Spectra of four new Graphs join based on Subdivision and Central Graph

Manash Protim Borah[1]*, Karam Ratan Singh[2] ,S. Prasad[3]

[1]Department of Mathematics, L.T.K. College, Azad, North Lakhimpur, 787031, Assam, India
[2]Department of Basic and Applied Science, National Institute of Technology Arunachal Pradesh, Papum pare, 791113, Arunachal Pradesh, India
[3] Department of Mathematics, National Institute of Technology Jamshedpur, 831014, Jharkhand, India
*Corresponding Author

Received: 15.04.2024 Revised : 16.05.2024 Accepted: 20.05.2024

ABSTRACT
We define four new graphs through the join operation of subdivision graph S(G) and central graph C(H), namely subdivision vertex-central vertex join $S(G)\ddot{\square}C(H)$, subdivision edge-central edge join $S(G)\overline{\overline{\square}}C(H)$, subdivision edge-central vertex join $S(G)\overset{\underset{\cdot}{-}}{\square}C(H)$, and subdivision vertex-central edge join $S(G)\overset{\overset{\cdot}{-}}{\square}C(H)$ graphs. We determine the adjacency and Laplacian spectra of these four graphs and generate a set of A-cospectral and L-cospectral non-regular graphs for these new graphs by choosing two pairs of regular cospectral graphs. Additionally, we compute the Kirchhoff indices and the number of spanning trees in these graphs.

Keywords: Adjacency spectrum, Laplacian spectrum, Cospectral graphs, Kirchhoff index, spanning trees

1. INTRODUCTION

Let G = (V_G ,E_G) be a simple and undirected graph with vertex set V_G and edge set E_G. Let A(G) be a n-th order adjacency matrix of G such that A = [a_{ij}] = 1 if the vertices $v_i \sim v_j$ and 0 otherwise. The Laplacian matrix of G denoted by L(G) is defined as L(G) = D(G) − A(G), where D(G) is the diagonal matrix. The complement of G, denoted by $\overline{G}$, is the graph with the same vertex set as G such that two vertices are adjacent in $\overline{G}$, if and only if they are not adjacent in G. The adjacency matrix of the complement graph, denoted by A($\overline{G}$) is defined as A($\overline{G}$) = $J_{n\times n}-I_n-A(G)$, where $J_{n\times n}$ is the n × n matrix with all entries '1' and I_n is the n × n unit matrix. The Laplacian matrix of the complement graph is defined as L($\overline{G}$) = $nI_n- J_{n\times n}- L(G)$. For any matrix $M_{n\times n}$, the polynomial associated with it is given by P(M;x) = det(xIn− M). Thus, $P_G(A;x)$ and $P_G(L;x)$ be the characteristic polynomial of A(G) and L(G), respectively. The roots of A(G) and L(G) matrices are adjacency eigenvalues and Laplacian eigenvalues of G, respectively. Denote the eigenvalues of A(G) and L(G), respectively, by $\lambda_j(G)$and $\mu_j(G)$, where j = 1,2,...,n. The collection of distinct eigenvalues of A(G) and L(G) and their corresponding multiplicities form the A-spectrum and L-spectrum of G, respectively. If two graphs have the same A and L spectrum, then they are said to be A-cospectral and L-cospectral, respectively. The number of the spanning trees of G of n vertices can be determined by $t(G) = \frac{\mu_2(G)\mu_3(G),...,\mu_n(G)}{n}$ and Kirchhoff index can be obtained by $Kf(G) = n\sum_{i=2}^{n}\frac{1}{\mu_i(G)}$

Spectra of different kinds of graphs operations have been computed by several types of research [3, 4, 7, 8, 9, 10, 11, 13]. The subdivision graph S(G) of a graph G is obtained by inserting a new vertex into every edge of G. The central graph C(G) of a graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices in G. Jahfar and Chithra [7, 8] defined central vertex join, central edge join, central vertex corona, central edge corona and central edge neighbourhood corona graphs and determine their adjacency, Laplacian and signless Laplacian spectra. Also, the Kirchhoff index determines the number of spanning trees and cospectral graphs' families. A. Das and P. Panigrahi [4] defined four new graphs join and determine their adjacency, Laplacian and normalized Laplacian spectra. As an application, they obtain pairs of simultaneous cospectral graphs for adjacency, Laplacian and normalized Laplacian matrices. In addition, the Kirchhoff index and the number of spanning trees are also determined in the paper.
Motivated by these above work, we define four new graphs join.

Definition 1.1. Let G and H be two vertex disjoint graphs with number of vertices n_1 and n_2, edges m_1 and m_2 respectively. Then

(i) The subdivision vertex-central vertex joinof G and H, represented by S(G)$\ddot{\square}$C(H), is the graph derived from S(G) and C(H) by joining each old vertex of G with every old vertex of H.
(ii) The subdivision edge- central edge join of G and H, represented by S(G) $\bar{\bar{\square}}$C(H), is the graph derived from S(G) and C(H) by joining each new vertex of G with every new vertex of H.
(iii) The subdivision edge-central vertex joinof G and H, represented by S(G)$\bar{\dot{\square}}$C(H),is the graph derived from S(G) andC(H) by joining each new vertex of G with every old vertex of H.
(iv) The subdivision vertex- central edge join of G and H, represented by S(G)$\dot{\bar{\square}}$C(H), is the graph derived from S(G) and C(H) by joining each old vertex of G with every new vertex of H.

