Deductive systems and filters of Sheffer stroke Hilbert algebras based on the bipolar-valued fuzzy set environment

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Abstract The notion of bipolar-valued fuzzy set is used to treat the filter and deductive system in Sheffer stroke Hilbert algebras. The concepts of bipolar-valued fuzzy filter and bipolar-valued fuzzy deductive system are introduced and related properties are investigated. Conditions under which the bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter are explored. Characterizations of the bipolar-valued fuzzy filter are examined. A bipolar-valued fuzzy filter is built using a filter. To consider the nomality of bipolar-valued fuzzy filter, the notion of normal bipolar-valued fuzzy filter is introduced and related properties are investigated. The method of normalizing the bipolar-valued fuzzy filter is addressed, and we will see what the normal bipolar-valued fuzzy filter looks like.

Keywords: Sheffer stroke Hilbert algebra, filter, (bipolar-valued fuzzy) deductive system, (normal) bipolar-valued fuzzy filter.

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1 Introduction

The shaper stroke, denoted by the symbol "|", is a logical operation for two inputs that produces false results only when both inputs are true, as shown in Table 1.

Table 1: The truth table for the Sheffer stroke "|"

P	Q	$P\|Q$
F	F	T
F	T	T
T	F	T
T	T	F

The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra, MV-algebra, BL-algebra, BCK-algebra, and ortholattices, etc., and it is also being dealt with in the fuzzy environment (see [3, 5, 7, 11, 12, 13, 14, 15]). In 2021, Oner et al. [12] applied the Sheffer stroke to Hilbert algebras. They introduced Sheffer stroke Hilbert algebra and investigated several properties. In [11], Oner et al. introduced the notion of deductive system and filter of Sheffer stroke Hilbert algebras, and dealt with their fuzzification. The bipolar-valued fuzzy set, which is introduced by Lee [9, 10] is a type of fuzzy set where the degree of membership to a set is represented by a value that can take on both positive and negative values, as opposed to traditional fuzzy sets where the degree of membership is represented by a value between 0 and 1. The value 0 in the bipolar-valued fuzzy set represents a lack of information about membership or a neutral position. Also, the negative values represent the degree of non-membership, while the positive values represent the degree of membership to the set. The bipolar-valued fuzzy set is useful for methods such as modeling complex and uncertain situations beyond traditional fuzzy sets. Therefore, the bipolar-valued fuzzy set has been applied in various fields, such as pattern recognition, decision making, and control systems etc. The bipolar-valued fuzzy set has also been widely applied in algebraic structures (see [1, 2, 4, 6, 8])

In this paper, we introduce the notion of the bipolar-valued fuzzy deductive system and the bipolar-valued fuzzy filter in Sheffer stroke Hilbert algebras, and investigate several properties. We first show that the bipolar-valued fuzzy deductive system and the bipolar-valued fuzzy filter are equivalent each other. We explore the conditions under which a bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter. We establish characterization of the bipolar-valued fuzzy filter. Using the filter of Sheffer stroke Hilbert algebra, we make a bipolar-valued fuzzy filter. We discuss the nomality of bipolar-valued fuzzy filter, and we deal with how to normalize the bipolar-valued fuzzy filter. We look into what the normal bipolar-valued fuzzy filter looks like.

2 Preliminaries

Definition 2.1 ([16]). Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation "$|$" is said to be *Sheffer stroke* or *Sheffer operation* if it satisfies:

(s1) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ (\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a})$,

(s2) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ ((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a})$,

(s3) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ (\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c})$,

(s4) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ ((\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a})$.

Definition 2.2 ([12]). A *Sheffer stroke Hilbert algebra* is a groupoid $\mathcal{L} := (L, |)$ with a Sheffer stroke "$|$" that satisfies:

(sH1) $(\mathfrak{a}|((A)|(A)))|(((B)|((C)|(C)))|((B)|((C)|(C)))) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a})$, where $A := \mathfrak{b}|(\mathfrak{c}|\mathfrak{c})$, $B := \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})$ and $C := \mathfrak{a}|(\mathfrak{c}|\mathfrak{c})$,

(sH2) $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{b}|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) \ \Rightarrow\ \mathfrak{a} = \mathfrak{b}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L$.

Let $\mathcal{L} := (L, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation "$\leq_L$" on L is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq_L \mathfrak{b} \ \Leftrightarrow\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = 1). \tag{2.1}$$

We observe that the relation "$\leq_L$" is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ (see [12]).

Proposition 2.3 ([12]). *Every Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ satisfies:*

$$(\forall \mathfrak{a} \in L)(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}) = 1), \tag{2.2}$$
$$(\forall \mathfrak{a} \in L)(\mathfrak{a}|(1|1) = 1), \tag{2.3}$$
$$(\forall \mathfrak{a} \in L)(1|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}), \tag{2.4}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq_L \mathfrak{b}|(\mathfrak{a}|\mathfrak{a})), \tag{2.5}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = (\mathfrak{b}|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|\mathfrak{a})), \tag{2.6}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)\,(((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))\,, \tag{2.7}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\,(\mathfrak{a}|((\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{b}|(\mathfrak{c}|\mathfrak{c}))) = \mathfrak{b}|((\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))|(\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))))\,, \tag{2.8}$$

Definition 2.4 ([11]). Let $(L, |)$ be a Sheffer stroke Hilbert algebra. A subset F of L is called

- a *deductive system* of $(L, |)$ if it satisfies:

$$1 \in F, \tag{2.9}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \in F,\ \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in F \ \Rightarrow\ \mathfrak{b} \in F), \tag{2.10}$$

- a *filter* of $(L, |)$ if it satisfies (2.9) and

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{b} \in F \;\Rightarrow\; \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \in F), \tag{2.11}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)(\mathfrak{b}, \mathfrak{c} \in F \;\Rightarrow\; (\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c}) \in F). \tag{2.12}$$

Definition 2.5 ([11])**.** Let $(L, |)$ be a Sheffer stroke Hilbert algebra. A fuzzy set f in L is called a *fuzzy filter* of $(L, |)$ if it satisfies:

$$(\forall \mathfrak{a} \in L)(f(1) \geq f(\mathfrak{a})), \tag{2.13}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(f(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \geq f(\mathfrak{b})), \tag{2.14}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)(f((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \geq \min\{f(\mathfrak{b}), f(\mathfrak{c})\}). \tag{2.15}$$

Denote by $FS(L)$ the collection of all fuzzy sets in L. Define a relation " $\subseteq$ " on $FS(L)$ by

$$(\forall f, g \in FS(L))(f \subseteq g \;\Leftrightarrow\; (\forall \mathfrak{a} \in L)(f(\mathfrak{a}) \leq g(\mathfrak{a}))).$$

Consider two maps f^- and f^+ on L (; a universe of discourse) as follows:

$$f^- : L \to [-1, 0] \text{ and } f^+ : L \to [0, 1],$$

respectively. A structure

$$\mathfrak{f} := \{(\mathfrak{a}; f^-(\mathfrak{a}), f^+(\mathfrak{a})) \mid \mathfrak{a} \in L\}$$

is called a *bipolar-valued fuzzy set* on L (see [9]), and is will be denoted by simply $\mathfrak{f} := (L;\, f^-,\, f^+)$.

