The required characteristics of this distribution such as probability mass function (pmf), statistical moments, estimation of parameters have been derived and discussed in systematic manner. Chi-square goodness-of -fit test has been applied to some over-dispersed secondary count data-seta which were early used by others.

Keywords: New Linear-Exponential distribution, Poisson-New Linear-exponential distribution (PNLED), Size-biased, Distribution, Probability distribution, Size-Biased Poisson-New Linear-exponential distribution (SBPNLED).

1. INTRODUCTION

Poisson-New Linear- exponential distribution (PNLED) of Sah and Sahani, [13], has been obtained by compounding Poisson distribution (PD) with New Linear-exponential distribution (NLED) of Sah, [14], where probability mass function (pmf) and Probability density function of PNLED and NLED are respectively mentioned as follows.

$$P_1(Z;\beta)=\left[\frac{\beta^2}{(1+\pi\beta)}\right]\left[\frac{1+\pi(1+\beta)+z}{(1+\beta)^{z+2}}\right] \tag{1}$$

where $z>0$ and $\beta>0$

$$f_1(z;\beta)=\frac{\beta^2}{(1+\pi\beta)}(\pi+\beta)e^{-\beta y} \;;\beta>0, z>0 \tag{2}$$

Let $P(Z;\beta)$ be the pmf of Size-biased Poisson-New Linear-exponential distribution (SBPNLED) which can be obtained by

$$P(Z;\beta)=\frac{z\,P_1(Z;\beta)}{Mean\,of\,PNLED} \tag{3}$$

It can also be obtained by mixing size-biased Poisson distribution (SBPD) with size-biased NLED (SBNLED), where SBPD is given by

$$g_1(z;\beta)=\frac{z\,P_2(Z;\lambda)}{Mean\,of\,PD}=\frac{e^{-\lambda}\lambda^{z-1}}{(z-1)!} \;;\lambda>0; z=1,2,3,... \tag{4}$$

SBNLED is given by

$$g_2(z;\beta)=\frac{z\,f(z;\beta)}{Mean\,of\,NLED}=\frac{\beta^3}{(2+\pi\beta)}(\pi z+z^2)e^{-\beta z} \;; z>0;\beta>0 \tag{5}$$

The expression (5) can be put into the following form

$$g_2(z;\beta)=p.g_{2*}(z;\beta)+q.g_{2**}(z;\beta) \tag{6}$$

Where $q=1-p$, $p=\frac{\pi\beta}{(2+\pi\beta)}$ and $q=\frac{2}{(2+\pi\beta)}$

$g_{2*}(z;\beta)=\beta^2 ze^{-\beta z}$ □ $Gamma(2,\beta)$ and $g_{2**}(z;\beta)=\frac{\beta^3}{2}z^2e^{-\beta z}\beta^2$ □ $Gamma(3,\beta)$

Proof

R.H.S.= $p.g_{2*}(z;\beta)+q.g_{2**}(z;\beta)=\frac{\pi\beta}{(2+\pi\beta)}.\beta^2 ze^{-\beta z}+\frac{2}{(2+\pi\beta)}.\frac{\beta^3}{2}z^2e^{-\beta z}$

$=\frac{\beta^3}{(2+\pi\beta)}(\pi z+z^2)e^{-\beta z}\ ; z>0;\beta>0$ =R.H.S ; Proved.

The parameter λ of PD follows NLED and hence the expression (5) can be written in the form

$$g_3(\lambda;\beta)=\frac{\beta^3}{(2+\pi\beta)}(\pi\lambda+\lambda^2)e^{-\beta\lambda}\ ;\lambda>0;\beta>0 \quad (7)$$

SBPNLED can be obtained by using

$$P(Z;\beta)=\int_0^\infty\{g_1(z;\lambda).g_3(\lambda;\beta)\}d\lambda \quad (8)$$

Nature is mysterious whose glory is unparalleled. No matter how much research we do, the result of all research is nature itself which can't be seen by everyone. It needs the eyes of the soul to see it. There exist many kinds of plants, trees, animals and many more characteristics in nature which differ in their shape and size. If we study a certain population of a distinct character contains different observations varying in their size, the probability of being selection of each item from population to sample may not be equal and their arises a case of size-biased distribution which is a special case of weighted probability distribution introduced by R.A. Fisher [6], later on, formulized by C.R. Rao [10].The following references show the pioneer contributors in the fields of size-biased Poisson-Continuous distribution and their applications [[1],[2],[8],[9],[11]].