Example 1.1. Let us consider G = P_4 and H= P_3be two path graphs. Then Figure 1, Figure 2, Figure 3 and Figure 4 represents $P_4\ddot{\square}P_3$, $P_4\ \bar{\bar{\square}}P_3$, $P_4\bar{\dot{\square}}P_3$ and $P_4\dot{\bar{\square}}P_3$ respectively.

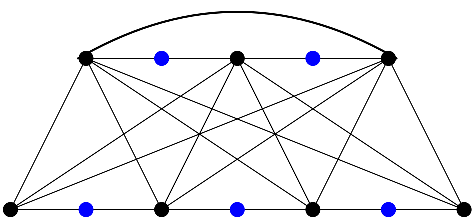

Figure 1. Subdivision-vertex-central vertex join of P_4 and P_3. i.e. $P_4\ddot{\square}P_3$

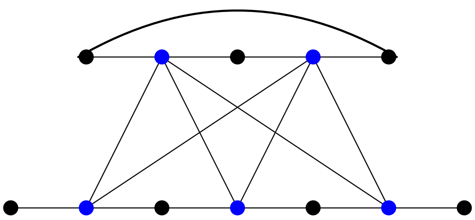

Figure 2. Subdivision edge-central edge join of P_4 and P_3. i.e. $P_4\ \bar{\bar{\square}}P_3$

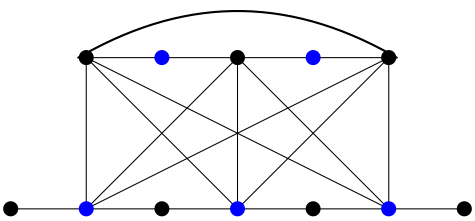

Figure 3. subdivision edge-central vertex join of P_4 and P_3. i.e. $P_4\bar{\dot{\square}}P_3$

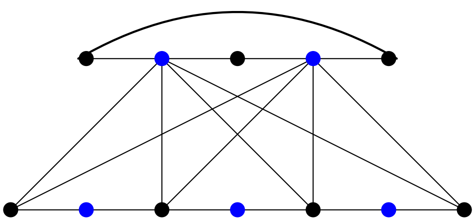

Figure 4. Subdivision vertex-central edge join of P_4 and P_3. i.e. $P_4\dot{\bar{\square}}P_3$

Four non-regular graphs are generated in this paper through the join operation of subdivision and central graphs. Subsequently we obtain the A-spectra and L-spectra of these joins. Moreover, we derive several cospectral graphs of A and L spectra. The number of spanning trees and Kirchhoff's indices are also determined.
To obtain our results, we need some basics useful results. Let B(G) = [b_{ij}] be the incidence matrix of order n×m such that b_{ij}= 1 if v_i is incident with e_j, where i,j= 1,2,...,n, otherwise 0. Let **L**(G) be the line graph, choosing B(G) = B, then B^TB = A(**L**(G)) + $2I_m$ and BB^T= A(G) + $2I_n$. The M-Coronal $\Gamma_M(x)$ [12] is defined on the n × n matrix of M such that $\Gamma_M(x) = J_n^T(xI_n - M)^{-1}J_n$, where J_n be the n × 1 matrix with all 1 entries. If 't'is the constant of each row sum of matrix M, then $\Gamma_M(x) = \frac{n}{x-t}$[12]. If L(G) is a Laplacian matrix, then $\Gamma_L(x) = \frac{n}{x}$ [12].

Also, det(M + γ$J_{n\times n}$) = det(M) +$\gamma J_{n\times 1}^T adj(M) J_{n\times 1}$[9], where adj(M) is the adjoint of M and γ is a real number. The following lemmas are also used to find our results.

Lemma 1.1.[9] If M is an real matrix of n × n, then
det(xI_n– M –γJ_n) = (1 –γΓ_M(x))det(xI_n– M)
Lemma 1.2.[3] Let N_1,N_2,N_3andN_4 be four matrices, where N_1 and N_4 are non-singular square matrices, then

$$det\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} = \det(N_1).\det(N_4 - N_3{N_1}^{-1}N_2) = \det(N_4).\det(N_1 - N_2{N_4}^{-1}N_3)$$

Lemma 1.3.[4] If for any real numbers c, d > 0, then

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}$$

2. Adjacency and Laplacian spectra of the graphs

In this section, we find adjacency and Laplacian spectra of subdivision vertex-central vertex join S(G)$\ddot{\square}$C(H), subdivision edge-central edge join S(G) $\overline{\overline{\square}}$C(H), subdivision edge-central vertex join S(G)$\bar{\square}$C(H) and subdivision vertex-central edgejoin S(G)$\dot{\square}$C(H) graphs.
First, we determine adjacency spectra of these graphs.