For a BVF-set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$L(f^-; s) := \{\mathfrak{a} \in L \mid f^-(\mathfrak{a}) \leq s\},$$

$$U(f^+; t) := \{\mathfrak{a} \in L \mid f^+(\mathfrak{a}) \geq t\}$$

which are called the *negative* s*-cut* and the *positive* t*-cut* of $\mathfrak{f} := (L;\, f^-,\, f^+)$, respectively.

3 Bipolar-valued fuzzy deductive systems and filters

In what follows, let $\mathcal{L} := (L, |)$ denote the Sheffer stroke Hilbert algebra unless otherwise specified.

Definition 3.1. A bipolar-valued fuzzy set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L is called

- a *bipolar-valued fuzzy deductive system* of $\mathcal{L} := (L, |)$ if it satisfies:

$$(\forall x \in L)(f^-(1) \leq f^-(x),\;\; f^+(1) \geq f^+(x)), \tag{3.1}$$

$$(\forall x, y \in L)\begin{pmatrix} f^-(y) \leq \max\{f^-(x), f^-(x|(y|y))\} \\ f^+(y) \geq \min\{f^+(x), f^+(x|(y|y))\} \end{pmatrix}. \tag{3.2}$$

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- a *bipolar-valued fuzzy filter* of $\mathcal{L} := (L, |)$ if it satisfies (3.1) and

$$(\forall x, y \in L)(f^-(x|(y|y)) \le f^-(y),\ f^+(x|(y|y)) \ge f^+(y)), \tag{3.3}$$

$$(\forall x, y, z \in L) \begin{pmatrix} f^-((x|(y|z))|(y|z)) \le \max\{f^-(y), f^-(z)\} \\ f^+((x|(y|z))|(y|z)) \ge \min\{f^+(y), f^+(z)\} \end{pmatrix}. \tag{3.4}$$

Example 3.2. Consider a set $L = \{0, 1, 2, 3, 4, 5, 6, 7\}$. The Hasse diagram and the Sheffer stroke "|" on L are given by Figure 1 and Table 2, respectively.

Figure 1: Hasse Diagram

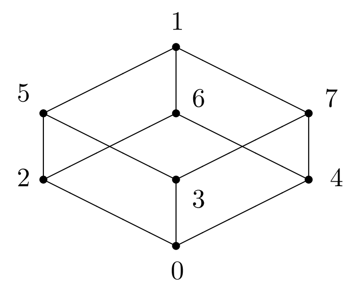

Table 2: Cayley table for the Sheffer stroke "|"

|	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

Then $\mathcal{L} := (L, |)$ is a Sheffer stroke Hilbert algebra (see [12]). Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ and $\mathfrak{g} := (L;\ g^-,\ g^+)$ be BVF-sets in L given by Table 3.
It is routine to verify that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy deductive system of $\mathcal{L} := (L, |)$, and $\mathfrak{g} := (L;\ g^-,\ g^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$.

Theorem 3.3. *Given a bipolar-valued fuzzy set $\mathfrak{f} := (L;\ f^-,\ f^+)$ in L, the following are equivalent to each other.*

(i) $\mathfrak{f} := (L;\ f^-,\ f^+)$ *is a bipolar-valued fuzzy deductive system of* $\mathcal{L} := (L, |)$.

Table 3: Tabular representation of $\mathfrak{f}$ and $\mathfrak{g}$

L	$f^-(x)$	$f^+(x)$	$g^-(x)$	$g^+(x)$
0	-0.42	0.49	-0.48	0.33
2	-0.56	0.68	-0.62	0.33
3	-0.42	0.49	-0.48	0.46
4	-0.42	0.49	-0.48	0.33
5	-0.64	0.79	-0.75	0.46
6	-0.56	0.68	-0.62	0.33
7	-0.42	0.49	-0.48	0.61
1	-0.72	0.83	-0.79	0.67

(ii) $\mathfrak{f} := (L;\, f^-,\, f^+)$ *is a bipolar-valued fuzzy filter of* $\mathcal{L} := (L, |)$.

Proof. Assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy deductive system of $\mathcal{L} := (L, |)$ and let $x, y, z \in L$. Note that $y|((x|(y|y))|(x|(y|y))) = 1$ by (2.1) and (2.5). The use of (3.1) and (3.2) leads to

$$\begin{aligned} f^-(x|(y|y)) &\le \max\{f^-(y), f^-(y|((x|(y|y))|(x|(y|y))))\} \\ &= \max\{f^-(y), f^-(1)\} = f^-(y) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} f^+(x|(y|y)) &\ge \min\{f^+(y), f^+(y|((x|(y|y))|(x|(y|y))))\} \\ &= \min\{f^+(y), f^+(1)\} = f^+(y). \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} y|(((y|z)|z)|((y|z)|z)) &\overset{(s2)}{=} y|(((y|z)|((z|z)|(z|z)))|((y|z)|((z|z)|(z|z)))) \\ &\overset{(2.8)}{=} (y|z)|((y|((z|z)|(z|z)))|(y|((z|z)|(z|z)))) \\ &\overset{(s2)}{=} (y|z)|((y|z)|(y|z)) \\ &\overset{(2.2)}{=} 1. \end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned} f^-((y|z)|z) &\le \max\{f^-(y), f^-(y|(((y|z)|z)|((y|z)|z)))\} \\ &= \max\{f^-(y), f^-(1)\} = f^-(y) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} f^+((y|z)|z) &\ge \min\{f^+(y), f^+(y|(((y|z)|z)|((y|z)|z)))\} \\ &= \min\{f^+(y), f^+(1)\} = f^+(y). \end{aligned} \tag{3.8}$$

Since $z|(((y|z)|(y|z))|((y|z)|(y|z))) \overset{(s2)}{=} z|(y|z) \overset{(s1)}{=} (y|z)|z$, we obtain

$$\begin{aligned} g^-((y|z)|(y|z)) &\leq \max\{g^-(z), g^-(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \max\{g^-(z), g^-((y|z)|z)\} \\ &\leq \max\{g^-(z), g^-(y)\} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} f^+((y|z)|(y|z)) &\geq \min\{f^+(z), f^+(z|(((y|z)|(y|z))|((y|z)|(y|z))))\} \\ &= \min\{f^+(z), f^+((y|z)|z)\} \\ &\geq \min\{f^+(z), f^+(y)\}. \end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned} & f^-((x|(y|z))|(y|z)) \\ & \overset{(s2)}{=} f^-((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z)))) \\ & \overset{(3.5)}{\leq} f^-((y|z)|(y|z)) \\ & \overset{(3.9)}{\leq} \max\{f^-(z), f^-(y)\} \end{aligned}$$

and

$$\begin{aligned} & f^+((x|(y|z))|(y|z)) \\ & \overset{(s2)}{=} f^+((x|(((y|z)|(y|z))|((y|z)|(y|z))))|(((y|z)|(y|z))|((y|z)|(y|z)))) \\ & \overset{(3.6)}{\geq} f^+((y|z)|(y|z)) \\ & \overset{(3.10)}{\geq} \min\{f^+(z), f^+(y)\}. \end{aligned}$$

Therefore $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$.