Ghitany and Al Mutairi [4] obtained a size-biased Poisson-Lindley distribution (SBPLD) by size biasing the Poisson-Lindley distribution (PLD) of Sankaran [12]. Probability mass function of SBPLD as well as PLD are respectively given by

$$P_2(Z;\beta)=\left[\frac{\beta^3}{(2+\beta)}\right]\left[\frac{z(z+\beta+2)}{(1+\beta)^{z+2}}\right];\beta>0;z=1,2,3,\ldots \quad (9)$$

$$P_3(Z;\beta)=\left[\frac{\beta^3(z+\beta+2)}{(1+\beta)^{z+3}}\right];z=0,1,2,\ldots;\beta>0 \quad (10)$$

Ghitany and Al Mutairi [4] gave a very significant contribution towards developing the theory of size-biased probability distribution.

The main objective of this paper is to develop a distribution, that works under the same conditions in the sense of structure and number of parameters in the distribution, which gives a better conclusion than SBPLD when we use the Chi-square goodness of fit test on over-dispersed count data of different fields but of similar in nature which were used by others. To get a better format of this paper, it is presented under the following heading and sub-headings

1.0 Introduction
2.0 Materials and Methos
3.0 Results
3.1 Size-Biased Poisson-New Linear-exponential Distribution (SBPNLED)
3.2 Moments of SBPNLED
3.3 Estimation of Parameter of SBPNLED
3.4 Applications and Chi-square goodness of fit test
4.0 Conclusions.

2. MATERIALS AND METHODS

In this paper we have developed a new theory which has been developed in the concept of size-biased continuous probability distribution and whose validity has been measured by using Chi-square-goodness-of fit test for which secondary data have been used.

3. RESULTS

The most important work required for this paper have been presented in systematic manner under different adequate sub-headings as follows.

3.1 Size-Biased Poisson-New Linear-exponential Distribution (SBPNLED)

The pmf of SBPNLED can be obtained by (a) Definition of size-biased probability distribution (b) Compounding size-biased Poisson distribution with size-biased NLED.

(a) By using the definition of size-biased probability distribution:

$$P(Z;\beta)=\frac{z\,P_1(Z;\beta)}{Mean\,of\,PNLED}=\frac{z\left[\frac{\beta^2}{(1+\pi\beta)}.\frac{(\pi(1+\beta)+z+1)}{(1+\beta)^{z+2}}.\right]}{\left[\frac{(\pi\beta+2)}{\beta(1+\pi\beta)}\right]}$$

$$=\frac{\beta^3}{(2+\pi\beta)}.\frac{z\{\pi(1+\beta)+z+1\}}{(1+\beta)^{z+2}}\;;\beta>0;z=1,2,3,... \qquad (11)$$

Using the expression (11) we can obtained probability at $z=1,2,3,...$ for SBPNLED.

(b) Compounding size-biased Poisson distribution with size-biased NLED:

It can be obtained as

$$P(Z;\beta)=\int_0^\infty\{g_1(z;\lambda).g_3(\lambda;\beta)\}d\lambda$$

$$=\frac{\beta^3}{(2+\pi\beta)(z-1)!}\int_0^\infty\{(\pi\lambda^z+\pi\lambda^{z+1}).e^{-\lambda(1+\beta)}\}d\lambda$$

$$=\frac{\beta^3}{(2+\pi\beta)(z-1)!}\left[\frac{\pi z!}{(1+\beta)^{z+1}}+\frac{(z+1)!}{(1+\beta)^{z+2}}\right]$$

$$=\frac{\beta^3}{(2+\pi\beta)}\left[\frac{z\{\pi(1+\beta)+z+1\}}{(1+\beta)^{z+2}}\right];\beta>0\;;z=1,2,3,... \qquad (12)$$

