

On the Restrained Geo Chromatic Number of a Graph

Dr. P. Kamal Devi

Associate Professor, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil-629003.

kdeviprasad2011@gmail.com

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirrunelveli – 627012, Tamilnadu, India

Abstract

A set S of vertices is a geodetic set if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the geodetic number of G and is denoted by $g(G)$. A set S of vertices in G is a restrained geodetic set if S is a geodetic set and the subgraph $G[V \setminus S]$ induced by $V \setminus S$ has no isolated vertex. A subset $S \subseteq V(G)$ is said to be a geo chromatic set if S is both a geodetic and a chromatic set if S is both a geodetic and a chromatic set of G . A subset $S \subseteq V(G)$ is a restrained geo chromatic set of G if S is a geo chromatic set and the subgraph $G[V \setminus S]$ induced by $V \setminus S$ has no isolated vertex. The minimum cardinality of a restrained geo chromatic number of G and is denoted by $\chi_{gr}(G)$. In this paper, we determine bounds for it and characterize graphs which realize these bounds for some known graphs.

Keywords:

Geo Chromatic set – Geo Chromatic number – Restrained Geo Chromatic set – Restrained Geo Chromatic number.

Mathematics Subject Classification (2010) : 05C12

1. Introduction

By a graph $G = (V, E)$, we mean a simple undirected connected graph. The order and size of G are denoted by m and n , respectively. For basic theoretic terminology, we refer to Harary [1,8]. For any two vertices u and v in a connected graph G , the distance $d(x, y)$ is the length of a shortest $u - v$ path in G . A $u - v$ path of length $d(x, y)$ is called $u - v$ geodesic P if v is a vertex of P including the vertices u and v . For any vertex u of G , the eccentricity of u is defined as $e(u) = \max\{d(u, v); v \in V(G)\}$. The radius $r(G)$ and diameter $diam(G)$ of G are defined as $r(G) = \min\{e(v); v \in V(G)\}$ and $diam(G) = \max\{e(v); v \in V(G)\}$ respectively. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v . A vertex v of G is called an extreme vertex of G if the subgraph induced by its neighbours is complete.

The closed interval $I[x, y]$ consists of all vertices lying on some $u - v$ geodesic of G while for $S \subseteq V(G)$, $I[S] = \bigcup_{x, y \in S} I[x, y]$. A set S of vertices of G is a geodetic set of G if $I[S] = V(G)$, and the minimum cardinality of a geodetic set of G is the geodetic number $g(G)$ of G . The geodetic number of a graph and its variants have been studied several authors in [2,9,10]. A set S of vertices is called chromatic set if S contains all vertices of distinct colours in G . The chromatic number of G is the chromatic sets in G and is denoted by $\chi(G)$. A set S of vertices in G is called a restrained geodetic set of G if S is a geodetic set and the subgraph $G[V \setminus S]$ induced by $V \setminus S$ has no isolated vertex. The minimum cardinality among all the restrained geodetic set of G is the restrained geodetic number of G and is denoted by $g_r(G)$. A set $S \subseteq V(G)$ is called a geo chromatic set of G if S is both a geodetic set and a chromatic set of G . It was introduced and studied in [1].

Throughout this paper G denotes a connected graph with at least two vertices.

2. Main Results:

Definition 2.1

A set S of geo chromatic set of G is a restrained geo chromatic set such that the subgraph induced by $V \setminus S$ has no isolated vertices. The minimum cardinality of a restrained geo chromatic number of G and is denoted by $\chi_{g_r}(G)$. A restrained geo chromatic set of size $\chi_{g_r}(G)$ is called χ_{g_r} -set of G .

Example 2.2

For the graph G given in Figure 2.1, $S = \{v_1, v_2, v_4, v_6, v_3\}$ is a minimum geo chromatic set of G . But the subgraph induced by $V(G) \setminus S$ has a isolated vertex. Therefore, that S is not a restrained geo chromatic set of G .

Now it is it clear that $S_1 = S \cup \{v_5\}$ is a minimum restrained geo chromatic set of G . Hence, $\chi_g(G) = |S| = 5$ and $\chi_{g_r}(G) = 6$.

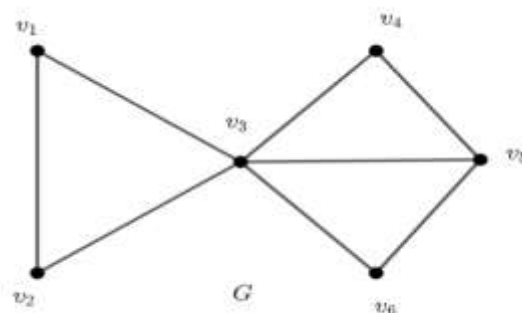


Figure 2.1

Here, the restrained geo chromatic number and the geo chromatic number are different.