Conversely, assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ and let $x, y, z \in L$. If we replace y, z, and x with x, $x|(y|y)$, and y, respectively, in (3.4), then

$$\begin{aligned} f^-(y) &= f^-(((x|x)|(1|1))|(y|y)) \\ &= f^-(((x|x)|((y|(y|y))|(y|(y|y))))|(y|y)) \\ &= f^-(((((x|x)|y)|((x|x)|y))|(y|y))|(y|y)) \\ &= f^-((y|((x|x)|y))|((x|x)|y)) \\ &= f^-(((((x|x)|y)|y)|y)|(((x|x)|y)|y)) \\ &= f^-((y|(x|(x|(y|y))))|(x|(x|(y|y)))) \\ &\leq \max\{f^-(x), f^-(x|(y|y))\} \end{aligned}$$

and

$$\begin{aligned} f^{+}(y) &= f^{-}(((x|x)|(1|1))|(y|y)) \\ &= f^{+}(((x|x)|((y|(y|y))|(y|(y|y))))|(y|y)) \\ &= f^{+}(((((x|x)|y)|((x|x)|y))|(y|y))|(y|y)) \\ &= f^{+}((y|((x|x)|y))|((x|x)|y)) \\ &= f^{+}(((((x|x)|y)|y)|y)|(((x|x)|y)|y)) \\ &= f^{+}((y|(x|(x|(y|y))))|(x|(x|(y|y)))) \\ &\geq \min\{f^{+}(x), f^{+}(x|(y|y))\} \end{aligned}$$

by (s1), (s2), (s3), (2.2), (2.3), (2.4) (2.6) and (2.7). Consequently, $\mathfrak{f} := (L; f^{-}, f^{+})$ is a bipolar-valued fuzzy deductive system of $\mathcal{L} := (L, |)$. □

By Theorem 3.3, it can be seen that all the results for the bipolar-valued fuzzy filter covered below can be handled in the same way using the bipolar-valued fuzzy deductive system.

Proposition 3.4. *Every bipolar-valued fuzzy filter* $\mathfrak{f} := (L; f^{-}, f^{+})$ *of* $\mathcal{L} := (L, |)$ *satisfies:*

$$(\forall x, y \in L)\left(\begin{array}{l} f^{-}((x|(y|y))|(y|y)) \leq f^{-}(x) \\ f^{+}((x|(y|y))|(y|y)) \geq f^{+}(x)\} \end{array}\right). \tag{3.11}$$

$$(\forall x, y \in L)\left(x \leq_L y \;\Rightarrow \left\{\begin{array}{l} f^{-}(x) \geq f^{-}(y) \\ f^{+}(x) \leq f^{+}(y) \end{array}\right.\right). \tag{3.12}$$

Proof. Let $\mathfrak{f} := (L; f^{-}, f^{+})$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then

$$\begin{aligned} f^{-}((x|(y|y))|(y|y)) &= f^{-}((y|(x|x))|(x|x)) \leq \max\{f^{-}(x), f^{-}(x)\} = f^{-}(x), \\ f^{+}((x|(y|y))|(y|y)) &= f^{+}((y|(x|x))|(x|x)) \geq \min\{f^{+}(x), f^{+}(x)\} = f^{+}(x) \end{aligned}$$

for all $x, y \in L$ by (2.6) and (3.4). Therefore, (3.11) is valid. Let $x, y \in L$ be such that $x \leq_L y$. Then $x|(y|y) = 1$, and so

$$f^{-}(y) = f^{-}(1|(y|y)) = f^{-}((x|(y|y))|(y|y)) \leq f^{-}(x)$$

and

$$f^{+}(y) = f^{+}(1|(y|y)) = f^{+}((x|(y|y))|(y|y)) \geq f^{+}(x)$$

by (2.4) and (3.11). □

We consider a bipolar-valued fuzzy set $\mathfrak{f} := (L; f^{-}, f^{+})$ in L satisfying the condition (3.12) and question whether it becomes a bipolar-valued fuzzy filter. But the example below shows that the answer to that is negative.

Figure 2: Hasse Diagram

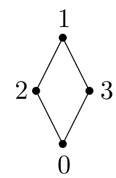

Table 4: Cayley table for the Sheffer stroke "|"

$\vert$	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Table 5: Tabular representation of $\mathfrak{f} := (L; f^-, f^+)$

L	$f^-(x)$	$f^+(x)$
0	-0.12	0.09
2	-0.37	0.16
3	-0.54	0.28
1	-0.81	0.62

Example 3.5. Consider a set $L = \{0, 1, 2, 3\}$. The Hasse diagram and the Sheffer stroke "|" on L are given by Figure 2 and Table 4, respectively.
Then $\mathcal{L} := (L, |)$ is a Sheffer stroke Hilbert algebra (see [12]). Let $\mathfrak{f} := (L; f^-, f^+)$ be a BVF-set in L given by Table 5.
Then $\mathfrak{f} := (L; f^-, f^+)$ satisfies the condition (3.12). But it is not a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ since

$$f^-((0|(3|2))|(3|2)) = f^-(0) = -0.12 \not\leq -0.37 = \max\{f^-(3), f^-(2)\}$$

and/or $f^+((0|(3|2))|(3|2)) = f^+(0) = 0.09 \not\geq 0.16 = \min\{f^+(3), f^+(2)\}$.

We explore the conditions under which a bipolar-valued fuzzy set can be a bipolar-valued fuzzy filter.

Theorem 3.6. *A bipolar-valued fuzzy set $\mathfrak{f} := (L; f^-, f^+)$ in L is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if it satisfies the condition (3.12) and*

$$(\forall x, y \in L) \left(\begin{array}{l} f^-((x|y)|(x|y)) \leq \max\{f^-(x), f^-(y)\} \\ f^+((x|y)|(x|y)) \geq \min\{f^+(x), f^+(y)\} \end{array} \right). \tag{3.13}$$

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Proof. Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then the condition (3.12) is valid by Proposition 3.4. Using (s1), (s2), (2.3), (2.4) and (3.4), we have $f^-((x|y)|(x|y)) = f^-(((1|1)|(x|y))|(x|y)) \leq \max\{f^-(x), f^-(y)\}$ and $f^+((x|y)|(x|y)) = f^+(((1|1)|(x|y))|(x|y)) \geq \min\{f^+(x), f^+(y)\}$ for all $x, y \in L$.

