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New Opial and Polya type inequalities over a spherical shell

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Abstract

Here we present general multivariate Opial and Polya type inequalities over spherical shells. The proofs derive by the use of some estimates coming out of some new trigonometric and hyperbolic Taylor's formulae ([1], [2]) and reducing the multivariate problem to a univariate one via general polar coordinates.

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1 Background

We need

Remark 1 *Let the spherical shell*

$$A := B(0, R_2) - \overline{B(0, R_1)},$$

$0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$; $r = |x|$, $r \in [R_1, R_2]$, $\forall\ x \in \overline{A}$, $|\cdot|$ *the Euclidean norm. Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see ([3], pp. 149-150 and [5], p. 421).*

Furthermore for $F : \overline{A} \to \mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega, \tag{1}$$

where $S^{N-1} := \left\{ x \in \mathbb{R}^N : |x| = 1 \right\}$.

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Let $d\omega$ be the element of surface measure on $\mathbb{S}^{N-1}$ with surface area

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{2}$$

Here it is volume of A,

$$vol\,(A) = \frac{\omega_N\left(R_2^N - R_1^N\right)}{N}. \tag{3}$$

Above it is $B(0,r) := \left\{x \in \mathbb{R}^N : |x| < r\right\}$, $r > 0$.

Here K is either $\mathbb{R}$ or $\mathbb{C}$, and $C_K^n(I)$ denotes functions n-times continuously differentiable on an interval $I \subset \mathbb{R}$ with values in K.

From [2] we need to mention the following Opial type inequalities.

Theorem 2 *Let $f \in C_K^2(I)$, with interval $I \subset \mathbb{R}$, $a, x \in I$, $a < x$, and $f(a) = f'(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)|\,|f''(w) + f(w)|\,dw \leq$$

$$2^{-\frac{1}{q}}\left(\int_a^x\left(\int_a^w |\sin(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x |f''(w) + f(w)|^q\,dw\right)^{\frac{2}{q}}. \tag{4}$$

Theorem 3 *Let $f \in C_K^2(I)$, $a, x \in I$, $a < x$, and $f(a) = f'(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)|\,|f''(w) - f(w)|\,dw \leq$$

$$2^{-\frac{1}{q}}\left(\int_a^x\left(\int_a^w |\sinh(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x |f''(w) - f(w)|^q\,dw\right)^{\frac{2}{q}}. \tag{5}$$

Theorem 4 *Let $f \in C_K^4(I)$, interval $I \subset \mathbb{R}$, let $a, x \in I$, $a < x$, $f(a) = f'(a) = f''(a) = f'''(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)|\left|f^{(iv)}(w) - f(w)\right|dw \leq$$

$$2^{-\left(1+\frac{1}{q}\right)}\left(\int_a^x\left(\int_a^w |\sinh(w-t) - \sin(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}} \tag{6}$$

$$\left(\int_a^x\left|f^{(iv)}(w) - f(w)\right|^q dw\right)^{\frac{2}{q}}.$$

Theorem 5 *All as in Theorem 4. Let* $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. *Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} |\alpha\beta| \left|\beta^2 - \alpha^2\right|} \left(\int_a^x \left(\int_a^w |\beta \sin(\alpha(w-t)) - \alpha \sin(\beta(w-t))|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{7}$$

Theorem 6 *All as in Theorem 4. Let* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1} |\alpha|^3} \left(\int_a^x \left(\int_a^w |\sin(\alpha(w-t)) - \alpha(w-t)\cos(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}} \tag{8}$$

$$\left(\int_a^x \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Theorem 7 *All as in Theorem 5. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} |\alpha\beta| \left|\beta^2 - \alpha^2\right|} \left(\int_a^x \left(\int_a^w |\alpha \sinh(\beta(w-t)) - \beta \sinh(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}} \tag{9}$$

$$\left(\int_a^x \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Theorem 8 *All as in Theorem 6. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1} |\alpha|^3} \left(\int_a^x \left(\int_a^w |\alpha(w-t)\cosh(\alpha(w-t)) - \sinh(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}} \tag{10}$$

$$\left(\int_a^x \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

We will use the above Opial type inequalities in the case of $p = q = 2$.
The motivation came from the following famous Opial's inequality

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Theorem 9 *(Z. Opial, 1960, [4]) Let $c > 0$ and $y(x)$ be real, continuously differentiable on $[0, c]$, with $y(0) = y(c) = 0$. Then*

$$\int_0^c |y(x)\, y'(x)|\, dx \leq \frac{c}{4} \int_0^c (y'(x))^2\, dx. \tag{11}$$

Equality holds for the function $y(x) = x$ on $\left[0, \frac{c}{2}\right]$ and $y(x) = c - x$ on $\left[\frac{c}{2}, c\right]$.