2.1. A-spectra of S(G)$\ddot{\square}$ C(H)

Theorem 2.1. LetG be a r_1- regular and H be a r_2 regular graph, then

$$P_{S(G)\ddot{\square}C(H)}(A:x) = x^{m_1+m_2-n_1-n_2}\prod_{j=2}^{n_2}(x^2+(1+\lambda_j(H))x-\lambda_j(H)-r_2)$$
$$[x^4-(n_2-1-r_2)x^3-(2r_1+2r_2+n_1n_2)x^2+(2r_1n_2-2r_1-2r_1r_2)x+4r_1r_2]$$
$$\prod_{i=2}^{n_1}(x^2-r_1-\lambda_i(G))$$

Proof:A-spectra of S(G)$\ddot{\square}$ C(H) can be expressed as

$$A(S(G)\ddot{\square}C(H)) = \begin{pmatrix} 0_{n_1\times n_1} & B(G) & J_{n_1\times n_2} & 0_{n_1\times m_2} \\ B(G)^T & 0_{m_1\times m_1} & 0_{m_1\times n_2} & 0_{m_1\times m_2} \\ J_{n_2\times n_1} & 0_{n_2\times m_1} & A(\overline{H}) & B(H) \\ 0_{n_2\times n_1} & 0_{m_1\times m_2} & B(H)^T & 0_{m_2} \end{pmatrix}$$

The characteristic polynomial is $P_{S(G)\ddot{\square}C(H)}$(A:x)

$$= det(xI_{n_1+n_2+m_1+m_2} - A(S(G)\ddot{\square}C(H)))$$
$$= det\begin{pmatrix} xI_{n_1} & -B(G) & -J_{n_1\times n_2} & 0_{n_1\times m_2} \\ -B(G)^T & xI_{m_1} & 0_{m_1\times n_2} & 0_{m_1\times m_2} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1} & xI_{n_2} - A(H) & -B(H) \\ 0_{m_2\times n_1} & 0_{m_1\times m_2} & -B(H)^T & xI_{m_2} \end{pmatrix}$$
$$= x^{m_2} det S$$

Where

$$S = \begin{pmatrix} xI_{n_1} & -B(G) & -J_{n_1\times n_2} \\ -B(G)^T & xI_{m_1} & 0_{m_1\times n_2} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1} & xI_{n_2} - A(\overline{H}) \end{pmatrix} - \begin{pmatrix} 0_{n_1\times m_2} \\ 0_{m_1\times m_2} \\ -B(H) \end{pmatrix}\frac{1}{x}\begin{pmatrix} 0_{m_2\times n_1} & 0_{m_2\times m_1} & -B(H)^T \end{pmatrix}$$
$$= \begin{pmatrix} xI_{n_1} & -B(G) & -J_{n_1\times n_2} \\ -B(G)^T & xI_{m_1} & 0_{m_1\times n_2} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1} & xI_{n_2} - A(\overline{H}) - \frac{1}{x}B(H)B(H)^T \end{pmatrix}$$

Hence,

$$\det S = \det(xI_{n_2} - A(\overline{H}) - \frac{1}{x}B(H)B(H)^T)\det(W)$$

$$= \det(xI_{n_2} - J_{n_2} + I_{n_2} + A(H) - \frac{1}{x}B(H)B(H)^T)\det(W)$$

Where

$$W = \begin{pmatrix} xI_{n_1} & -B(G) \\ -B(G)^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} -J_{n_1\times n_2} \\ 0_{m_1\times n_2} \end{pmatrix}((xI_{n_2} - A(\overline{\mathcal{H}}) - \frac{1}{x}B(H)B(H)^T)^{-1})$$

$$\begin{pmatrix} J_{n_2\times n_1} & 0_{n_2\times m_1} \end{pmatrix}$$

$$= \begin{pmatrix} xI_{n_1} - \Gamma_{A(H)+\frac{1}{x}B(H)B(H)^T}(x)J_{n_1\times n_1} & -B(G) \\ -B(G)^T & xI_{m_1} \end{pmatrix}$$

Then,

$$\det W = x^{m_1}\det\left(xI_{n_2} - \Gamma_{A(\overline{H})+\frac{1}{x}B(H)B(H)^T}(x)J_{n_1\times n_1} - \frac{B(G)B(G)^T}{x}\right)$$

$$= x^{m_1}\det\left(xI_{n_1} - \frac{B(G)B(G)^T}{x}\right)\left[1 - \Gamma_{A(\overline{H})+\frac{B(H)B(H)^T}{x}}(x)\Gamma_{\frac{B(G)B(G)^T}{x}}(x)\right]$$

$$= x^{m_1}\det\left(xI_{n_1} - \frac{B(G)B(G)^T}{x}\right)\left[1 - \Gamma_{A(\overline{H})+\frac{B(H)B(H)^T}{x}}(x)\Gamma_{\frac{B(G)B(G)^T}{x}}(x)\right]$$

So,

$$\det W = x^{m_1-n_1}\prod_{i=1}^{n_1}(x^2 - r_1 - \lambda_i(G))[1 - \frac{n_2}{x-(n_2-1-r_2+\frac{2r_2}{x})}.\frac{n_1}{x-\frac{2r_1}{x}}]$$

$$= x^{m_1-n_1}\prod_{i=1}^{n_1}(x^2 - r_1 - \lambda_i(G))[x^4 - (n_2-1-r_2)x^3 - (2r_1+2r_2+n_1n_2)x^2$$

$$+ (2r_1n_2 - 2r_1 - 2r_1r_2)x + 4r_1r_2]$$

Therefore,

$$P_{(S(G)\ddot{\square}C(H))}(A:x) = x^{m_1+m_2-n_1-n_2}\prod_{j=2}^{n_2}(x^2 + (1+\lambda_j(H))x - \lambda_j(H) - r_2)$$