Conversely, assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ satisfies (3.12) and (3.13). Since $x \leq_L 1$ and $y \leq_L x|(y|y)$ for all $x, y \in L$, we have $f^-(1) \leq f^-(x)$, $f^+(1) \geq f^+(x)$, $f^-(x|(y|y)) \leq f^-(y)$, and $f^+(x|(y|y)) \geq f^+(y)$ by (3.12). Using (2.5), (s2), (3.12) and (3.13), we have

$$f^-((x|(y|z))|(y|z)) \leq f^-((y|z)|(y|z)) \leq \max\{f^-(y), f^-(z)\}$$

and $f^+((x|(y|z))|(y|z)) \geq f^+((y|z)|(y|z)) \geq \min\{f^+(y), f^+(z)\}$ for all $x, y \in L$. Therefore $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. □

Theorem 3.7. *A bipolar-valued fuzzy set $\mathfrak{f} := (L;\, f^-,\, f^+)$ in L is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if its negative s-cut and positive t-cut are filters of $\mathcal{L} := (L, |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$.*

Proof. Assume that $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ and $L(f^-; s) \neq \emptyset \neq U(f^+; t)$ for all $(s, t) \in [-1, 0] \times [0, 1]$. It is clear that $1 \in L(f^-; s) \cap U(f^+; t)$. Let $y, \mathfrak{b} \in L$ be such that $(y, \mathfrak{b}) \in L(f^-; s) \times U(f^+; t)$. Then $f^-(y) \leq s$ and $f^+(\mathfrak{b}) \geq t$. It follows from (3.3) that $f^-(x|(y|y)) \leq f^-(y) \leq s$ and $f^+(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \geq f^+(\mathfrak{b}) \geq t$ for all $x, \mathfrak{a} \in L$. Hence $(x|(y|y), \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \in L(f^-; s) \times U(f^+; t)$. Let $y, \mathfrak{b}, z, \mathfrak{c} \in L$ be such that $(y, \mathfrak{b}) \in L(f^-; s) \times U(f^+; t)$ and $(z, \mathfrak{c}) \in L(f^-; s) \times U(f^+; t)$. Then $f^-(y) \leq s$, $f^-(z) \leq s$, $f^+(\mathfrak{b}) \geq t$, and $f^+(\mathfrak{c}) \geq t$. Using (3.4), we get $f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\} \leq s$ and $f^+((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \geq \min\{f^+(\mathfrak{b}), f^+(\mathfrak{c})\} \geq t$, and so

$$((x|(y|z))|(y|z), (\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \in L(f^-; s) \times U(f^+; t).$$

Therefore $L(f^-; s)$ and $U(f^+; t)$ are filters of $\mathcal{L} := (L, |)$.

Conversely, let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a bipolar-valued fuzzy set in L for which its negative s-cut and positive t-cut are filters of $\mathcal{L} := (L, |)$ whenever they are nonempty for all $(s, t) \in [-1, 0] \times [0, 1]$. If $f^-(1) > f^-(\mathfrak{a})$ or $f^+(1) < f^+(x)$ for some $x, \mathfrak{a} \in L$, then $\mathfrak{a} \in L(f^-; f^-(\mathfrak{a}))$ and $x \in U(f^+; f^+(x))$, but $1 \notin L(f^-; f^-(\mathfrak{a})) \cap U(f^+; f^+(x))$. This is a contradiction, and thus $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$ for all $x \in L$. If $f^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) > f^-(\mathfrak{b})$ for some $\mathfrak{a}, \mathfrak{b} \in L$, then $\mathfrak{b} \in L(f^-; f^-(\mathfrak{b}))$ but $\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) \notin L(f^-; f^-(\mathfrak{b}))$ which is a contradiction. Hence $f^-(x|(y|y)) \leq f^-(y)$ for all $x, y \in L$. If $f^+(x|(y|y)) < f^+(y)$ for some $x, y \in L$, then $y \in U(f^+; f^+(y))$ but $x|(y|y) \notin U(f^+; f^+(y))$, a contadiction. Thus $f^+(x|(y|y)) \geq f^+(y)$ for all $x, y \in L$. Suppose that

$$f^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))(\mathfrak{b}|\mathfrak{c})) > \max\{f^-(\mathfrak{b}), f^-(\mathfrak{c})\}$$

or $f^+((x|(y|z))(y|z)) < \min\{f^+(y), f^+(z)\}$ for some $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, x, y, $z \in L$. Then $\mathfrak{b}, \mathfrak{c} \in L(f^-; s)$ or $y, z \in U(f^+; t)$ where $s := \max\{f^-(\mathfrak{b}), f^-(\mathfrak{c})\}$ and $t :=$

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$\min\{f^+(y), f^+(z)\}$. But $(\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))(\mathfrak{b}|\mathfrak{c}) \notin L(f^-; s)$ or $(x|(y|z))(y|z) \notin U(f^+; t)$, a contradiction. Therefore $f^-((x|(y|z))(y|z)) \leq \max\{f^-(y), f^-(z)\}$ and

$$f^+((x|(y|z))(y|z)) \geq \min\{f^+(y), f^+(z)\}$$

for all $x, y, z \in L$. Consequently, $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. □

Theorem 3.8. *A bipolar-valued fuzzy set $\mathfrak{f} := (L;\ f^-,\ f^+)$ in L is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if the fuzzy sets f_c^- and f^+ are fuzzy filters of $\mathcal{L} := (L, |)$, where $f_c^- : L \to [0, 1],\ x \mapsto 1 - f^-(x)$.*

Proof. Assume that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. It is clear that f^+ is a fuzzy filter of $\mathcal{L} := (L, |)$. For every $x, y, z \in L$, we have $f_c^-(1) = 1 - f^-(1) \geq 1 - f^-(x) = f_c^-(x)$,

$$f_c^-(x|(y|y)) = 1 - f^-(x|(y|y)) \geq 1 - f^-(y) = f_c^-(y),$$

and

$$\begin{aligned} f_c^-((x|(y|z))|(y|z)) &= 1 - f^-((x|(y|z))|(y|z)) \\ &\geq 1 - \max\{f^-(y), f^-(z)\} \\ &= \min\{1 - f^-(y), 1 - f^-(z)\} \\ &= \min\{f_c^-(y), f_(^-z)\}. \end{aligned}$$

Hence f_c^- is a fuzzy filter of $\mathcal{L} := (L, |)$.