2 Results

First we present a collection of Opial type inequalities on the spherical shell A.

Theorem 10 *Let $F : \overline{A} \to \mathbb{R}$ Lebesgue integrable function with $F(\cdot\omega) \in C^2([R_1, R_2])$, with $F(R_1\omega) = \frac{\partial F}{\partial r}(R_1\omega) = 0$, $\forall\ \omega \in S^{N-1}$. Then*

$$\int_A |F(x)| \left| F(x) + \frac{\partial^2 F(x)}{\partial r^2} \right| dx \leq$$

$$2^{-\frac{1}{2}} \left(\frac{R_2}{R_1}\right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\sin(r-t))^2\, dt \right) dr \right)^{\frac{1}{2}} \int_A \left(F(x) + \frac{\partial^2 F(x)}{\partial r^2} \right)^2 dx. \tag{12}$$

Proof. Here we apply Theorem 2 to $F(\cdot\omega)$ for $p = q = 2$. So for every $\omega \in S^{N-1}$ we have that

$$\int_{R_1}^{R_2} |F(r\omega)| \left| \frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega) \right| dr \leq$$

$$2^{-\frac{1}{2}} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\sin(r-t))^2\, dt \right) dr \right)^{\frac{1}{2}} \left(\int_{R_1}^{R_2} \left(\frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega) \right)^2 dr \right). \tag{13}$$

We have $R_1 \leq r \leq R_2$ and $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, and $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$.

We observe the following

$$R_2^{1-N} \int_{R_1}^{R_2} |F(r\omega)| \left| \frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega) \right| r^{N-1} dr \leq \tag{14}$$

$$\int_{R_1}^{R_2} |F(r\omega)| \left| \frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega) \right| r^{N-1} r^{1-N} dr =$$

$$\int_{R_1}^{R_2} |F(r\omega)| \left| \frac{\partial^2 F(r\omega)}{\partial r^2} + F(r\omega) \right| dr \overset{(13)}{\leq}$$

$$2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}}$$
$$\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}r^{1-N}dr\right)\leq$$
$$R_1^{1-N}2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}} \tag{15}$$
$$\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}dr\right).$$

Therefore it holds

$$\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right|r^{N-1}dr\leq$$
$$\left(\frac{R_1}{R_2}\right)^{1-N}2^{-\frac{1}{2}}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}} \tag{16}$$
$$\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}dr\right).$$

Consequently we obtain

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2}|F(r\omega)|\left|\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right|r^{N-1}dr\right)d\omega\leq$$
$$2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sin(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}} \tag{17}$$
$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2}\left(\frac{\partial^2 F(r\omega)}{\partial r^2}+F(r\omega)\right)^2 r^{N-1}dr\right)d\omega.$$

Applying (1) we obtain (12). ∎

Next, we present more Opial type inequalities on spherical shell. Their proofs are similar to the proof of Theorem 10 and are based on Theorems 3-8. Use also of (1).

Theorem 11 *Same assumptions as in Theorem 10. Then*

$$\int_A |F(x)|\left|F(x)-\frac{\partial^2 F(x)}{\partial r^2}\right|dx\leq$$
$$2^{-\frac{1}{2}}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\int_{R_1}^{R_2}\left(\int_{R_1}^{r}(\sinh(r-t))^2\,dt\right)dr\right)^{\frac{1}{2}}\int_A\left(F(x)-\frac{\partial^2 F(x)}{\partial r^2}\right)^2dx. \tag{18}$$

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Proof. Based on (5). ■

Theorem 12 *Let $F : \overline{A} \to \mathbb{R}$ Lebesgue integrable function with $F(\cdot\omega) \in C^4([R_1, R_2])$, with $F(R_1\omega) = \frac{\partial^{(i)} F}{\partial r^{(i)}}(R_1\omega) = 0$, $i = 1, 2, 3$; $\forall\ \omega \in S^{N-1}$. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} - F(x) \right| dx \leq$$

$$2^{-\frac{3}{2}} \left(\frac{R_2}{R_1}\right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\sinh(r-t) - \sin(r-t))^2 \, dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} - F(x) \right)^2 dx. \tag{19}$$

Proof. Based on (6). ■

Theorem 13 *All as in Theorem 12. Let $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} + \left(\alpha^2 + \beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x) \right| dx \leq$$

$$\frac{1}{\sqrt{2}\left|\alpha\beta\left(\beta^2 - \alpha^2\right)\right|} \left(\frac{R_2}{R_1}\right)^{N-1}$$