Graphical representation of the pmf of SBPNLED

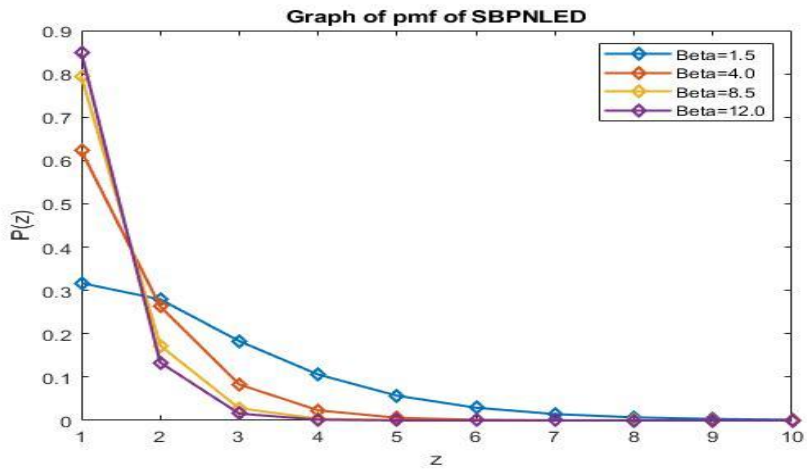

Figure.1

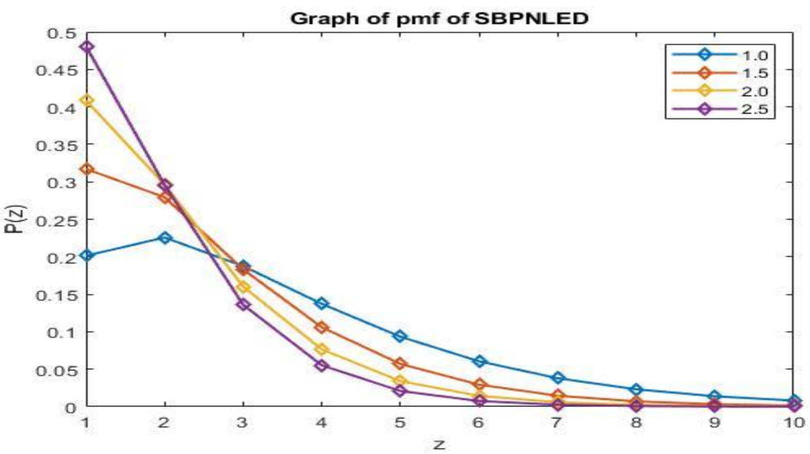

Figure.2

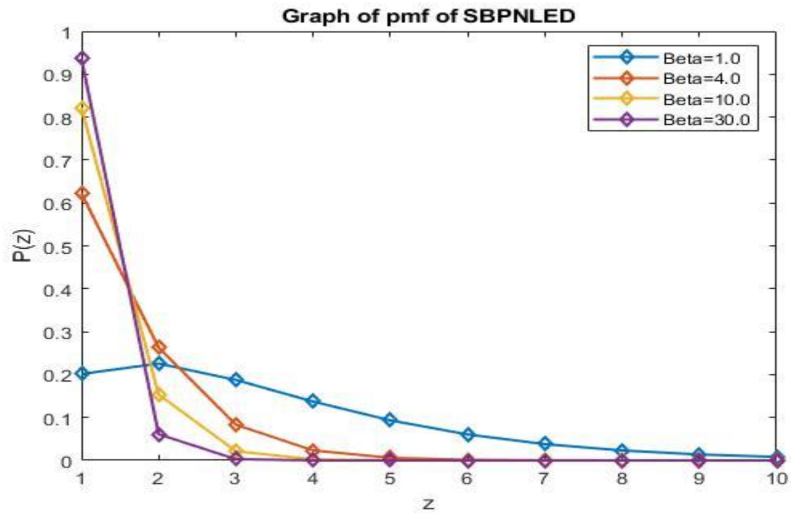

Figure.3

3.2 Moments of SBPNLED

All the four moments of SBPNLED about origin as well as the mean have been derived under this sub-heading. To obtain the first moments about origin as well as the mean, at first, we have to derive the general expression for Factorial moments of order r of SBPNLED as follows.

$$\mu'_{(r)} = E\left(E(Z^{(r)} / \lambda\right) \tag{13}$$

Where $Z^{(r)} = z(z-1)(z-2)...(z-r+1)$

$$\mu'_{(r)} = \frac{\beta^3}{(2+\pi\beta)}\int_0^\infty \left[\sum_{z=0}^{\infty} \frac{z^{(r)} e^{-\lambda}\lambda^{z-1}}{(z-1)!}\right](\pi\lambda+\lambda^2)e^{-\beta\lambda}d\lambda$$

$$= \frac{\beta^3}{(2+\pi\beta)}\int_0^\infty \left[\lambda^{r-1}\sum_{z+r=r}^{\infty} \frac{(z+r)e^{-\lambda}\lambda^{z+r-r}}{(z+r-r)!}\right](\pi\lambda+\lambda^2)e^{-\beta\lambda}d\lambda$$

$$= \frac{\beta^3}{(2+\pi\beta)}\int_0^\infty \left[(\lambda+r)\right](\pi\lambda+\lambda^2)e^{-\beta\lambda}d\lambda$$

$$= \left[\frac{r!}{(2+\pi\beta)\beta^r}\right]\left\{r\beta(r\beta+r+1)+(r+1)(\pi\beta+r+2\right\} \tag{14}$$