$$[x^4 - (n_2-1-r_2)x^3 - (2r_1+2r_2+n_1n_2)x^2 + (2r_1n_2 - 2r_1 - 2r_1r_2)x + 4r_1r_2]$$

$$\prod_{i=2}^{n_1}(x^2 - r_1 - \lambda_i(G))$$

2.2. A-spectra of S(G) $\bar{\bar{\square}}$C(H)

Theorem 2.2. Let G be a r_1- regular and H be a r_2 regular graph, then A-spectra of S(G) $\bar{\bar{\square}}$C(H) can be expressed as

$$P_{S(G)\bar{\bar{\square}}C(H)}(A:x) = x^{m_1+m_2-n_1-n_2}\prod_{j=2}^{n_2}(x^2 - (n_2-1-\lambda_j(H))x - \lambda_j(H) - r_2)$$

$$\prod_{i=2}^{n_1}(x^2 - r_1 - \lambda_i(G))[x^4 - (n_2+1-r_1)x^3 - (4r_1 - m_2n_1)x^2 + (2r_1n_2 - 2r_1 +$$

$$2r_1^2 + m_2n_1n_2 - m_2n_1 + m_2n_1r_1)x + 4r_1^2 + 2r_1m_2n_1 - n_1n_2r_2^2]$$

Proof: A(S(G) $\bar{\bar{\square}}$C(H)) can be written as

$$A(S(G)\bar{\bar{\square}}C(H)) = \begin{pmatrix} 0_{n_1\times n_1} & B(G) & 0_{n_1\times n_2} & 0_{n_1\times m_2} \\ B(G)^T & 0_{m_1\times m_1} & 0_{m_1\times n_2} & J_{m_1\times m_2} \\ 0_{n_2\times m_1} & 0_{n_2\times m_1} & A(\overline{H}) & B(H) \\ 0_{m_2\times n_1} & J_{m_2\times m_1} & B(H)^T & 0_{m_2} \end{pmatrix}$$

The characteristic polynomial is

$$P_{S(G)\,\bar{\square}\,C(H)}(A:x) = det\begin{pmatrix} xI_{n_1} & -B(G) & 0_{n_1\times n_2} & 0_{n_1\times m_2} \\ -B(G)^T & xI_{m_1} & 0_{m_1\times n_2} & J_{m_1\times m_2} \\ 0_{n_2\times m_1} & 0_{n_2\times m_1} & xI_{n_2} - A(\bar{H}) & -B(H) \\ 0_{m_2\times n_1} & -J_{m_2\times m_1} & -B(H)^T & xI_{m_2} \end{pmatrix}$$
$$= x^{n_1} det\, S$$

Where,

$$S = \begin{pmatrix} xI_{m_1} & 0_{m_1\times n_2} & -J_{m_1\times m_2} \\ 0_{n_2\times m_1} & xI_{n_2} - A(\overline{H}) & -B(H) \\ -J_{m_2\times m_1} & -B(H)^T & xI_{m_2} \end{pmatrix} - \begin{pmatrix} -B(G)^T \\ 0_{n_1\times m_1} \\ 0_{m_2\times n_1} \end{pmatrix} \frac{1}{x} \begin{pmatrix} -B(G) & 0_{n_1\times n_2} & 0_{n_1\times m_2} \end{pmatrix}$$

$$= \begin{pmatrix} xI_{m_1} - \frac{1}{x}B(G)^T B(G) & 0_{m_1\times n_2} & -J_{m_1\times m_2} \\ 0_{n_2\times m_1} & xI_{n_2} - A(\bar{H}) & -B(H) \\ -J_{m_2\times m_1} & -B(H)^T & xI_{m_2} \end{pmatrix}$$

Hence,

$$det\, S = det\left(\left(xI_{m_1} - \frac{1}{x}B(G)B(G)^T\right) det\, W\right.$$
$$= det\left(xI_{m_1} - \frac{1}{x}\left(A(L_G) + 2I_{m_1}\right)\right) det\, W$$
$$= x^{m_1 - n_1} \prod_{i=1}^{n_1} \left(x - \frac{r_1}{x} - \frac{\lambda_i(G)}{x}\right) det\, W$$

Where

$$W = \begin{pmatrix} xI_{n_2} - A(\overline{H}) & -B(H) \\ -B(H)^T & xI_{m_2} \end{pmatrix} - \begin{pmatrix} 0_{n_2\times m_1} \\ 0_{m_2\times m_1} \end{pmatrix} \left(xI_{m_1} - \frac{1}{x}B(H)B(H)^T\right)^{-1}$$
$$\begin{pmatrix} 0_{m_1\times n_2} & -J_{m_1\times m_2} \end{pmatrix}$$
$$= \begin{pmatrix} xI_{n_2} - A(\overline{H}) & -B(H) \\ -B(H)^T & xI_{m_2} - \Gamma_{\frac{B(G)B(G)^T}{x}}(x) J_{m_2\times m_2} \end{pmatrix}$$