$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\beta \sin(\alpha(r-t)) - \alpha \sin(\beta(r-t)))^2 \, dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} + \left(\alpha^2 + \beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2\beta^2 F(x) \right)^2 dx. \tag{20}$$

Proof. Based on (7). ■

Theorem 14 *All as in Theorem 12. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} + 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right| dx \leq$$

$$\frac{1}{2^{\frac{3}{2}}\left|\alpha^3\right|} \left(\frac{R_2}{R_1}\right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} (\sin(\alpha(r-t)) - \alpha(r-t)\cos(\alpha(r-t)))^2 \, dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} + 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right)^2 dx. \tag{21}$$

Proof. Based on (8). ■

Theorem 15 *All as in Theorem 13. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} - \left(\alpha^2 + \beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2 \beta^2 F(x) \right| dx \leq$$

$$\frac{1}{\sqrt{2} \left| \alpha\beta \left(\beta^2 - \alpha^2\right) \right|} \left(\frac{R_2}{R_1} \right)^{N-1}$$

$$\left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\alpha \sinh\left(\beta (r-t)\right) - \beta \sinh\left(\alpha (r-t)\right)\right)^2 dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} - \left(\alpha^2 + \beta^2\right) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2 \beta^2 F(x) \right)^2 dx. \tag{22}$$

Proof. By (9). ■

Finally we give the following Opial type inequality.

Theorem 16 *All as in Theorem 14. Then*

$$\int_A |F(x)| \left| \frac{\partial^4 F(x)}{\partial r^4} - 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right| dx \leq$$

$$\frac{1}{2^{\frac{3}{2}} |\alpha|^3} \left(\frac{R_2}{R_1} \right)^{N-1} \left(\int_{R_1}^{R_2} \left(\int_{R_1}^{r} \left(\alpha (r-t) \cosh\left(\alpha (r-t)\right) - \sinh\left(\alpha (r-t)\right)\right)^2 dt \right) dr \right)^{\frac{1}{2}}$$

$$\int_A \left(\frac{\partial^4 F(x)}{\partial r^4} - 2\alpha^2 \frac{\partial^2 F(x)}{\partial r^2} + \alpha^4 F(x) \right)^2 dx. \tag{23}$$

Proof. Based on (10). ■

We need the following results.

Theorem 17 *([1]) For $f \in C_K^2([a,b])$ and $x \in [a,b] : f(a) = f'(a) = 0$, we have that*

$$f(x) = \int_a^x \left(f''(t) + f(t)\right) \sin(x-t)\, dt, \tag{24}$$

and

$$f(x) = \int_a^x \left(f''(t) - f(t)\right) \sinh(x-t)\, dt. \tag{25}$$

Theorem 18 *([1]) For $f \in C_K^4([a,b])$ and $x \in [a,b] : f(a) = f'(a) = f''(a) = f'''(a) = 0$, we have that*

$$f(x) = \int_a^x \left(f''''(t) - f(t)\right) \left(\frac{\sinh(x-t) - \sin(x-t)}{2} \right) dt. \tag{26}$$

Theorem 19 *([1]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\beta^2 - \alpha^2\right) \neq 0$, and $f \in C_K^4([a,b])$, $x \in [a,b] : f(a) = f'(a) = f''(a) = f'''(a) = 0$. Then*

$$f(x) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left(f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right)$$

$$(\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t)))\, dt. \tag{27}$$

Theorem 20 *([1]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$, and $f \in C_K^4([a,b])$, $x \in [a,b] : f(a) = f'(a) = f''(a) = f'''(a) = 0$. Then*

$$f(x) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left(f''''(t) - \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right)$$

$$(\alpha \sinh(\beta(x-t)) - \beta \sinh(\alpha(x-t)) -)\, dt. \tag{28}$$

We will use

$$|\sin x| \leq |x|, \quad \forall\, x \in \mathbb{R}, \tag{29}$$

$$|\sinh x| \leq \cosh(b-a)\,|x|, \quad \forall\, x \in [-(b-a), b-a]. \tag{30}$$

Both of the above come by applications of mean value theorem.

We give the following Polya type univariate inequalities.