Conversely, let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a bipolar-valued fuzzy set in L for which f_c^- and f^+ are fuzzy filters of $\mathcal{L} := (L, |)$. Then $1 - f^-(1) = f_c^-(1) \geq f_c^-(x) = 1 - f^-(x)$,

$$1 - f^-(x|(y|y)) = f_c^-(x|(y|y)) \geq f_c^-(y) = 1 - f^-(y)$$

and

$$\begin{aligned} &1 - f^-((x|(y|z))|(y|z)) = f_c^-((x|(y|z))|(y|z)) \\ &\geq \min\{f_c^-(y), f_c^-(z)\} \\ &= \min\{1 - f^-(y), 1 - f^-(z)\} \\ &= 1 - \max\{f^-(y), f^-(z)\} \end{aligned}$$

for all $x, y, z \in L$. Hence $f^-(1) \leq f^-(x)$, $f^-(x|(y|y)) \leq f^-(y)$ and

$$f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\}$$

for all $x, y, z \in L$. Therefore, $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. □

Theorem 3.9. *Given a nonempty subset F of L, let $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ be a bipolar-valued fuzzy set in L defined as follows:*

$$f_F^- : L \to [-1, 0],\ \mathfrak{a} \mapsto \begin{cases} s^- & \text{if } \mathfrak{a} \in F, \\ t^- & \text{otherwise,} \end{cases}$$

and

$$f_F^+ : L \to [0, 1],\ x \mapsto \begin{cases} s^+ & \text{if } x \in F, \\ t^+ & \text{otherwise,} \end{cases}$$

where $s^- < t^-$ in $[-1, 0]$ and $s^+ > t^+$ in $[0, 1]$. Then $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if F is a filter of $\mathcal{L} := (L, |)$. Moreover, we have $F = L_{\mathfrak{f}_F} := \{x \in L \mid f_F^-(x) = f_F^-(1),\, f_F^+(x) = f_F^+(1)\}$.

Proof. Assume that $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then $f_F^-(1) = s^-$ and $f_F^+(1) = s^+$, and so $1 \in F$. Let $x, y \in L$ be such that $y \in F$. Then $f_F^-(y) = s^-$ and $f_F^+(y) = s^+$. It follows from (3.3) that $s^- = f_F^-(y) \geq f_F^-(x|(y|y))$ and $s^+ = f_F^+(y) \leq f_F^+(x|(y|y))$. Hence $f_F^-(x|(y|y)) = s^-$ and $f_F^+(x|(y|y)) = s^+$, from which $x|(y|y) \in F$ is derived. Let $x, y, z \in L$ be such that $y, z \in F$. Using (3.4), we have:

$$f_F^-((x|(y|z))|(y|z)) \leq \max\{f_F^-(y), f_F^-(z)\} = s^-,$$
$$f_F^+((x|(y|z))|(y|z)) \geq \min\{f_F^+(y), f_F^+(z)\} = s^+,$$

and so $f_F^-((x|(y|z))|(y|z)) = s^-$ and $f_F^+((x|(y|z))|(y|z)) = s^+$. This shows that $(x|(y|z))|(y|z) \in F$. Therefore F is a filter of $\mathcal{L} := (L, |)$.

Conversely, let F be a filter of $\mathcal{L} := (L, |)$. Since $1 \in F$, we get $f_F^-(1) = s^- \leq f_F^-(\mathfrak{a})$ and $f_F^+(1) = s^+ \geq f_F^+(x)$ for all $(\mathfrak{a}, x) \in L \times L$. Let $x, y \in L$. If $y \in F$, then $x|(y|y) \in F$, and thus $f_F^-(x|(y|y)) = s^- = f_F^-(y)$ and $f_F^+(x|(y|y)) = s^+ = f_F^+(y)$. If $y \notin F$, then $f_F^-(y) = t^- > f_F^-(x|(y|y))$ and $f_F^+(y) = t^+ < f_F^+(x|(y|y))$. For every $x, y, z \in L$, if $y, z \in F$ then $(x|(y|z))|(y|z) \in F$ which implies that $f_F^-((x|(y|z))|(y|z)) = s^- = \max\{f_F^-(y), f_F^-(z)\}$ and $f_F^+((x|(y|z))|(y|z)) = s^+ = \min\{f_F^+(y), f_F^+(z)\}$. If $y \notin F$ or $z \notin F$, then

$$f_F^-((x|(y|z))|(y|z)) \leq t^- = \max\{f_F^-(y), f_F^-(z)\},$$
$$f_F^+((x|(y|z))|(y|z)) \geq t^+ = \min\{f_F^+(y), f_F^+(z)\}.$$

Therefore, $\mathfrak{f}_F := (L;\, f_F^-,\, f_F^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Since F is a filter of $\mathcal{L} := (L, |)$, we get

$$\begin{aligned} L_{\mathfrak{f}_F} &= \{x \in L \mid f_F^-(x) = f_F^-(1),\, f_F^+(x) = f_F^+(1)\} \\ &= \{x \in L \mid f_F^-(x) = s^-,\, f_F^+(x) = s^+\} \\ &= \{x \in L \mid x \in F\} = F. \end{aligned}$$

This completes the proof. □

4 Normality of bipolar-valued fuzzy filters

Definition 4.1. A bipolar-valued fuzzy filter $\mathfrak{f} := (L;\ f^-,\ f^+)$ of $\mathcal{L} := (L, |)$ is said to be *normal* if there exists $(\mathfrak{a}, x) \in L \times L$ such that $f^-(\mathfrak{a}) = -1$ and $f^+(x) = 1$.

Example 4.2. Consider the Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ in Example 3.2. Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a BVF-set in L given by Table 6.

Table 6: Tabular representation of $\mathfrak{f} := (L;\ f^-,\ f^+)$

L	$f^-(x)$	$f^+(x)$
0	-0.42	0.36
2	-0.42	0.36
3	-0.42	0.76
4	-0.57	0.36
5	-0.42	1.00
6	-1.00	0.36
7	-0.57	0.76
1	-1.00	1.00

Then $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$.

Theorem 4.3. *A bipolar-valued fuzzy filter $\mathfrak{f} := (L;\ f^-,\ f^+)$ of $\mathcal{L} := (L, |)$ is normal if and only if $f^-(1) = -1$ and $f^+(1) = 1$.*

Proof. Suppose that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then $f^-(\mathfrak{a}) = -1$ and $f^+(x) = 1$ for some $(\mathfrak{a}, x) \in L \times L$. It follows from (3.1) that $f^-(1) \leq f^-(\mathfrak{a}) = -1$ and $f^+(1) \geq f^+(x) = 1$. Hence $f^-(1) = -1$ and $f^+(1) = 1$. The sufficiency is clear. □

Given two bipolar-valued fuzzy sets $\mathfrak{f} := (L;\ f^-,\ f^+)$ and $\mathfrak{g} := (L;\ g^-,\ g^+)$ in L, the inclusion " $\Subset$" between them is defined as follows:

$$\mathfrak{f} \Subset \mathfrak{g} \ \Leftrightarrow\ (\forall x \in L)(f^-(x) \geq g^-(x),\ f^+(x) \leq g^+(x)).$$

In this case we say that $\mathfrak{g} := (L;\ g^-,\ g^+)$ is larger than $\mathfrak{f} := (L;\ f^-,\ f^+)$.

