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# Edge Domination Decomposition of Brick Product Graphs

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**Abstract**

A decomposition $\{G_1, G_2, \cdots, G_n\}$ is said to be an edge domination decomposition of a connected graph $G$ if $\gamma'(G_j) = \mathrm{j}, 1 \leq \mathrm{j} \leq \mathrm{n}$ where $\gamma'(G_j)$ is called the Edge Domination number of $G_j$. In this paper, we introduce the concept of Edge Domination Decomposition of graphs and study its behavior in Brick Product graphs, establishing several structural properties.

**Keywords:** Edge Dominating Set, Edge Domination number, Edge Domination Decomposition.

**AMS Subject Classification 05C31, 05C70**

## 1. Introduction

We consider only finite, simple and undirected graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $p = |V(G)|$ and $q = |E(G)|$. For standard terminologies and notations, we refer [4]. The concept of edge domination was introduced by Mitchell and Hedetniemi [6]. Ebin Raja Merly and Jeya Jothi [2] introduced Connected Domination Decomposition of Graphs. Motivated by the above results, we introduce a new concept, namely the edge domination decomposition of graphs. The necessary definitions and preliminary theorems are presented below.

**Definition 1.1.** [6] A subset F of E is called an edge dominating set of G if every edge not in F is adjacent to some edge in F. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G.

**Definition 1.2.** [2] A decomposition of a graph G is a collection of edge disjoint subgraphs $\{G_1, G_2, \cdots, G_n\}$ of G such that every edge of G belongs to exactly one $G_j, 1 \leq \mathrm{j} \leq \mathrm{n}$.

**Definition 1.3.** [3] A decomposition $\{G_1, G_2, \ldots G_n\}$ is said to be an edge domination decomposition (EDD) of a connected graph G if $\gamma'(G_j) = \mathrm{j}, 1 \leq \mathrm{j} \leq \mathrm{n}$.

Obviously $\sum_{j=1}^{n} \gamma'(G_j) = \frac{n(n+1)}{2}$.

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**Definition 1.4.** [3] If G admits EDD$\{G_1, G_2, \dots, G_n\}$ then G is said to be an edge domination decomposable graph and is denoted by $G_{EDD}$.

**Definition 1.5.** [8] Let $m, n$ and $r$ be positive integers. Let $C_{2n} = a_0, a_1, a_2, \dots, a_{2n-1}, a_0$ denote a cycle of order $2n$. The ( $m, r$ )-brick product of $C_{2n}$, denoted by $C(2n, m, r)$ is defined in two cases as follows:

(1) For $m = 1$, we require that $r$ be odd and greater than 1. Then $C(2n, 1, r)$ is obtained from $C_{2n}$ by adding chords $a_{2k}a_{2k+r}$, where $k = 1,2, \dots, n$, and the indices are computed modulo $2n$.
(2) For $m > 1$, we require that $m + r$ be even. Then $C(2n, m, r)$ is obtained as follows.

First, take the disjoint union of $m$ copies of $C_{2n}$, namely

$C_{2n}^{(1)}, C_{2n}^{(2)}, \dots, C_{2n}^{(m)}$,where for each $i = 1,2, \dots, m, V\left(C_{2n}^{(i)}\right) = \{(i, 0), (i, 1), \dots, (i, 2n - 1)\}$.

Next, for each odd $i = 1,3,5, \dots, m - 1$ and each even $k = 0,2,4, \dots, 2n - 2$, add an edge (called a brick edge) joining $(i, k)$ to $(i + 1, k)$.

Similarly, for each even $i = 2,4,6, \dots, m - 1$ and each odd $k = 1,3,5, \dots, 2n - 1$, add an edge (also called a brick edge) joining $(i, k)$ to $(i + 1, k)$.

Finally, for each odd $k = 1,3,5, \dots, 2n - 1$, add an edge (called a hooking edge) joining $(1, k)$ to $(m, k + r)$, where the second coordinate is taken modulo $2n$.

An edge in $C(2n, m, r)$ which is neither a brick edge nor a hooking edge is called a flat edge.

**Theorem 1.6.** [3] G admits EDD$\{G_1, G_2, \dots, G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n} \gamma'(G_j)$.

**Theorem 1.7.** [8] Let $G = C(2p, 1,3)$. Then $\gamma'(G) = \left\lceil \frac{2p}{3} \right\rceil$, for $p > 3$, where $2p \equiv k(\text{mod}3)$ and $k = 1,2$.