Theorem 21 *For $f \in C_K^2([a,b])$ and $x \in [a,b] : f(a) = f'(a) = 0$, it holds*

$$\int_a^b |f(x)|\, dx \leq \frac{(b-a)^2}{2} \int_a^b |f''(t) + f(t)|\, dt, \tag{31}$$

and

$$\int_a^b |f(x)|\, dx \leq \cosh(b-a) \frac{(b-a)^2}{2} \int_a^b |f''(t) - f(t)|\, dt. \tag{32}$$

Proof. (i) By (24) we have that

$$|f(x)| \leq \int_a^x |f''(t) + f(t)|\, |\sin(x-t)|\, dt \leq$$

$$\int_a^x |f''(t) + f(t)|\, (x-t)\, dt \leq \tag{33}$$

$$(x-a) \int_a^x |f''(t) + f(t)|\, dt \leq (x-a) \int_a^b |f''(t) + f(t)|\, dt.$$

Therefore, it holds

$$\int_a^b |f(x)|\, dx \leq \left(\int_a^b (x-a)\, dx\right) \int_a^b |f''(t) + f(t)|\, dt = \tag{34}$$

$$\frac{(b-a)^2}{2} \int_a^b |f''(t) + f(t)|\, dt.$$

(ii) By (25) we have that

$$|f(x)| \leq \int_a^x |f''(t) - f(t)|\, |\sinh(x-t)|\, dt \leq$$

$$\cosh(b-a) \int_a^x |f''(t) - f(t)|\, (x-t)\, dt \leq \tag{35}$$

$$\cosh(b-a)(x-a) \int_a^x |f''(t) - f(t)|\, dt \leq$$

$$\cosh(b-a)(x-a) \int_a^b |f''(t) - f(t)|\, dt.$$

Therefore, we get

$$\int_a^b |f(x)|\, dx \leq \cosh(b-a) \left(\int_a^b (x-a)\, dx \right) \int_a^b |f''(t) - f(t)|\, dt = \tag{36}$$

$$\cosh(b-a) \frac{(b-a)^2}{2} \int_a^b |f''(t) - f(t)|\, dt.$$

■

Theorem 22 *All as in Theorem 18. Then*

$$\int_a^b |f(x)|\, dx \leq (\cosh(b-a) + 1) \frac{(b-a)^2}{4} \int_a^b |f''''(t) - f(t)|\, dt. \tag{37}$$

Theorem 23 *All as in Theorem 19. Then*

$$\int_a^b |f(x)|\, dx \leq \frac{(b-a)^2}{|\beta^2 - \alpha^2|} \int_a^b \left| f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2 \beta^2 f(t) \right| dt. \tag{38}$$

Theorem 24 *All as in Theorem 20, plus* $|\alpha|, |\beta| < 1$. *Then*

$$\int_a^b |f(x)|\, dx \leq \frac{\cosh(b-a)(b-a)^2}{|\beta^2 - \alpha^2|} \int_a^b \left| f''''(t) - \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2 \beta^2 f(t) \right| dt. \tag{39}$$

Next comes a collection of Polya type inequalities on the spherical shell. Their proofs are based on Theorems 21-24, (1) and they are similar to the proof of Theorem 10, and as such details are omitted.

Theorem 25 *Same assumptions as in Theorem 10. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \frac{(R_2 - R_1)^2}{2} \int_A \left| \frac{\partial^2 F(x)}{\partial r^2} + F(x) \right| dx, \tag{40}$$

and

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \cosh(R_2 - R_1) \frac{(R_2 - R_1)^2}{2} \int_A \left| \frac{\partial^2 F(x)}{\partial r^2} - F(x) \right| dx. \tag{41}$$

Proof. Based on Theorem 21. ■

Theorem 26 *Same assumptions as in Theorem 12. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} (\cosh(R_2 - R_1) + 1) \frac{(R_2 - R_1)^2}{4} \int_A \left| \frac{\partial^4 F(x)}{\partial r^4} - F(x) \right| dx. \tag{42}$$

Proof. Based on Theorem 22. ■

Theorem 27 *All as in Theorem 13. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \frac{(R_2 - R_1)^2}{|\beta^2 - \alpha^2|} \int_A \left| \frac{\partial^4 F(x)}{\partial r^4} + (\alpha^2 + \beta^2) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2 \beta^2 F(x) \right| dx. \tag{43}$$

Proof. Based on Theorem 23. ■

We finish with

Theorem 28 *All as in Theorem 13, plus $|\alpha|, |\beta| < 1$. Then*

$$\int_A |F(x)| \leq \left(\frac{R_2}{R_1}\right)^{N-1} \frac{\cosh(R_2 - R_1)(R_2 - R_1)^2}{|\beta^2 - \alpha^2|} \int_A \left| \frac{\partial^4 F(x)}{\partial r^4} - (\alpha^2 + \beta^2) \frac{\partial^2 F(x)}{\partial r^2} + \alpha^2 \beta^2 F(x) \right| dx. \tag{44}$$

Proof. Based on Theorem 24. ■

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