Theorem 4.4. *Given a bipolar-valued fuzzy set $\mathfrak{f} := (L;\ f^-,\ f^+)$ in L, let $\mathfrak{f}_* := (L;\ f_*^-,\ f_*^+)$ be a bipolar-valued fuzzy set in L defined by $f_*^-(\mathfrak{a}) = f^-(\mathfrak{a}) - 1 - f^-(1)$ and $f_*^+(x) = f^+(x) + 1 - f^+(1)$ for all $(\mathfrak{a}, x) \in L \times L$. Then $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ if and only if $\mathfrak{f}_* := (L;\ f_*^-,\ f_*^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Moreover, $\mathfrak{f}_* := (L;\ f_*^-,\ f_*^+)$ is normal which is larger than $\mathfrak{f} := (L;\ f^-,\ f^+)$.*

Proof. Assume that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ and let $x, y \in L$ be such that $x \leq_L y$. Then

$$f_*^-(x) = f^-(x) - 1 - f^-(1) \geq f^-(y) - 1 - f^-(1) = f_*^-(y)$$

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and

$$f_*^+(x) = f^+(x) + 1 - f^+(1) \leq f^+(y) + 1 - f^+(1) = f_*^+(y).$$

For every $x, y \in L$, we have:

$$\begin{aligned} f_*^-((x|y)|(x|y)) &= f^-((x|y)|(x|y)) - 1 - f^-(1) \\ &\leq \max\{f^-(x), f^-(y)\} - 1 - f^-(1) \\ &= \max\{f^-(x) - 1 - f^-(1), f^-(y) - 1 - f^-(1)\} \\ &= \max\{f_*^-(x), f_*^-(y)\} \end{aligned}$$

and

$$\begin{aligned} f_*^+((x|y)|(x|y)) &= f^+((x|y)|(x|y)) + 1 - f^+(1) \\ &\geq \min\{f^+(x), f^+(y)\} + 1 - f^+(1) \\ &= \min\{f^+(x) + 1 - f^+(1), f^+(y) + 1 - f^+(1)\} \\ &= \min\{f_*^+(x), f_*^+(y)\}. \end{aligned}$$

Hence $\mathfrak{f}_* := (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 3.6. Suppose that $\mathfrak{f}_* := (L; f_*^-, f_*^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Since $f^-(1) - 1 - f^-(1) = f_*^-(1) \leq f_*^-(\mathfrak{a}) = f^-(\mathfrak{a}) - 1 - f^-(1)$ and $f^+(1) + 1 - f^+(1) = f_*^+(1) \geq f_*^+(x) = f^+(x) + 1 - f^+(1)$ for all $(\mathfrak{a}, x) \in L \times L$, we have $f^-(1) \leq f^-(x)$ and $f^+(1) \geq f^+(x)$ for all $x \in L$. Since

$$f^-(\mathfrak{b}) - 1 - f^-(1) = f_*^-(\mathfrak{b}) \geq f_*^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) = f^-(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) - 1 - f^-(1)$$

and $f^+(y) + 1 - f^+(1) = f_*^+(y) \leq f_*^+(x|(y|y)) = f^+(x|(y|y)) + 1 - f^+(1)$ for all $(\mathfrak{a}, x), (\mathfrak{b}, y) \in L \times L$, it follows that $f^-(y) \geq f^-(x|(y|y))$ and $f^+(y) \leq f^+(x|(y|y))$ for all $x, y \in L$. Since

$$\begin{aligned} & f^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) - 1 - f^-(1) = f_*^-((\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c})) \\ & \leq \max\{f_*^-(\mathfrak{b}), f_*^-(\mathfrak{c})\} \\ & = \max\{f^-(\mathfrak{b}) - 1 - f^-(1), f^-(\mathfrak{c}) - 1 - f^-(1)\} \\ & = \max\{f^-(\mathfrak{b}), f^-(\mathfrak{c})\} - 1 - f^-(1) \end{aligned}$$

and

$$\begin{aligned} & f^+((x|(y|z))|(y|z)) + 1 - f^+(1) = f_*^+((x|(y|z))|(y|z)) \\ & \geq \min\{f_*^+(y), f_*^+(z)\} \\ & = \min\{f^+(y) + 1 - f^+(1), f^+(z) + 1 - f^+(1)\} \\ & = \min\{f^+(y), f^+(z)\} + 1 - f^+(1) \end{aligned}$$

for all $(\mathfrak{a}, x), (\mathfrak{b}, y), (\mathfrak{c}, z) \in L \times L$, we have

$$f^-((x|(y|z))|(y|z)) \leq \max\{f^-(y), f^-(z)\}$$

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and $f^+((x|(y|z))|(y|z)) \geq \min\{f^+(y), f^+(z)\}$ for all $x, y, z \in L$. Therefore, $\mathfrak{f} := (L;\, f^-,\, f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Since $f_*^-(1) = f^-(1) - 1 - f^-(1) = -1$ and $f_*^+(1) = f^+(1) + 1 - f^+(1) = 1$, we know that $\mathfrak{f}_* := (L;\, f_*^-,\, f_*^+)$ is normal. Also, we have $f_*^-(x) = f^-(x) - 1 - f^-(1) \leq f^-(x)$ and $f_*^+(x) = f^+(x) + 1 - f^+(1) \geq f^+(x)$ for all $x \in L$. This shows that $\mathfrak{f}_* := (L;\, f_*^-,\, f_*^+)$ is larger than $\mathfrak{f} := (L;\, f^-,\, f^+)$. □

Theorem 4.5. *Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then it is normal if and only if $\mathfrak{f}_* = \mathfrak{f}$, that is, $f^-(x) = f_*^-(x)$ and $f^+(x) = f_*^+(x)$ for all $x \in L$.*

Proof. Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then $\mathfrak{f}_* := (L;\, f_*^-,\, f_*^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 4.4. Hence it is clear that if $\mathfrak{f}_* = \mathfrak{f}$, then $\mathfrak{f} := (L;\, f^-,\, f^+)$ is normal.

Conversely, if $\mathfrak{f} := (L;\, f^-,\, f^+)$ is normal, then $f_*^-(x) = f^-(x) - 1 - f^-(1) = f^-(x)$ and $f_*^+(x) = f^+(x) + 1 - f^+(1) = f^+(x)$ for all $x \in L$. Hence $\mathfrak{f}_* = \mathfrak{f}$. □

Proposition 4.6. *Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ and $\mathfrak{g} := (L;\, g^-,\, g^+)$ be bipolar-valued fuzzy filters of $\mathcal{L} := (L, |)$ with $\mathfrak{f} \Subset \mathfrak{g}$. If $f^-(1) = g^-(1)$ and $f^+(1) = g^+(1)$, then $L_{\mathfrak{f}_F} \subseteq L_{\mathfrak{g}_F}$.*

Proof. Straightforward. □

The example below shows that there are bipolar-valued fuzzy filters $\mathfrak{f} := (L;\, f^-,\, f^+)$ and $\mathfrak{g} := (L;\, g^-,\, g^+)$ of $\mathcal{L} := (L, |)$ that satisfy $L_{\mathfrak{f}_F} \subseteq L_{\mathfrak{g}_F}$ and $\mathfrak{f} \not\Subset \mathfrak{g}$.