**Theorem 1.8.** [8] Let $G = C(2p, 1, r)$. Then $\gamma'(G) = \left\lceil \frac{2p}{3} \right\rceil$ for the following cases:

(i) $r = 5$ and $p > 3$,
(ii) $r = 7$ and $p > 4$,
(iii) $r = n$ and $p > 4$,

where $2p \equiv k(\text{mod}3)$ and $k = 1,2$.

**Theorem 1.9.** [8] Let $G = C(2p, 1,3)$. Then $\gamma'(D_{sd}(G, \{2\})) = \begin{cases} 2p - 2, & p = 4,5,7 \\ 2p - 4, & p \geq 8, \end{cases}$ where $2n \equiv k(\text{mod}3)$ and $k = 1,2$.

**Theorem 1.10.** [8] Let $G = C(2p, 1,5)$. Then Then $\gamma'(D_{sd}(G, \{2\})) = 2p - 2$ for $n > 3$.

**Theorem 1.11.** [8] Let $G = C(2p, 1,5)$. Then $\gamma'(D_{sd}(G, \{3\})) = 2p$ for $p > 3$.

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**Theorem 1.12.** [8] Let $G = C(2p, 1,7)$. Then $\gamma'(D_{sd}(G, \{2\})) = 2p - 2$ for $p > 4$.

**Theorem 1.13.** [8] Let $G = C(2p, 1,7)$. Then $\gamma'(D_{sd}(G, \{3\})) = 2p$ for $p > 3$.

## 2. Edge Domination Decomposition of Brick product Graphs

**Theorem 2.1.** The graph $C(2p, 1,3)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $n = \frac{-1+\sqrt{1+8s/3}}{2}$ where $s = \begin{cases} 2p+2 & \text{if } p \equiv 2(\text{mod}3), \\ 2p+1 & \text{if } p \equiv 1(\text{mod}3) \end{cases}$

**Proof.** By Theorem 1.6, G admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n} \gamma'(G_j)$. By the definition of EDD, the graph $C(2p, 1,3)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(C(2p, 1,3)) = \frac{n(n+1)}{2}$. By theorem 1.7$\gamma'(C(2p, 1,3)) = \left\lceil \frac{2p}{3} \right\rceil$,

**Case (i):** $p \equiv 1(\text{mod}3)$

$\gamma'(C(2p, 1,3)) = \frac{2p+1}{3}$. So we have $\frac{n(n+1)}{2} = \frac{2p+1}{3}$. This implies that $p = \frac{3n^2+3n-2}{4}$. Therefore $n = \frac{-1\pm\sqrt{1+8(2p+1)/3}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+8(2p+1)/3}}{2}$.

| **p** | **n** | $\boldsymbol{\gamma'(C(2p, 1, 3))}$ |
|---|---|---|
| 4 | 3 | 6 |
| 22 | 5 | 15 |
| 31 | 6 | 21 |

**Case (ii):** $p \equiv 2(\text{mod}3)$

We have $\frac{n(n+1)}{2} = \frac{2p+2}{3}$. This implies that $p = \frac{3n^2+3n-4}{4}$. Therefore $n = \frac{-1\pm\sqrt{1+8p/3}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+8(2p+2)/3}}{2}$.

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| p | n | $\gamma'(C(2p,1,3))$ |
|---|---|---|
| 8 | 3 | 6 |
| 14 | 4 | 10 |
| 41 | 7 | 28 |