The expression (14) is the generalised factorial moment of order r of SBPNLED. The first four factorial moments are given respectively in the expression (15), (16), (17) and (18).

$$\mu'_{(1)} = 1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)} \tag{15}$$

$$\mu'_{(2)} = \frac{4(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{6(\pi\beta+4)}{\beta^2(2+\pi\beta)} \tag{16}$$

$$\mu'_{(3)} = \frac{18(\pi\beta+4)}{\beta^2(2+\pi\beta)}+\frac{24(\pi\beta+5)}{\beta^3(2+\pi\beta)} \tag{17}$$

$$\mu'_{(4)} = \frac{96(\pi\beta+5)}{\beta^3(2+\pi\beta)}+\frac{120(\pi\beta+6)}{\beta^4(2+\pi\beta)} \tag{18}$$

Conversion of factorial moments into moments about origin:

$$\mu_1' = \mu'_{(1)}$$

$$\mu_1' = 1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)} \tag{19}$$

It is the mean of SBPNLED.

$$\mu_2' = \mu'_{(1)} + \mu'_{(2)}$$

$$= 1+\frac{6(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{6(\pi\beta+4)}{\beta^2(2+\pi\beta)} \tag{20}$$

$$\mu'_3 = \mu'_{(3)} + 3\mu'_{(2)} + \mu'_{(1)}$$

$$= 1 + \frac{14(\pi\beta+3)}{\beta(2+\pi\beta)} + \frac{36(\pi\beta+4)}{\beta^2(2+\pi\beta)} + \frac{24(\pi\beta+5)}{\beta^3(2+\pi\beta)} \tag{21}$$

$$\mu'_4 = \mu'_{(4)} + 6\mu'_{(3)} + 7\mu'_{(2)} + \mu'_{(1)}$$

$$= 1 + \frac{30(\pi\beta+3)}{\beta(2+\pi\beta)} + \frac{150(\pi\beta+4)}{\beta^2(2+\pi\beta)} + \frac{240(\pi\beta+5)}{\beta^3(2+\pi\beta)} + \frac{120(\pi\beta+6)}{\beta^4(2+\pi\beta)} \tag{22}$$

The expression (19), (20), (21) and (22) are respectively the first four moments about origin. Graphical representation of the mean of SBPNLED is given as

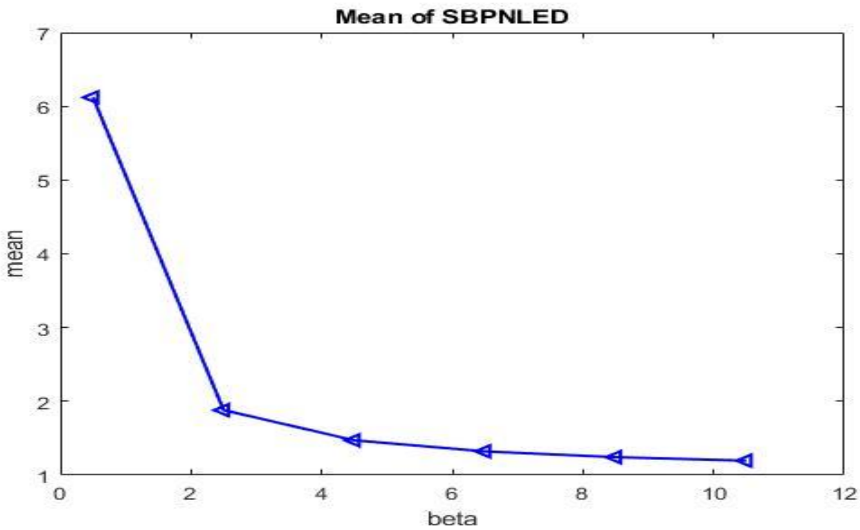

Figure.4

Conversion of μ'_r into μ_r :

The second, third and fourth moments about the mean of SBPNLED are given by the expression (23), (24) and (25) respectively.