Then,

$$det\, W = det\left(xI_{m_2} - \Gamma_{\frac{B(G)B(G)^T}{x}}(x) J_{m_2\times m_2}\right)$$
$$det\left(xI_{n_2} - A(\overline{H}) - B(H)\left(xI_{m_2} - \Gamma_{\frac{H(G)B(G)^T}{x}}(x) J_{m_2\times m_2}\right)^{-1} B(H)^T\right)$$
$$= x^{m_2}\left(1 - \Gamma_{\frac{B(G)B(G)^T}{x}}(x)\frac{m_2}{x}\right) det\left[xI_{n_2} - A(\overline{H}) - B(H)\left\{\frac{1}{x}I_{m_2} + \right.\right.$$
$$\left.\left.\frac{B(G)B(G)^T(x)}{x\left(x - m_2\Gamma_{\frac{B(G)B(G)^T}{x}}(x)\right)} J_{m_2\times m_2}\right\} B(H)^T\right]$$
$$= x^{m_2}\left(1 - \Gamma^{x}_{\frac{B(G)B(G)^T}{x}}(x)\frac{m_2}{x}\right) det\left(xI_{n_2} - A(\overline{H}) - \frac{1}{x}B(H)B(H)^T - \right.$$
$$\left. r_2^2 \frac{\Gamma_{\frac{B(H)B(H)^T}{x}}(x)}{x\left(x - m_2\Gamma_{\frac{B(G)B(G)^T}{x}}\right.} J_{n_2\times n_2}\right)$$

So,

$$P_{S(G)\bar{\bar{\Box}}C(H)}(A:x)=x^{m_1+m_2-n_1-n_2}\prod_{j=2}^{n_2}(x^2-(n_2-1-\lambda_j(H))x-\lambda_j(H)-r_2)$$
$$\prod_{i=2}^{n_1}(x^2-r_1-\lambda_i(G))[x^4-(n_2+1-r_1)x^3-(4r_1-m_2n_1)x^2+(2r_1n_2-$$
$$2r_1+2r_1^2+m_2n_1n_2-m_2n_1+m_2n_1r_1)x+4r_1^2+2r_1m_2n_1-n_1n_2r_2^2]$$

and the result follows.

2.3. A-spectra of S(G)$\bar{\dot{\Box}}$C(H)

Theorem 2.3. Let G be a r_1- regular and H be a r_2 regular graph, then A-spectra of S(G)$\bar{\dot{\Box}}$C(H) can be expressed as

$$P_{S(G)\Box C(H)}(A:x)=x^{m_1+m_2-n_1-n_2}\prod_{j=2}^{n_2}\left(x^2-\left(n_2-1-\lambda_j(H)\right)x-\lambda_j(H)-r_2\right)$$
$$\prod_{i=2}^{n_1}(x^2-r_1-\lambda_i(G))[x^4-(n_2-1-r_2)x^3-(2r_1+2r_2+m_1n_2)x^2+(2r_1n_2-$$
$$2r_1-2r_1r_2)x+4r_1r_2]$$

Proof: A(S(G)$\bar{\dot{\Box}}$C(H)) can be written as

$$A(S(G)\bar{\dot{\Box}}C(H))=\begin{pmatrix}0_{n_1\times n_1} & B(G) & 0_{n_1\times n_2} & 0_{n_1\times m_2}\\ B(G)^T & 0_{m_1\times m_1} & J_{m_1\times n_2} & 0_{m_1\times m_2}\\ 0_{n_2\times m_1} & J_{n_2\times m_1} & A(\overline{H}) & B(H)\\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & B(H)^T & 0_{m_2}\end{pmatrix}$$

The rest of the proof of the Theorem is same as Theorem (2.2)

2.4. A-Spectra of S(G)$\dot{\bar{\Box}}$C(H)

Theorem 3.1. LetG be a r_1- regular and H be a r_2 regular graph, then A-spectra of S(G)$\Box^{\bar{\cdot}}$ C(H) can be expressed as

$$P_{S(G)\hat{\Box}C(H)}(A:x)=x^{m_1+m_2-n_1-n_2}\prod_{j=2}^{n_2}(x^2-(n_2-1-\lambda_j(H))x-\lambda_j(H)-r_2)$$
$$\prod_{i=2}^{n_1}(x^2-r_1-\lambda_i(G))[x^4-(n_2-1-r_2)x^3-(2r_1+2r_2+n_1)x^2+(2r_1n_2+$$
$$2r_1+2r_1r_2+n_1n_2+n_1+n_1r_2)x+4r_1r_2+2n_1r_2-n_1n_2r_2^2]$$

Proof: A(S(G)$\dot{\bar{\Box}}$C(H)) can be written as

$$A(S(G)\dot{\bar{\Box}}C(H))=\begin{pmatrix}0_{n_1\times n_1} & B(G) & 0_{n_1\times n_2} & J_{n_1\times m_2}\\ B(G)^T & 0_{m_1\times m_1} & 0_{m_1\times n_2} & 0_{m_1\times m_2}\\ 0_{n_2\times m_1} & 0_{n_2\times m_1} & A(\overline{\mathcal{H}}) & B(H)\\ J_{m_2\times n_1} & 0_{m_2\times m_1} & B(H)^T & 0_{m_2}\end{pmatrix}$$

The remaining part of the proof of the Theorem is same as above.

