Amazing Results Concerning Zeta Function at (-2) and Bernoulli3^{rd} Number

Ahmed Adnan Tuma

Department of Mathematics, College of Education for Pure Science (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq, Email: ahmedadnantuma@gmail.com

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ABSTRACT
In this paper two amazing theorems in number theory concerning $\zeta(-2)$ and third Bernoulli number B_3 showing that Zeta function $\zeta(-2) = \frac{i}{84}$and Bernoulli third number is not zero but $B_3 = \frac{i}{28}$, where $i = \sqrt{-1}$.

Keywords: Zeta function, Bernoulli numbers.

1. INTRODUCTION

Bernard Riemann extended the scope of the zeta function to complex numbers, and the nontrivial zeros at critical line arose. When using the zeta function with Bernoulli's numbers,they are called axiomatic zeros (trivial zeros) appear at negative even numbers [1] .
$\zeta(-2k) = 0,$ when k is natural number .
We know the individual Bernoulli numbers are all zeros except for the first number[1] .
Remember that:
$B_{2n+1} = 0 ;\ \forall\ n = 1,2,3, \ldots$
$B_1 = \frac{-1}{2}\ ,\ B_3 = B_5 = B_7 = \cdots = 0$
These numbers were discovered by the Swiss mathematician Jacob Bernoulli, where,published and named after his death in 1713. Independently, the Japanese mathematician Seki Kawa also discovered these numbers and published Seki Kawa's discovery in 1712 after his death too [3].

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$

Leonhard Eulerfirst defined zeta function, but it was in the range of real numbers, where zeta function defined as[3].
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, where sis realin Euler definition and complex in Riemann definition.
Euler studied zeta function in the first half of the seventeenth century, and he was able to prove that [1]:$\zeta(2) = \frac{\pi^2}{6}$
Euler was able to calculate the values of zeta function for positive even numbers using the formula [1].$\zeta(2k) = \frac{(-1)^{k+1}\cdot(2\pi)^{2k}\cdot B_{2k}}{2(2k)!}$; $k = 1,2,3, \ldots$ $\cdots\cdots\cdot (1)$
And he was also able to calculate the values of zeta function in the range of negative numbers by [1].
$\zeta(-k) = \frac{(-1)^k \cdot B_{k+1}}{k+1};\ k = 0,1,2,3, \ldots$ $\cdots\cdots\cdot (2)$
The two formulas (1) and (2) are considered his most important works in number theory depending on Euler's formula (2) then zeta function for the even negative numbers will be zero because all odd Bernoulli numbers are zeros except for the first Bernoulli's number B_1 which is equal $\frac{-1}{2}$
In 1859,Bernard Riemann extended the domain of the zeta function to complex numbers .

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \qquad \text{where } s = a + bi$$

2. The Main Results

In this section we prove two theorems concerning $\zeta(-2)$ &B_3 .

To prove that the axiomatic zero of the zeta function at $s = -2$ is not equal zero, we need to know$\zeta(0)$ and $\zeta(-1)$, where,

if$s = 0$, then $\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = 1 + 1 + 1 + 1 + \cdots$

To find the value ofζat $s = 0$, we rely on Euler's law in (2).

when $k = 0$

$$\zeta(-k) = \frac{(-1)^k \cdot B_{k+1}}{k+1}$$

$$\zeta(0) = \frac{B_1}{1} = \frac{-1}{2}$$

Now,

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n$$

Let

$S = \zeta(-1) = 1 + 2 + 3 + 4 + \cdots$

From [2] we reshowing that $S = \frac{-1}{12}$ as follows .

consider A & Bgiven by

$A = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

$B = 1 - 2 + 3 - 4 + \cdots$

$1 - A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots = A$

$$\therefore \quad 1 = 2A \implies A = \frac{1}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now we compute B as follows

$A - B = (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - \ldots)$

$\quad = 1 - 2 + 3 - 4 + \cdots = B$

$$B = \frac{1}{2} A = \frac{1}{4} \quad \ldots\ldots\ldots\ldots\ldots \ (4)$$

To evaluateS we do the following

$S - B = (1 + 2 + 3 + 4 + \cdots) - (1 - 2 + 3 - 4 + \cdots)$

$\quad = 4 + 8 + 16 + \cdots = 4S$

$$\mathbf{S} = \frac{-1}{3} B = \frac{-1}{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Then, $\zeta(-1) = \frac{\mathbf{-1}}{\mathbf{12}}$

Now, we show an interesting value of zeta function at (-2) .