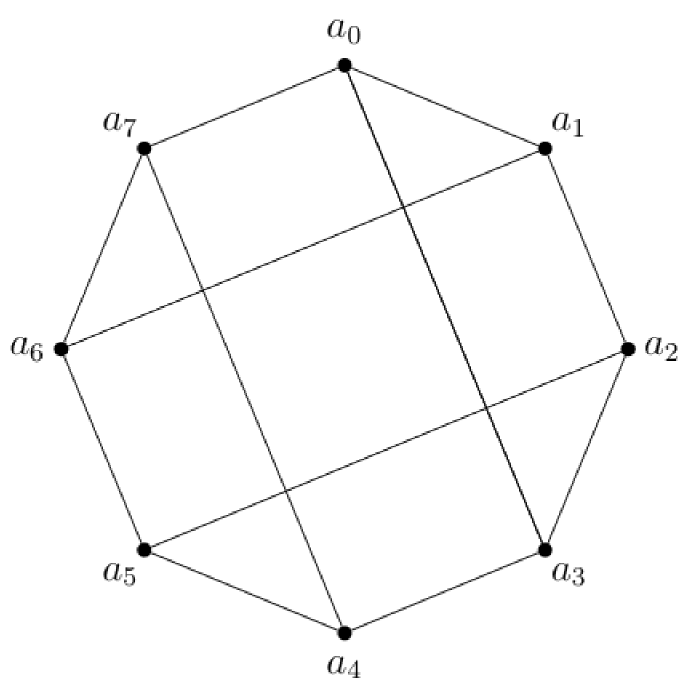


**Figure 1: $C(8,1,3)$**

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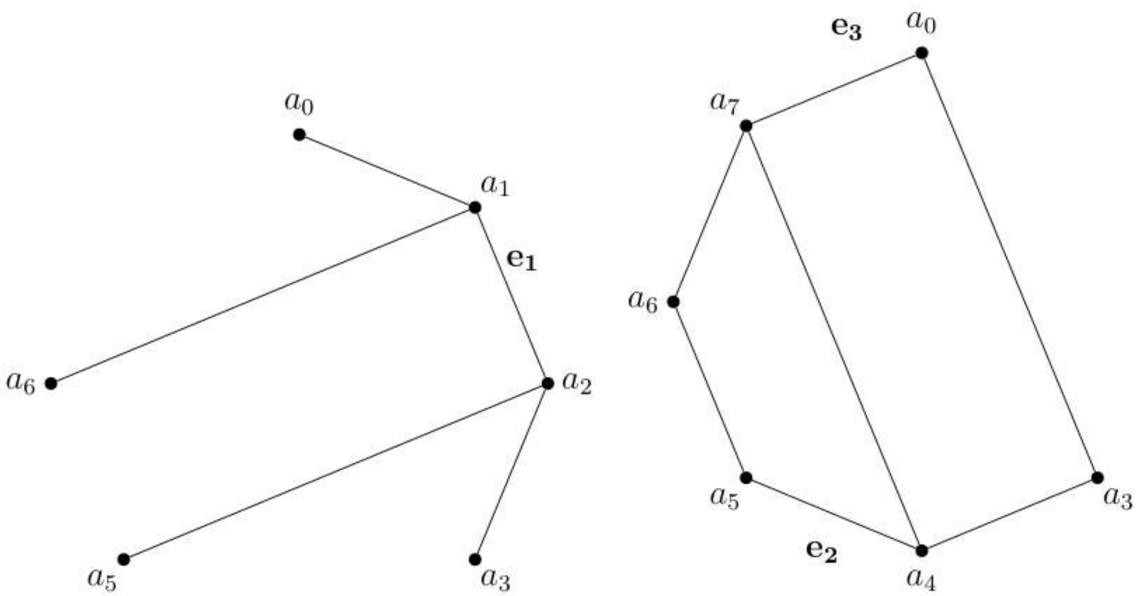


**Figure 2: EDD$\{G_1, G_2\}$**

**Theorem 2.2.** The graph $D_{sd}(G,\{2\}), G = C(2p,1,3)$ admits EDD$\{G_1, G_2, \dots G_n\}$ if and only if $n = \frac{-1+\sqrt{1+16(p-2)}}{2}$.

**Proof.** By Theorem 1.6, G admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n} \gamma'(G_j)$. By the definition of EDD, the graph $C(2p,1,3)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(D_{sd}(G,\{2\})) = \frac{n(n+1)}{2}$. By Theorem 1.9 $\gamma'(D_{sd}(G,\{2\})) = \begin{cases} 2p-2, & p = 4,5,7 \\ 2p-4, & p \geq 8, \end{cases}$ for $p \geq 8, \gamma'(C(2p,1,3)) = 2p-4$. So we have $\frac{n(n+1)}{2} = 2p-4$. This implies that $p = \frac{n^2+n+8}{4}$.

Therefore $n = \frac{-1\pm\sqrt{1+16(p-2)}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+16(p-2)}}{2}$.

| **p** | **n** | $\boldsymbol{\gamma'(C(2p,1,3))}$ |
|---|---|---|
| 16 | 7 | 28 |
| 20 | 8 | 36 |
| 35 | 11 | 66 |
| 41 | 12 | 78 |

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**Theorem 2.3.** The graph $D_{sd}(G,\{2\}), G = C(2p,1,5)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $n = \frac{-1+\sqrt{1+16(p-1)}}{2}$.