Example 4.7. Consider the Sheffer stroke Hilbert algebra $\mathcal{L} := (L, |)$ in Example 3.5. Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ and $\mathfrak{g} := (L;\, g^-,\, g^+)$ be bipolar-valued fuzzy sets in L defined by the Table 7.

Table 7: Tabular representation of $\mathfrak{f}$ and $\mathfrak{g}$

L	$f^-(x)$	$f^+(x)$	$g^-(x)$	$g^+(x)$
0	−0.42	0.43	−0.36	0.33
2	−1.00	1.00	−1.00	1.00
3	−0.42	0.43	−0.36	0.33
1	−1.00	1.00	−1.00	1.00

Then $L_{\mathfrak{f}_F} = \{1, 2\} = L_{\mathfrak{g}_F}$ but $\mathfrak{f} \not\Subset \mathfrak{g}$ since $f^-(3) = -0.42 < -0.36 = g^-(3)$ and/or $f^+(0) = 0.43 > 0.33 = g^+(0)$.

Theorem 4.8. *Let $\mathfrak{f} := (L;\, f^-,\, f^+)$ be a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Then it is normal if and only if there is a bipolar-valued fuzzy filter $\mathfrak{g} := (L;\, g^-,\, g^+)$ of $\mathcal{L} := (L, |)$ such that $\mathfrak{g}_* \Subset \mathfrak{f}$.*

Proof. The necessity is straightforward because if $\mathfrak{f} := (L;\, f^-,\, f^+)$ is normal, then $\mathfrak{f}_* = \mathfrak{f}$.

Conversely, assume that there is a bipolar-valued fuzzy filter $\mathfrak{g} := (L;\ g^-,\ g^+)$ of $\mathcal{L} := (L, |)$ such that $\mathfrak{g}_* \Subset \mathfrak{f}$. Then $-1 = g_*^-(1) \geq f^-(1)$ and $1 = g_*^+(1) \leq f^+(1)$. Thus $f^-(1) = -1$ and $f^+(1) = 1$, and so $\mathfrak{f} := (L;\ f^-,\ f^+)$ is normal. $\square$

Theorem 4.9. *Given a bipolar-valued fuzzy set $\mathfrak{f} := (L;\ f^-,\ f^+)$ in L, consider an increasing mapping $\ell := (\ell^-, \ell^+) : [-1, f^-(1)] \times [0, f^+(1)] \to [-1, 0] \times [0, 1]$. If $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$, then the bipolar-valued fuzzy set $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ in L defined by $f_\ell^-(\mathfrak{a}) = \ell^-(f^-(\mathfrak{a}))$ and $f_\ell^+(x) = \ell^+(f^+(x))$ for all $(\mathfrak{a}, x) \in L \times L$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Moreover, if $f_\ell^-(1) = -1$ and $f_\ell^+(1) = 1$, then $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ is normal, and*

$$(\forall (s,t) \in [-1, f^-(1)] \times [0, f^+(1)])(\ell^-(s) \leq s,\ \ell^+(t) \geq t \ \Rightarrow\ \mathfrak{f} \Subset \mathfrak{f}_\ell).$$

Proof. Assume that $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$. Let $x, y \in L$ be such that $x \leq_L y$. Then $f_\ell^-(x) = \ell^-(f^-(x)) \geq \ell^-(f^-(y)) = f_\ell^-(y)$ and $f_\ell^+(x) = \ell^+(f^+(x)) \leq \ell^+(f^+(y)) = f_\ell^+(y)$. For every $x, y, z \in L$, we have

$$\begin{aligned} f_\ell^-((x|y)|(x|y)) &= \ell^-(f^-((x|y)|(x|y))) \\ &\leq \ell^-(\max\{f^-(x), f^-(y)\}) \\ &= \max\{\ell^-(f^-(x)), \ell^-(f^-(y))\} \\ &= \max\{f_\ell^-(x), f_\ell^-(y)\} \end{aligned}$$

and

$$\begin{aligned} f_\ell^+((x|y)|(x|y)) &= \ell^+(f^+((x|y)|(x|y))) \\ &\geq \ell^+(\min\{f^+(x), f^+(y)\}) \\ &= \min\{\ell^+(f^+(x)), \ell^+(f^+(y))\} \\ &= \min\{f_\ell^+(x), f_\ell^+(y)\}. \end{aligned}$$

Therefore, $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 3.6. If $f_\ell^-(1) = -1$ and $f_\ell^+(1) = 1$, then $\mathfrak{f}_\ell := (L;\ f_\ell^-,\ f_\ell^+)$ is normal by Theorem 4.3. Let $(s,t) \in [-1, f^-(1)] \times [0, f^+(1)]$ be such that $\ell^-(s) \leq s$ and $\ell^+(t) \geq t$. Then $f_\ell^-(x) = \ell^-(f^-(x)) \leq f^-(x)$ and $f_\ell^+(x) = \ell^+(f^+(x)) \geq f^+(x)$ for all $x \in L$. Hence $\mathfrak{f} \Subset \mathfrak{f}_\ell$. $\square$

Theorem 4.10. *Let $\mathfrak{f} := (L;\ f^-,\ f^+)$ be a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ such that $f^-(\mathfrak{a}) \neq f^-(1)$ and $f^+(x) \neq f^+(1)$ for some $(\mathfrak{a}, x) \in L \times L$. If $\mathfrak{f} := (L;\ f^-,\ f^+)$ is a maximal element of $(\mathcal{N}_F(L), \Subset)$, then it is described as follows:*

$$\begin{aligned} &f^- : L \to [-1, 0],\ \mathfrak{a} \mapsto \begin{cases} -1 & \text{if } \mathfrak{a} = 1, \\ 0 & \text{otherwise,} \end{cases} \\ &f^+ : L \to [0, 1],\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4.1}$$

where $\mathcal{N}_F(L)$ is the set of all normal bipolar-valued fuzzy filters of $\mathcal{L} := (L, |)$.

Proof. Clearly, $(\mathcal{N}_F(L), \Subset)$ is a poset. Assume that $\mathfrak{f} := (L; f^-, f^+)$ is a maximal element of $(\mathcal{N}_F(L), \Subset)$. It is clear that $f^-(1) = -1$ and $f^+(1) = 1$ since $\mathfrak{f} := (L; f^-, f^+)$ is normal. Let $(\mathfrak{a}, x) \in L \times L$ be such that $f^-(\mathfrak{a}) \neq f^-(1)$ and $f^+(x) \neq f^+(1)$. If $f^-(\mathfrak{a}) \neq 0$ and $f^+(x) \neq 0$, then $-1 < f^-(\mathfrak{c}) < 0$ and $0 < f^+(z) < 1$ for some $(\mathfrak{c}, z) \in L \times L$. Let $\mathfrak{g} := (L; g^-, g^+)$ be a bipolar-valued fuzzy set in L defined by

$$g^- : L \to [-1, 0],\ \mathfrak{a} \mapsto \tfrac{1}{2}(f^-(\mathfrak{a}) + f^-(\mathfrak{c})),$$
$$g^+ : L \to [0, 1],\ x \mapsto \tfrac{1}{2}(f^+(x) + f^+(z)).$$