$$\mu_2 = \left\{1 + \frac{6(\pi\beta+3)}{\beta(2+\pi\beta)} + \frac{6(\pi\beta+4)}{\beta^2(2+\pi\beta)}\right\} - \left\{1 + \frac{2(\pi\beta+3)}{\beta(2+\pi\beta)}\right\}^2$$

$$= \frac{[2\pi^2\beta^3 + 2\pi^2\beta^2 + 10\pi\beta^2 + 12\pi\beta + 12\beta + 12]}{[\beta(\pi\beta+2)]^2} \tag{23}$$

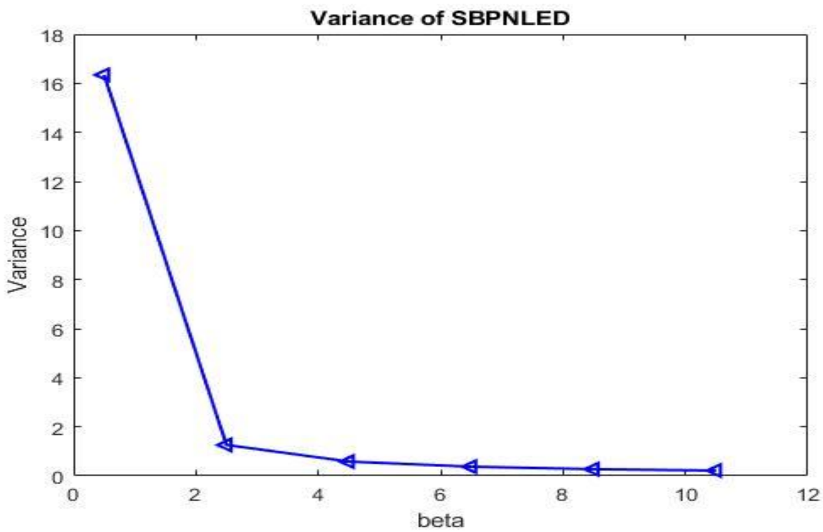

Figure.5

$$\mu_3=\left\{1+\frac{14(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{36(\pi\beta+4)}{\beta^2(2+\pi\beta)}+\frac{24(\pi\beta+5)}{\beta^3(2+\pi\beta)}\right\}-3\left\{1+\frac{6(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{6(\pi\beta+4)}{\beta^2(2+\pi\beta)}\right\}\left\{1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)}\right\}$$
$$+2\left\{1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)}\right\}^3$$
$$=\frac{(2\pi^3\beta^5+6\pi^3\beta^4+14\pi^2\beta^4+4\pi^3\beta^3+48\pi^2\beta^3+32\pi\beta^3+36\pi^2\beta^2+108\pi\beta^2+72\pi\beta+24\beta^2+72\beta+48)}{[\beta(2+\pi\beta)]^3} \quad (24)$$

$$\mu_4=\left\{1+\frac{30(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{150(\pi\beta+4)}{\beta^2(2+\pi\beta)}+\frac{240(\pi\beta+5)}{\beta^3(2+\pi\beta)}+\frac{120(\pi\beta+6)}{\beta^4(2+\pi\beta)}\right\}-4\left\{1+\frac{14(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{36(\pi\beta+4)}{\beta^2(2+\pi\beta)}+\frac{24(\pi\beta+5)}{\beta^3(2+\pi\beta)}\right\}$$
$$\left\{1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)}\right\}+6\left\{1+\frac{6(\pi\beta+3)}{\beta(2+\pi\beta)}+\frac{6(\pi\beta+4)}{\beta^2(2+\pi\beta)}\right\}\left\{1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)}\right\}^2-3\left\{1+\frac{2(\pi\beta+3)}{\beta(2+\pi\beta)}\right\}^4$$

$$=\frac{(2\pi^4\beta^7+26\pi^4\beta^6+18\pi^3\beta^6+48\pi^4\beta^5+260\pi^3\beta^5+60\pi^2\beta^5+24\pi^4\beta^4+528\pi^3\beta^4+920\pi^2\beta^4+88\pi\beta^4+288\pi^3\beta^3+1472\pi^2\beta^3++1392\pi\beta^3+48\beta^3+1008\pi^2\beta^2+2736\pi\beta^2+768\beta^2+1440\pi\beta+1440\beta+720)}{[\beta(2+\pi\beta)]^4} \quad (25)$$

Properties of SBPNLED

Property (1) $\frac{P(z+1;\beta)}{P(z;\beta)}=\left(\frac{1}{(1+\beta}\right)\left(1+\frac{1}{z}\right)\left(1+\frac{1}{(z+\pi+\pi\beta+1)}\right)$ (26)

- The expression (26) is decreasing function of z.
- $P(z;\beta)$ is log-concave.
-SBPNLED is unimodal and has increasing failure rate [IFR].
-It has increasing failure rate average [IFRA].
-It is new better than used [UNB] and new better than used in expectation (NBUE).
-For details account [see, Barlow and Pros Chan (1981), Ghitany and Al-Mutairi (2008).