**Proof.** By Theorem 1.6, G admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n} \gamma'(G_j)$. By the definition of EDD, the graph $C(2p,1,5)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma' D_{sd}(G,\{2\}) = \frac{n(n+1)}{2}$. By theorem 1.10$\gamma' D_{sd}(G,\{2\}) = 2p - 2$ for $p \geq 3$, So we have $\frac{n(n+1)}{2} = 2p - 2$. This implies that $p = \frac{n^2+n+4}{4}$. Therefore $n = \frac{-1\pm\sqrt{1+16(p-1)}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+16(p-1)}}{2}$.

| **p** | **n** | $\boldsymbol{\gamma' D_{sd}(G,\{2\})}$ |
|---|---|---|
| 4 | 3 | 6 |
| 6 | 4 | 10 |
| 15 | 7 | 28 |
| 19 | 8 | 36 |

**Theorem 2.4.** The graph $D_{sd}(G,\{3\}), G = C(2p,1,5)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $n = \frac{-1+\sqrt{1+16p}}{2}$.

**Proof.** By Theorem 1.6, G admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n} \gamma'(G_j)$. By the definition of EDD, the graph $C(2p,1,5)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(D_{sd}(G,\{3\})) = \frac{n(n+1)}{2}$. By theorem 1.11$\gamma'(D_{sd}(G,\{3\})) = 2p$ for $p > 3$. So we have $\frac{n(n+1)}{2} = 2p$. This implies that $p = \frac{n^2+n}{4}$. Therefore $n = \frac{-1\pm\sqrt{1+16p}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+16p}}{2}$.

| **p** | **n** | $\boldsymbol{\gamma'(D_{sd}(G,\{3\}))}$ |
|---|---|---|
| 5 | 4 | 10 |
| 14 | 7 | 28 |
| 18 | 8 | 36 |
| 33 | 11 | 66 |

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**Theorem 2.5.** The graph $D_{sd}(G,\{2\}), G = C(2p,1,7)$ admits EDD$\{G_1, G_2, \dots G_n\}$ if and only if $n = \frac{-1+\sqrt{1+16(p-1)}}{2}$.

**Proof.** By Theorem 1.6, G admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n}\gamma'(G_j)$. By the definition of EDD, the graph $C(2p,1,7)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(D_{sd}(G,\{2\})) = \frac{n(n+1)}{2}$. By theorem 1.12$\gamma'(D_{sd}(G,\{2\})) = 2p - 2$ for $p \geq 3$, So we have $\frac{n(n+1)}{2} = 2p - 2$. This implies that $p = \frac{n^2+n+4}{4}$. Therefore $n = \frac{-1\pm\sqrt{1+16(p-1)}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+16(p-1)}}{2}$.

| **p** | **n** | $\boldsymbol{\gamma'(D_{sd}(G,\{2\}))}$ |
|---|---|---|
| 4 | 3 | 6 |
| 6 | 4 | 10 |
| 15 | 7 | 28 |
| 19 | 8 | 36 |

**Theorem 2.6.** The graph $D_{sd}(G,\{3\}), G = C(2p,1,7)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $n = \frac{-1+\sqrt{1+16p}}{2}$.

**Proof.** By Theorem 1.6, G admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(G) = \sum_{j=1}^{n}\gamma'(G_j)$. By the definition of EDD, the graph $C(2p,1,7)$ admits EDD $\{G_1, G_2, \dots G_n\}$ if and only if $\gamma'(D_{sd}(G,\{3\})) = \frac{n(n+1)}{2}$. By theorem 1.13$\gamma'(D_{sd}(G,\{3\})) = 2p$ for $p > 3$. So we have $\frac{n(n+1)}{2} = 2p$. This implies that $p = \frac{n^2+n}{4}$. Therefore $n = \frac{-1\pm\sqrt{1+16p}}{2}$. But n is always positive. Hence $n = \frac{-1+\sqrt{1+16p}}{2}$.

| **p** | **n** | $\boldsymbol{\gamma'(D_{sd}(G,\{3\}))}$ |
|---|---|---|
| 5 | 4 | 10 |
| 14 | 7 | 28 |
| 18 | 8 | 36 |
| 33 | 11 | 66 |

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## 3. Conclusion

In this paper, we introduce the concept of Edge Domination Decomposition of graphs and investigate its structural properties in Brick Product graphs. Moreover, this concept can be further extended to obtain additional results concerning the Edge Domination Decomposition of graphs.

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