Theorem 2.3

Each extreme vertex of a connected graph G belongs to every restrained geo chromatic set of G .

Proof

Let G be a connected graph and let S be a restrained geo chromatic set of G . Suppose $x \notin S$. Then by Theorem 1.1, that S is not a geodetic set of G . This implies that S is not a restrained geo chromatic set of G , which is a contradiction. Hence $x \in S$ and so every extreme vertex of G belongs to every restrained geo chromatic set of G .

Theorem 2.4

Let G be any connected graph and if the set of all extreme vertices S of G is a restrained geo chromatic set of G , then S is the unique minimum restrained geo chromatic of G .

Proof

Let G be a connected graph and S be the set of all extreme vertices of G , which is a restrained geo chromatic set of G . Then, $\chi_{g_r}(G) \leq |S|$. But by Theorem 2.3, $\chi_{g_r}(G) \geq |S|$. Hence, that S is a minimum restrained geo chromatic set of G . Moreover, by Theorem 2.3, every minimum restrained geo chromatic set contains S and so that S is the unique minimum restrained geo chromatic set of G .

Observation 2.5

Let G be a connected graph. Then the following are true.

- (i) A minimum restrained geodetic set S of G which is also a chromatic set of G , then S is a χ_{g_r} -set of G .
- (ii) A minimum chromatic set of G which is also a restrained geodetic set of G , then S is a χ_{g_r} -set of G .

Theorem 2.6

Let G be a connected graph with cut vertices and let S be a restrained geo chromatic set of G . If x is a cut vertex of G , then every component of $G - x$ contains an element of S .

Proof

Let x be a cut vertex of G and let S be a restrained geo chromatic set of G . Suppose there exists a component, say G_1 of $G - x$ such that G_1 contains no vertices of S . Then by Theorem 2.3, S contains all the extreme vertices of G . It follows that G_1 contains no extreme vertices of G . let $y \in V(G_1)$. (This is possible because G is connected). Since S is a restrained geo chromatic set of G , there exists $u, v \in S$ with x is an interval vertex of some $u - v$ geodesic path $P: u = x_0 x_1 x_2 \dots x_l = v$ in G . Since x is a cut vertex of G , the $u - x$ subpath of P and $x - v$ subpath of P both contains y . It follow that P is not a path, which is a contradiction. Hence, every component of $G - x$ contains an element of S .

Corollary 2.7

Let G be a connected graph with cut vertices and let S be a restrained geo chromatic set of G . Then, every branch of G contains an element of S .

Proof

Let G be a connected graph and let S be a restrained geo chromatic set of G . Let x be a cut vertex of G . Since every branch of G at x is a component of $G - x$ with the vertex x together all the edges joining at x to $V(G_1)$, by Theorem 2.6, we conclude that every branch of G contains an element of S .

Theorem 2.8

Let G be any connected graph and let $S \subseteq V(G)$ with S is a clique in G . Then S is a restrained geo chromatic set of G if and only if $S = V(G)$.

Proof

Let G be any connected graph and $S \subseteq V(G)$ be a complete subgraph S is a clique in G . First assume S is a restrained geo chromatic set of G . To prove $S = V(G)$. Suppose on the contrary, that $S \neq V(G)$. This means that $S \subset V(G)$. Let $x \in V(G) \setminus S$. Since the subgraph induced by S is complete, $d(u, v) = 1$ for every $u, v \in S$. It follow that x does not lie on a any geodesic path between any pair of vertices from S . It follows that S is not a restrained geo chromatic set of G , which is a contradiction. Hence $S = V(G)$. Conversely assume that $S = V(G)$. The set of all vertices of G form a restrained geo chromatic set of G . Thus, S is a restrained geo chromatic set of G .

Corollary 2.9

Let G be a non-complete connected graph. Then every restrained geo chromatic set of G contains at least two non-adjacent vertices.

Proof

Let G be a non-complete connected graph. Since G is non-complete, that G has at least two non-adjacent vertices. Let S be a restrained geo chromatic set of G . To show that S contains at least two non-adjacent vertices. If $S = V(G)$, then by hypothesis that S contains at least two non-adjacent vertices. Now suppose $S \neq V(G)$. To prove S contains at least two non-adjacent vertices. Suppose on the contrary that S does not contain any non-adjacent vertices of G . It follows that the subgraph of G induced by S is complete. Thus S is a clique in G . So by Theorem 2.8, that $S = V(G)$, which is a contradiction. Hence, every restrained geo chromatic set of G has at least two non-adjacent vertices.

Theorem 2.10

Let G be any connected graph of order $n \geq 2$. Let F be the set of all full degree vertices of G and S be a restrained geo chromatic set of G . Then, $F \subseteq S$.