Property (2) $f(\beta)=(\pi^2\beta^4+4\pi\beta^3-2\pi^2\beta^2-4\beta^2-12\pi\beta-12)$. It is obtained from the expression $\mu_2>\mu_1'$.

- If $\beta<1.57866$ then $f(\beta)<0$. So, SBPNLED will be over-dispersed.
- If $\beta=1.57866$ then $f(\beta)=0$. So, SBPNLED will be equal-dispersed.
- If $\beta>1.57866$ then $f(\beta)>0$. So, SBPNLED will be under-dispersed.

Property (3) SBPNLED is positively skewed because $(2/\sqrt{3})<\gamma_1<\infty$, where γ_1 is co-efficient of skewness based on moments and it can be given as

$$\gamma_1=\frac{(\pi^3\beta^5+3\pi^3\beta^4+7\pi^2\beta^4+2\pi^3\beta^3+24\pi^2\beta^3+16\pi\beta^3+18\pi^2\beta^2+54\pi\beta^2+12\beta^2+36\pi\beta+36\beta+24)}{(\sqrt{2})(\pi^2\beta^3+\pi^2\beta^2+5\pi\beta^2+6\pi\beta+6\beta+6)^{3/2}} \quad (27)$$

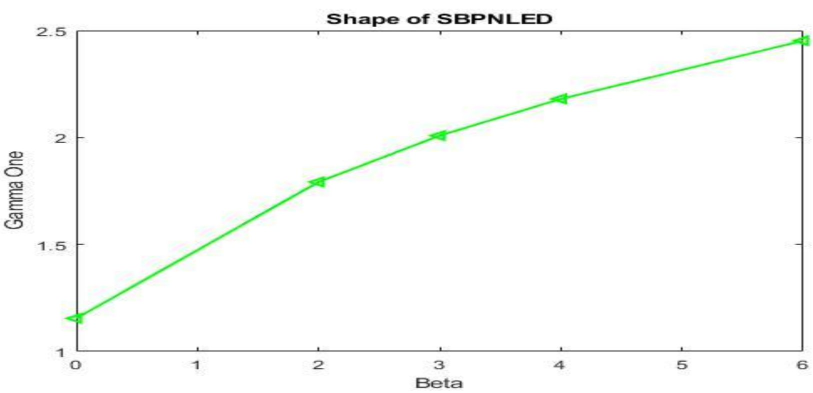

Figure.6

Property (4) SBPNLED is leptokurtic because $5<\beta_2<\infty$, where β_2 is co-efficient of kurtosis based on moments given by

$$\beta_2=\frac{\begin{array}{l}(\pi^4\beta^7+13\pi^4\beta^6+9\pi^3\beta^6+24\pi^4\beta^5+130\pi^3\beta^5+30\pi^2\beta^5+12\pi^4\beta^4\\ +264\pi^3\beta^4+460\pi^2\beta^4+44\pi\beta^4+144\pi^3\beta^3+736\pi^2\beta^3+\\ +696\pi\beta^3+24\beta^3+504\pi^2\beta^2+1368\pi\beta^2+384\beta^2+720\pi\beta+720\beta+360)\end{array}}{(2)(\pi^2\beta^3+\pi^2\beta^2+5\pi\beta^2+6\pi\beta+6\beta+6)^2} \tag{28}$$