3.1. L-Spectra of (S(G) $\ddot{\Box}$ C(H)**)**

Theorem 3.2. Let G be a r_1- regular and H be a r_2 regular graph, then

$$L(S(G) \ddot{\square} C(H)) = (x-2)^{m_1+m_2-n_1-n_2} \prod_{i=2}^{n_1} (x^2-(r_1+n_2+2)x+2n_2+\mu_i(G))[x^4 - (r_1+n_1+n_2+r_2+4)x^3+(r_1r_2+r_2n_2+n_1r_1+4n_1+4n_2+2r_1+2r_2+4)x^2-2(r_2n_2+2n_1+2n_2+n_1r_1+n_1n_2)x] \prod_{j=2}^{n_2} x^2-\left(r_2+n_1+2+\mu_j(H)\right)x+2n_1-\mu_j(H)$$

Proof: L-spectra of subdivision-vertex central vertex join can be expressed as

$$L(S(G) \ddot{\square} C(H)) = \begin{pmatrix} (r_1+n_2)I_{n_1} & -B(G) & -J_{n_1\times n_2} & 0_{n_1\times m_2} \\ -B(G)^T & 2I_{m_1\times m_1} & 0m_1\times n_2 & 0_{m_1\times m_2} \\ -J_{m_2\times n_1} & 0_{n_2\times m_1} & (r_2+n_1)I_{m_2}+L(\overline{H}) & -B(H) \\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & -B(H)^T & 2I_{m_2} \end{pmatrix}$$

The characteristic polynomial is $P_{(S(G) \ddot{\square} C(H))}$(L :x)

$$= det(xI_{n_1+m_2+m_1+m_2} - L(S(G) \ddot{\square} C(H))$$
$$= det(x-2)I_{m_2} det S$$

Where,

$$S = \begin{pmatrix} (x-r_1-n_2)I_{n_1} & -B(G) & -J_{n_1\times n_2} \\ -B(G)^T & (x-2)I_{m_1} & 0_{m_1\times n_2} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1} & (x-r_2-n_1)I_{n_2}-L(\overline{\mathcal{H}}) \end{pmatrix} - \begin{pmatrix} 0_{n_1\times m_2} \\ 0_{m_1\times m_2} \\ -B(H) \end{pmatrix} \frac{1}{x-2} \begin{pmatrix} 0_{m_2\times n_1} & 0_{m_2\times m_1} & -B(H)^T \end{pmatrix}$$

$$= \begin{pmatrix} (x-r_1-n_2)I_{n_1} & -B(G) & -J_{n_1\times n_2} \\ -B(G)^T & (x-2)I_{m_1} & 0_{m_1\times n_2} \\ -J_{n_2\times n_1} & 0_{n_2\times m_1} & (x-r_2-n_1)I_{n_2}-L(\overline{\mathcal{H}})-\frac{B(H)B(H)^T}{x-2} \end{pmatrix}$$

$$= \det\{(x-r_2-n_1)I_{n_2}-L(\overline{\mathcal{H}})-\frac{B(H)B(H)^T}{x-2}\} \det W$$

Where,

$$W = \begin{pmatrix} (x-r_1-n_2)I_{n_1} & -B(G) \\ -B(G)^T & (x-2)I_{m_1} \end{pmatrix} - \begin{pmatrix} -J_{n_1\times n_2} \\ 0_{m_1\times n_2} \end{pmatrix} ((x-r_2-n_1)I_{n_2}-L(\overline{\mathcal{H}}) - \frac{B(H)B(H)^T}{x-2})^{-1} \begin{pmatrix} J_{n_2\times n_1} & 0_{n_2\times m_1} \end{pmatrix}$$

$$= \begin{pmatrix} (x-r_1-n_2)I_{n_1} - \Gamma_{L(\overline{\mathcal{H}})+\frac{1}{x-2}B(H)B(H)^T}(x-r_2-n_1)J_{n_1\times n_1} & -B(G) \\ -B(G)^T & (x-2)I_{m_1} \end{pmatrix}$$

Then,

$$\det W = (x-2)^{m_1} \det((x-r_1-n_2)I_{n_1} - \frac{B(G)B(G)^T}{x-2}) [1-\Gamma_{L(\overline{\mathcal{H}})+\frac{1}{x-2}B(H)B(H)^T}(x-r_2-n_1)\Gamma_{\frac{B(G)B(G)^T}{x-2}}(x-r_1-n_2)]$$

Putting the coronal values and simplifying above gives the desired result.

3.2. L-Spectra of S(G) ⊡C(H)

Theorem 3.3. Let G be a r_1- regular and H be a r_2 regular graph, then

$$L(S(G)\,\bar{\bar{\square}}C(H)) = (x-2-m_2)^{m_1-n_1}(x-2-m_1)^{m_2-n_2}\prod_{i=2}^{n_1} x^2-(r_1+m_2+2)x$$
$$+r_1m_2+\mu_i(G)\prod_{j=2}^{n_2}\left(x^2-(r_2+m_1+2-\mu_i(H))x+m_1r_2-m_1\mu_j(H)-\mu_j(H)\right)$$
$$[x^4-(r_1+n_1+m_1+m_2+4)x^3+(r_1r_2+2m_1+2m_2+2r_1+2r_2+r_1m_1$$
$$+r_2m_2+r_1m_2+r_2m_1+4)x^2-(2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2)x]$$

Proof: L(S(G) $\bar{\bar{\square}}$C(H))matrix can be written as

$$L(S(G)\,\bar{\bar{\square}}C(H)) = \begin{pmatrix} r_1I_{n_1} & -B(G) & -0_{n_1\times n_2} & 0_{n_1\times m_2} \\ -B(G)^T & (2+m_2)I_{m_1\times m_1} & 0_{m_1\times n_2} & -J_{m_1\times m_2} \\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & r_2In_2+L(H) & -B(H) \\ 0_{m_2\times n_1} & -J_{m_2\times m_1} & -B(H)^T & (2+m_1)I_{m_2} \end{pmatrix}$$

The remaining part of the proof of the Theorem is same as Theorem 2.2.