Theorem (1) : $\zeta(-2) = \frac{i}{84}$

Proof

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(-2) = \sum_{n=1}^{\infty} \frac{1}{n^{-2}} = \sum_{n=1}^{\infty} n^2$$

$\zeta(-2) = 1 + 4 + 9 + 16 + \cdots$

$A_1 = 1 - 1 + 1 - 1 + \cdots$

$B_1 = 1 - 4 + 9 - 16 + \cdots$

$\zeta(-2) - B_1 = 8 + 32 + 72 + \cdots = 8\,\zeta(-2)$

$$\zeta(-2) = \frac{-\mathbf{B_1}}{\mathbf{7}} \qquad \ldots\ldots\ldots\ldots\ldots \ (\mathbf{6})$$

Consider the partial sums of B_1: $1\,,1\;-\;4\,,1-\;4\;+\;9\,,\;1-\;4\;+\;9\;-16\,,\;\ldots$

- $1\;=\;1\;\left((-1)^1*(-\;1)\right)$
- $1-\;4=-3=3*-1=(1+\;2)((-1)^{1+1}*\;(-1))$
- $1-4+9=6=6*1=(1+2+3)\left((-1)^{1+1+1}*(-1)\right)$
- $1-4+9-16=-10=10*-1=(1+2+3+4)\left((-1)^{1+1+1+1}*(-1)\right)$

and so on,

$B_1=(1+4+9+16+\cdots)\;((-1)^{1+1+1+1\ldots}*(-1))$

$\quad=\;\boldsymbol{\zeta}\,(-\mathbf{1})(-1)^{\boldsymbol{\zeta}(\mathbf{0})}(-1)=\boldsymbol{\zeta}\,(-\mathbf{1})(-1)^{\frac{-\mathbf{1}}{\mathbf{2}}}\;(-1)$

$=\;\left(\frac{-1}{12}\right)\left(\frac{1}{(-1)^{1/2}}*-1\right)=\;\left(\frac{-1}{12}\right)\left(\left(\frac{1}{i}*\frac{-i}{-i}\right)(-1)\right)=\;\frac{-i}{12}$

$\therefore\,B_1=\;\frac{-i}{12}$

From (6) , $\boldsymbol{\zeta}(-\mathbf{2})=\frac{-\mathbf{B_1}}{\mathbf{7}}$

$\therefore\,\boldsymbol{\zeta}(-\mathbf{2})=\frac{-\frac{-i}{12}}{7}=\;\frac{\mathbf{i}}{\mathbf{84}}$

which is amazing result of the axiomatic zero of $\zeta(-2)$ is not zero

$\boldsymbol{\zeta}(-\mathbf{2})=\sum_{n=1}^{\infty}\,n^2=\;\frac{i}{84}.$

Theorem $(\mathbf{2})$: Third Bernoulli number B_3 not zero but $\frac{i}{28}$

Proof

from Euler's second law (2)**:**

$\boldsymbol{\zeta}(-\mathbf{k})=\frac{(-\mathbf{1})^{\mathbf{k}}\cdot\mathbf{B_{k+1}}}{\mathbf{k}+\mathbf{1}}$

$\boldsymbol{\zeta}(-\mathbf{2})=\frac{\mathbf{B_3}}{\mathbf{3}}$

$B_3\;=3\;\zeta(-2)=B_3=\frac{\mathbf{3i}}{\mathbf{84}}=\;\frac{\mathbf{i}}{\mathbf{28}}$

which is another amazing result

3. CONCLUSIONS

It has been proved that there is no axiomatic zero, and that the zeta function at (-2) is equal to an imaginary value, which is (i/84)

$\boldsymbol{\zeta}(-\mathbf{2})=\mathbf{1}+\mathbf{4}+\mathbf{9}+\mathbf{16}+\cdots=\frac{\mathbf{i}}{\mathbf{84}}$

And the third Bernoulli's number is not equal zero, but equals i/28

$\mathbf{B_3}=\frac{\mathbf{i}}{\mathbf{28}}$

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