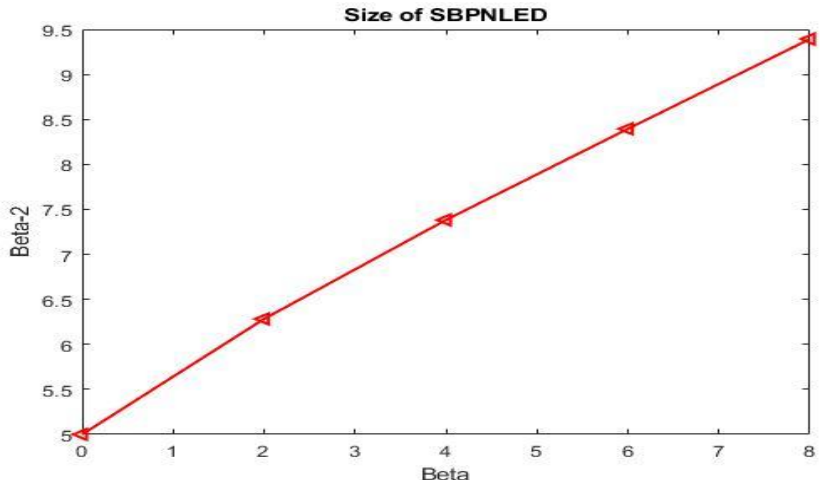

Figure.7

Property (5) The variance of SBPNLED increases as the mean increases.
Property (6) The co-efficient of skewness and co-efficient of kurtosis of SBPNLED decrease as the mean increases.

3.3 Estimation of Parameter of SBPNLED

This model has a single parameter β.To estimate value of β, we use (a) Method of Moments (MoM), and (b) Method of Maximum Likelihood (MML).

(a) Method of Moments (MoM): Replacing the population mean by the sample mean and solving the equation (19), we get a quadratic equation in terms of sample mean as

$$\pi(\bar{z}-1)\beta^2+(2\bar{z}-2-2\pi)\beta-6=0 \tag{29}$$

in terms of β.Solving the equation (29), we get an estimator $\hat{\beta}$ of β given by

$$\hat{\beta}=\frac{(1+\pi-\bar{z})+\sqrt{\left\{\bar{z}^2+2(2\pi-1)\bar{z}+(\pi^2-4\pi+1)\right\}}}{2\pi(\bar{z}-1)};\bar{z}>1 \tag{30}$$

(b) Method of Maximum Likelihood (MML):

Let us choose a random sample $(z_1,z_2,z_3,\ldots,z_n)$ of size n from population which follows SBPNLED with pmf (11), the maximum likelihood estimate (MLE) $\hat{\beta}$ of β is the solution of

$$\frac{3}{\beta}-\frac{\pi}{(2+\pi\beta)}-\frac{(\bar{z}+2)}{(1+\beta)}+\frac{\pi}{n}\sum_{i=1}^{n}\frac{1}{(1+\pi+\pi\beta+z_i)}=0 \tag{31}$$

Proof:

$$L=\left[\frac{\beta^3}{(2+\pi\beta)}\right]^n\left[\frac{1}{(1+\beta)^{\sum_{i=1}^{n}(z_i+2)}}\right]\prod_{i=1}^{n}\left\{(1+\pi+\pi\beta)z+z^2\right\}$$

Or, $\log L=n.\log\left[\frac{\beta^3}{(2+\pi\beta)}\right]-\left[\sum_{i=1}^{n}(z_i+2)\right]\left[\log(1+\beta)\right]+\sum_{i=1}^{n}\left\{(1+\pi+\pi\beta)z+z^2\right\}$

Or, $\frac{\delta(\log L)}{\delta\beta}=\left(\frac{3n}{\beta}\right)-\left(\frac{\pi n}{2+\pi\beta}\right)-\left(\frac{n(\bar{z}+2)}{1+\beta}\right)+\left[\sum_{i=1}^{n}\frac{\pi z_i}{z_i(1+\pi+\pi\beta+z_i)}\right]=0$

Finally, we get

Or, $\left(\frac{3}{\beta}\right)-\left(\frac{\pi}{2+\pi\beta}\right)-\left(\frac{(\bar{z}+2)}{1+\beta}\right)+\left[\frac{\pi}{n}\sum_{i=1}^{n}\frac{1}{(1+\pi+\pi\beta+z_i)}\right]=0$

Solving this equation, we, get an estimator $\hat{\beta}$ of β of SBPNLED.

Remarks:

(1) The MoM estimator $\hat{\beta}$ of β of SBPNLED is positively biased.

(2) The MoM estimator $\hat{\beta}$ of β of SBPNLED is consistent and asymptotically normal.

3.4 Applications and Chi-square goodness of fit test:

Size-biased distributions arise in several context in forestry, ecology, thunderstorm modelling, etc. Following are the examples used by previous researchers in which we have applied Chi square goodness of fit test. The first example is due to Cullen at al [3]. The second example is due to Keith and Meslow [7]. The third, fourth and fifth examples are due to Falls at al [5] related to number of thunderstorm activities in the months of September, August and for the season fall respectively.