3.3. L-Spectra of S(G)$\bar{\dot{\square}}$C(H)

Theorem 3.4. LetG be a r_1- regular and H be a r_2 regular graph, then

$$L(S(G)\,\bar{\dot{\square}}\,C(H)) = (x-2-n_2)^{m_1-n_1}(x-2)^{m_2-n_2}\prod_{i=2}^{n_1} x^2-(r_1+n_2+2)x+\mu_i(G)+n_2r_1$$
$$\prod_{j=2}^{n_2}\left(x^2-\left(r_2+m_1-2-\mu_j(H)\right)x+2m_1-\mu_j(H)[x^4-(r_1+m_1+r_2+n_2+4)x^3+(4+4m_1\right.$$
$$+2r_1+2r_2+n_2r_2+n_2m_1+2n_2+r_1r_2+r_1m_1+r_1n_2-m_1n_2)x^2-(4m_1+4m_1n_2+2m_1r_1$$
$$+r_1r_2n_2+r_1n_2m_1+2r_1n_2+m_1r_1n_2)x].$$

Proof: L(S(G)$\bar{\dot{\square}}$C(H)) can be written as

$$L(S(G)\bar{\dot{\square}}C(H)) = \begin{pmatrix} r_1I_{n_1} & -B(G) & -0_{n_1\times n_2} & 0_{n_1\times m_2} \\ -B(G)^T & (2+n_2)I_{m_1\times m_1} & -J_{m_1\times n_2} & 0_{m_1\times m_2} \\ 0_{n_2\times n_1} & -J_{n_2\times m_1} & (r_2+m_1)In_2+L(\overline{H}) & -B(H) \\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & -B(H)^T & (2)I_{m_2} \end{pmatrix}$$

The remaining part of the proof of the above theorem is as same as above.

3.4. L-Spectra of S(G)$\dot{\bar{\square}}$C(H)

Theorem 3.5. Let G be a r_1- regular and H be a r_2 regular graph, then

$$L(S(G)\,\dot{\bar{\square}}\,C(H)) = (x-2)^{m_1-n_1}(x-2-n_1)^{m_2-n_2}\prod_{i=2}^{n_1}(x^2-(r_1+m_2+2)x+\mu_i(G)+2m_2)$$
$$\prod_{j=2}^{n_2}\left(x^2-\left(r_2+n_1+2-\mu_j(H)x\right)+n_1r_2-n_1\mu_j(H)-\mu_j(H)[x^4-(r_1+r_2+m_2+n_1)x^3+\right.$$
$$(4+2r_1+2r_2+4m_2+2n_1+r_1r_2+r_2m_2+r_1n_1+r_2n_1)x^2-(4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1)x]$$

Proof: S(G)$\dot{\bar{\square}}$C(H) can be expressed as

$$L(\mathcal{G}\dot{\bar{\square}}\mathcal{H}) = \begin{pmatrix} r_1I_{n_1} & -B(G) & -0_{n_1\times n_2} & J_{n_1\times m_2} \\ -B(G)^T & 2I_{m_1\times m_1} & 0_{m_1\times n_2} & 0_{m_1\times m_2} \\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & r_2In_2+L(\overline{H}) & -B(H) \\ J_{m_2\times n_1} & 0_{m_2\times m_1} & -B(H)^T & (2+n_1)I_{m_2} \end{pmatrix}$$

The remaining part of the proof of the theorem is as same as above.

4. Spanning trees and Kirchhoff Index

Kirchhoff's indices and spanning trees of subdivision vertex-central vertex join S(G) $\ddot{\square}$ C(H), subdivision edge-central edge join S(G)$\bar{\bar{\square}}$C(H), subdivision edge-central vertex join S(G)$\bar{\dot{\square}}$C(H) and subdivision vertex-central edge join S(G)$\dot{\bar{\square}}$C(H) are determined by using Laplacian spectra..

The Kirchhoff's indices of these graphs are

1.

$$Kf(S(G)\ddot{\square}\,\mathrm{C}(H)) = (m_1+n_1+m_2+n_2)\left(\frac{m_1-n_1+m_2-n_2}{2}+\sum_{i=2}^{n_1}\frac{2+r_1+n_2}{2n_2+\mu_i(G)}+\right.$$
$$+\frac{4+2r_1+2r_2+4n_1+4n_2+r_1r_2+n_1r_1+r_2n_2}{2(2n_1+2n_2+n_1n_2+n_1r_1+n_2r_2)}$$
$$\left.+\sum_{j=2}^{n_2}\frac{2+n_1+r_2+\mu_j(H)}{2n_1-\mu_j(H)}\right)$$

2.