Let $x, y \in L$ be such that $x \leq_L y$. Then

$$g^-(x) = \tfrac{1}{2}(f^-(x) + f^-(\mathfrak{c})) \geq \tfrac{1}{2}(f^-(y) + f^-(\mathfrak{c})) = g^-(y)$$

and $g^+(x) = \frac{1}{2}(f^+(x)+f^+(z)) \leq \frac{1}{2}(f^+(y)+f^+(z)) = g^+(y)$. For every $x, y \in L$, we have

$$\begin{aligned} g^-((x|y)|(x|y)) &= \tfrac{1}{2}(f^-((x|y)|(x|y)) + f^-(\mathfrak{c})) \\ &\leq \tfrac{1}{2}(\max\{f^-(x), f^-(y)\} + f^-(\mathfrak{c})) \\ &= \tfrac{1}{2}\max\{f^-(x) + f^-(\mathfrak{c}), f^-(y) + f^-(\mathfrak{c})\} \\ &= \max\{\tfrac{1}{2}(f^-(x) + f^-(\mathfrak{c})), \tfrac{1}{2}(f^-(y) + f^-(\mathfrak{c}))\} \\ &= \max\{g^-(x), g^-(y)\} \end{aligned}$$

and

$$\begin{aligned} g^+((x|y)|(x|y)) &= \tfrac{1}{2}(f^+((x|y)|(x|y)) + f^+(z)) \\ &\geq \tfrac{1}{2}(\min\{f^+(x), f^+(y)\} + f^+(z)) \\ &= \tfrac{1}{2}\min\{f^+(x) + f^+(z), f^+(y) + f^+(z)\} \\ &= \min\{\tfrac{1}{2}(f^+(x) + f^+(z)), \tfrac{1}{2}(f^+(y) + f^+(z))\} \\ &= \min\{g^+(x), g^+(y)\}. \end{aligned}$$

Hence $\mathfrak{g} := (L; g^-, g^+)$ is a bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 3.6, and $\mathfrak{g}_* := (L; g_*^-, g_*^+)$ is a normal bipolar-valued fuzzy filter of $\mathcal{L} := (L, |)$ by Theorem 4.4. We can observe that

$$\begin{aligned} g_*^-(x) &= g^-(x) - 1 - g^-(1) \\ &= \tfrac{1}{2}(f^-(x) + f^-(\mathfrak{c})) - 1 - \tfrac{1}{2}(f^-(1) + f^-(\mathfrak{c})) \\ &= \tfrac{1}{2}(f^-(x) - 1) \leq f^-(x) \end{aligned}$$

and

$$\begin{aligned} g_*^+(x) &= g^+(x) + 1 - g^+(1) \\ &= \tfrac{1}{2}(f^+(x) + f^+(z)) + 1 - \tfrac{1}{2}(f^+(1) + f^+(z)) \\ &= \tfrac{1}{2}(f^+(x) + 1) \geq f^+(x) \end{aligned}$$

for all $x \in L$. Hence $\mathfrak{f} \Subset \mathfrak{g}_*$, and so $\mathfrak{f} := (L;\, f^-,\, f^+)$ is not a maximal element of $(\mathcal{N}_F(L), \Subset)$. This is a contradiction, and therefore $(f^-(\mathfrak{a}), f^+(x)) = (0, 0)$ for all $(\mathfrak{a}, x) \in L \times L$ with $f^-(\mathfrak{a}) \neq -1$ and $f^+(x) \neq 1$. Consequently, $\mathfrak{f} := (L;\, f^-,$ $f^+)$ is described as (4.1). □

References

[1] S. Bashir, R. Mazhar, H. Abbas and M. Shabir, Regular ternary semirings in terms of bipolar fuzzy ideals, omputational and Applied Mathematics, **39**, 319 (2020). https://doi.org/10.1007/s40314-020-01319-z

[2] A. Borumand Saeid, Bipolar-valued Fuzzy BCK/BCI-algebras, World Applied Sciences Journal, **7**(11) (2009), 1404–1411.

[3] I. Chajad, Sheffer operation in ortholattices, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **44**(1) (2005), 19–23. http://dml.cz/dmlcz/133381

[4] M. S. Kang, Bipolar fuzzy hyper MV-deductive systems of hyper MV-algebras, Communications of the Korean Mathematical Society **26**(2) (2011), 169–182. DOI10.4134/CKMS.2011.26.2.169

[5] T. Katican. Branchesm and obstinate SBE-filters of Sheffer stroke BE-algebras, Bulletin of the International Mathematical Virtual Insttitute, **12**(1) (2022), 41–50. DOI:10.7251/BIMVI2201041K

[6] P. Khamrot and M. Siripitukdet, Some types of subsemigroups characterized in terms of inequalities of generalized bipolar fuzzy subsemigroups, Mathematics 2017, **5**(4), 71. doi:10.3390/math5040071

[7] V. Kozarkiewicz and A. Grabowski, Axiomatization of Boolean algebras based on Sheffer stroke, Formalized Mathematics **12**(3) (2004), 355–361.

[8] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bulletin of the Malaysian Mathematical Sciences Society, (2) **32**(3) (2009), 361–373. http://math.usm.my/bulletin

[9] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proceedings of International Conference on Intelligent Technologies, Bangkok, Thailand (2000) 307–312.

[10] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets and bipolar-valued fuzzy sets, Journal of Korean institute of intelligent systems, **14**(2) (2004), 125–129. DOI:10.5391/JKIIS.2004.14.2.125

[11] T. Oner, T. Katican and A. Borumand Saeid, Fuzzy filters of Sheffer stroke Hilbert algebras, Journal of Intelligent & Fuzzy Systems **40**(1) (2021), 759–772. DOI:10.3233/JIFS-200760

[12] T. Oner, T. Katican and A. Borumand Saeid, Relation between Sheffer Stroke and Hilbert Algebras, Categories and General Algebraic Structures with Applications **14**(1) (2021), 245–268. https://doi.org/10.29252/cgasa.14.1.245

[13] T. Oner, T. Katican and A. Borumand Saeid, BL-algebras defined by an operator, Honam Mathematical J. **44**(2) (2022), 18–31. https://doi.org/10.5831/HMJ.2022.44.2.18

[14] T. Oner, T. Katican and A. Borumand Saeid, Class of Sheffer stroke BCK-algebras, Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa **30**(1) (2022), 247–269. DOI:10.2478/auom-2022-0014

[15] T. Oner, T. Katican, A. Borumand Saeid and M. Terziler, Filters of strong Sheffer stroke non-associative MV-algebras, Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa **29**(1) (2021), 143–164. DOI:10.2478/auom-2021-0010

[16] H. M. Sheffer, A set of five independent postulates for Boolean algebras, Transactions of the American Mathematical Society **14**(4) (1913), 481–488.