Data (1)

z	1	2	3	4	5
Observed frequency	122	50	18	4	4

Data (2)

z	1	2	3	4	5
Observed frequency	184	55	14	4	4

Data (3)

z	1	2	3	4	5
Observed frequency	122	35	5	4	2

Data (4)

z	1	2	3	4	5
Observed frequency	201	60	10	3	2

Data (5)

z	1	2	3	4	5
Observed frequency	170	47	7	4	2

Table 1. Chi-square goodness of fit test to the data (1)

z_i	Observed Frequency	Theoretical frequency due to		
		SBPD	SBPLD	SBPNLED
1	122	111.3	119.0	119.3
2	50	64.1	53.8	53.5
3	18	18.5	18.0	17.9
4	4	3.5	5.3	5.3
5	4	0.6	1.9	2.0
	198	198.0	198.0	198.0
Mean	1.576			
$\hat{\beta}$		0.576	4.051	3.7256
$d.f.$		1	2	2
$\chi_{(d.f.)}^{2}$		4.642	0.433	0.358
$P-value$		0.031	0.805	0.836

Table 2. Chi-square goodness of fit test to the data (2)

z_i	Observed Frequency	Theoretical frequency due to		
		SBPD	SBPLD	SBPNLED
1	184	170.6	177.3	177.4
2	55	72.5	62.5	62.3
3	14	15.4	16.4	16.4
4	4	2.2	3.8	3.8
5	4	0.3	1.0	1.1
	261	261.0	261.0	261.0
Mean	1.425			
$\hat{\beta}$		0.425	5.351	4.9697
$d.f.$		1	1	1
$\chi_{(d.f.)}^{2}$		6.216	1.183	1.124
$P-value$		0.031	0.277	0.289

Table 3. Chi-square goodness of fit test to the data (3)

z_i	Observed Frequency	Theoretical frequency due to		
		SBPD	SBPLD	SBPNLED
1	122	114.0	117.9	117.9
2	35	44.0	38.4	38.3
3	5	8.5	9.3	9.3
4	4	1.1	2.0	2.0
5	2	0.3	0.4	0.5
	168	168.0	168.0	168.0
Mean	1.386905			
$\hat{\beta}$		0.386905	5.839181	5.440006
$d.f.$		1	1	1
$\chi_{(d.f.)}^{2}$		2.561	0.485	0.481
$P-value$		0.1095	0.486	0.488

Table 4. Chi-square goodness of fit test to the data (4)

z_i	Observed Frequency	Theoretical frequency due to		
		SBPD	SBPLD	SBPNLED
1	201	194.2	199.5	199.7
2	60	68.5	59.9	59.7
3	10	12.0	13.3	13.4
4	3	1.4	2.6	2.7
5	2	0.1	0.7	0.5
	276	276.0	276.0	276.0
Mean	1.351449			
$\hat{\beta}$		0.351449	6.374771	5.965111
$d.f.$		1	2	2
$\chi_{(d.f.)}^{2}$		1.414	0.165	0.164
$P-value$		0.2344	0.6846	0.6855

Table 5. Chi-square goodness of fit test to the data (5)

z_i	Observed Frequency	Theoretical frequency due to		
		SBPD	SBPLD	SBPNLED
1 2 3 4 5	170 47 7 4 2	161.7 56.9 10.0 1.2 0.2	166.2 49.9 11.1 2.2 0.6	166.3 49.8 11.2 2.2 0.5
	230	230.0	230.0	230.0
Mean $\hat{\beta}$ $d.f.$ $\chi_{(d.f.)}^{2}$ $P-value$	1.352174	 0.352174 1 2.372 0.124	 6.365473 1 0.313 0.576	 5.953321 1 0.298 0.585

In all the tables, along with the theoretical frequency obtained using the SBPNLE model, the theoretical frequencies obtained using SBP and SBPL models have been kept which makes comparison easy and simple. If we apply definition of size-biased distribution on the references [[15],[16],[17]] it is expected to get better fit to the similar kind of data than SBPLD.

CONCLUSION

- P- value obtained by using SBPNLE model is greater than those obtained by SBP and SBPL modelsHence, it is recommended to apply the proposed model instead of SBP and SBPL models to the similar nature of over-dispersed count data.
- SBPNLE model will be over-dispersed, equal-dispersed and under-dispersed as $\beta < 1.57866$, and $\beta = 1.57866$ and $\beta > 1.57866$ respectively. It is positively skewed and Leptokurtic by shape and size respectively.
- The estimator $\hat{\beta}$ of β is positively biased and it is consistent.

CONFLICT OF INTEREST

The authors of this paper have not their personal opinion in this paper, everything is based on facts.

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