Proof

Let $F = \{x_1, x_2, \dots, x_{k(k \leq n)}\}$ be the set of all full degree vertices of G . Then by definition, every vertices of that F is adjacent to all the vertices of G . Therefore, for a proper colouring vertices of F receive different colours say $c_1, c_2, \dots, c_k$. Let S be a restrained geo chromatic set of G . To prove $F \subseteq S$. Suppose $F \not\subseteq S$. Without loss of generality, we assume that $x_i \notin S$. Since x_i receive colour c_i , that c_i colours does not assign any vertices of S . It follows that S is not a restrained geo chromatic set of G , which is a contradiction. Hence $F \subseteq S$.

Theorem 2.11

Let G be any connected graph of order n with $g(G) = n - 1$. Then $\chi_{g_r}(G) = n$.

Proof

Let G be a connected graph of order n with $g(G) = n - 1$. Then by Theorem we have $G = K_1 + (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_r})$, where $n_1, n_2, \dots, n_r$ and r are positive integers with $n_1 + n_2 + \dots + n_r = n - 1$. Let $x \in V(K_1)$. Let $S = V(K_{n_1}) \cup V(K_{n_2}) \cup \dots \cup V(K_{n_r})$ be a geodetic set of G . Since $x \notin S$ and $\deg(x) = n - 1$, by Theorem 2.10, x in every restrained geo chromatic set of G . This show that S is not a chromatic set of G and so $\chi_{g_r}(G) \geq V(G)$. Thus, $V(G)$ is the only restrained geo chromatic set of G and so $\chi_{g_r}(G) \leq n$. Hence, $\chi_{g_r}(G) = n$.

Theorem 2.12

For any positive integer $2 \leq a \leq n$, there exists are connected graph G of order n and $\chi_{g_r}(G) = n$.

Proof

For $a = n$, we take G as the complete graph of n vertices. Then clearly G contains only the extreme vertices and so by Theorem 2.4, $\chi_{g_r}(G) = n$.

For $2 \leq a \leq n$, we consider the connected graph G as shown in Figure 2.2.

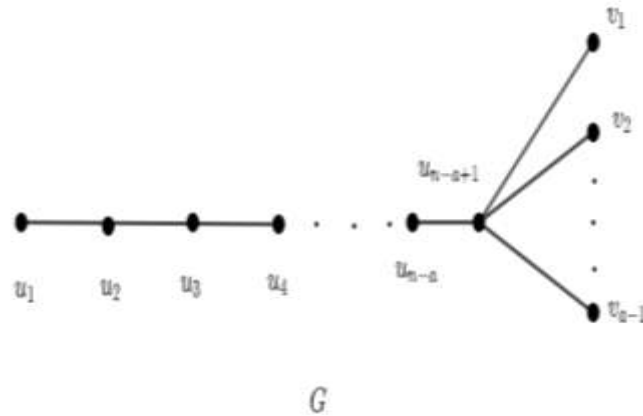


Figure 2.2

It is clear that the set of all extreme vertices of G form a restrained geo chromatic set of G and so $\chi_{gr}(G) = a$.

References

- [1] Beulah, S.S., & Robinson Chellathurai, S. (2018), *Geo Chromatic Number of Graphs*, International Journal of Scientific Research in Mathematics and Statistical Sciences, 5, 259-264.
- [2] Bondy, J.A., & Murthy, U.S.R., (1976), *Graph Theory with Application*, Macmillan London and Elsevier, Newyork.
- [3] Buckley, F., Harary, F., (1990), *Distance in Graphs*, Addition- Wesley, Red –wood city, CA,
- [4] Buckley, F., Harary, F., & Quintas, L.V., (1988), *Extremal Results on the Geodetic Number of a Graph*, Scientia A2, 17-26.
- [5] Chartrand, G., & Zhang., P, (1999), *The Forcing Geodetic Number of a Graph*, Discussiones Mathematicae Graph Theory 19, 45-58.
- [6] Chartrand, G., Harary, F., & Zhang., P, (2000), *Geodetic sets in Graphs*, Discussiones Mathematicae Graph Theory 20, 129-138.
- [7] Chartrand, G., & Zhang., P, (2000), *The Geodetic Number of an oriented Graph*, European J. Combin. 21, 181-189.
- [8] Chartrand, G., Harary, F., & Zhang., P, (2002), *On the Geodetic Number of a Graph*, Network, an International Journal, Vol.39, Issue 1, January.
- [9] D B West, (2015), *Introduction to Graph Theory*, Pearson Publishing.

- [10] Harary, F., (1969), *Graph Theory*, (Addison – Wesley, Reading, MA).
- [11] Abdollahzadeh Ahangar, H., Najimi. M., *Total Restrained Geodetic Number of Graphs*, Iran. J. Sci. Technol. Trans. A, Sci. (2017) in press.