$$Kf(S(G)\bar{\bar{\square}}C(H)) = (m_1+n_1+m_2+n_2)\left(\frac{m_1-n_1}{2+m_2}+\frac{m_2-n_2}{2+m_1}+\sum_{i=2}^{n_1}\frac{2+r_1+m_2}{m_1r_1+\mu_i(G)}\right.$$
$$+\frac{r_1r_2+2m_1+2m_2+2r_1+2r_2+r_1m_1+r_2m_2+r_1m_2+r_2m_1+4}{2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2}$$
$$\left.+\sum_{j=2}^{n_2}\frac{2+m_1+r_2-\mu_j(H)}{m_1r_2-m_1\mu_J(H)-\mu_j(H)}\right)$$

3.

$$Kf(S(G)\,\bar{\dot{\square}}\,C(H)) = (m_1+n_1+m_2+n_2)\left(\frac{m_2-n_2}{2}+\frac{m_1-n_1}{2+n_2}+\sum_{i=2}^{n_1}\frac{2+r_1+n_2}{2m_2+\mu_i(G)}\right.$$
$$+\frac{4+4m_1+2r_1+2r_2+n_2r_2+n_2m_1+2n_2+r_1r_2+r_1m_1+r_1n_2-m_1n_2}{4m_1+4m_1n_2+2_1r_1+r_1r_2n_2+r_1m_1n_2+2r_1n_2+m_1r_1n_2}$$
$$\left.+\sum_{j=2}^{n_2}\frac{2+m_1+r_2-\mu_j(H)}{2m_1-\mu_j(H)}\right)$$

4.

$$Kf(S(G)\,\dot{\bar{\square}}\,C(H)\,) = (m_1+n_1+m_2+n_2)\left(\frac{m_1-n_1}{2}+\frac{m_2-n_2}{2+n_1}+\sum_{i=2}^{n_1}\frac{2+r_1+m_2}{2m_2+\mu_i(G)}\right.$$
$$+\frac{4+2r_1+2r_2+4m_2+2n_1+r_1r_2+r_2m_2+r_1n_1+r_2n_1}{4m_2+2r_2m_2+2r_2n_1+1n_1r_2}$$
$$\left.+\sum_{j=2}^{n_2}\frac{2+n_1+r_2-\mu_j(H)}{n_1r_2-n_1\mu_j(H)-\mu_j(H)}\right)$$

The number of the spanning trees of these graphs are

1.

$$t(S(G)\ddot{\square}\,\mathrm{C(H)}) = \frac{1}{\mathrm{n_1+m_1+n_2+m_2}}2^{\mathrm{m_1-n_1+m_2-n_2}}2(\mathrm{r_2n_2+2n_1+2n_2+n_1r_1+n_1n_2})$$
$$\prod_{\mathrm{i=2}}^{\mathrm{n_1}}\left(2\mathrm{n_2}+\mu_{\mathrm{i}}(\mathrm{G})\right)\prod_{\mathrm{j=2}}^{\mathrm{n_2}}2\mathrm{n_1}-\mu_{\mathrm{j}}(\mathrm{H})$$

2.

$$\mathrm{t(S(G)}\bar{\bar{\square}}\mathrm{C(H))} = \frac{1}{\mathrm{n_1+m_1+m_2+m_2}}(2+\mathrm{m_2})^{\mathrm{m_1-n_1}}(2+\mathrm{m_1})^{\mathrm{m_2-n_2}}\prod_{\mathrm{i=2}}^{\mathrm{n_1}}\left(\mathrm{r_1m_2}+\mu_{\mathrm{i}}(\mathrm{G})\right)$$
$$(\mathrm{2r_1m_2+2r_2m_1+r_1r_2m_1+r_1r_2m_2})\prod_{\mathrm{j=2}}^{\mathrm{n_2}}\mathrm{m_1r_2-m_1}\mu_{\mathrm{j}}(\mathrm{H})-\mu_{\mathrm{j}}(\mathrm{H})$$

3.

$$t(S(G)\,\bar{\dot{\square}}\,C(H)) = \frac{1}{n_1+m_1+n_2+m_2}(2+n_2)^{m_1-n_1}(2)^{m_2-n_2}\prod_{i=2}^{n_1}\left(r_1n_2+\mu_i(G)\right)\prod_{j=2}^{n_2}\Big(2m_1 - \mu_j(H)\Big)(4m_1+4m_1n_2+2m_1r_1+r_1r_2n_2+r_1n_2m_1+2r_1n_2+m_1r_1n_2)$$

4.

$$t(S(G)\,\dot{\bar{\square}}\,C(H)) = \frac{1}{n_1+m_1+n_2+m_2}(2)^{m_1-n_1}(2+n_1)^{m_2-n_2}\prod_{i=2}^{n_1}\left(2m_2+\mu_i(G)\right)\prod_{j=2}^{n_2}\left(n_1r_2-n_1\mu_j(H)-\mu_j(H)\right)(4m_2+2r_2m_2+2r_2n_1+r_1r_2n_1)$$

5. Non-regular simultaneous cospectral graphs

We obtained adjacency and Laplacian spectra of these graph joins $S(G)\,\ddot{\square}\,C(H)$, $S(G)\,\bar{\bar{\square}}C(H)$, $S(G)\bar{\dot{\square}}C(H)$ and $S(G)\dot{\bar{\square}}C(H)$. All of these graphs are non-regular. We find cospectral graphs of these non-regular graphs. The subsequent lemmas are used to determine the cospectral graphs.

Lemma 5.1.1. If G is an r-regular graph then $L(G) = rI_n - A(G)$
2. If G and H are A-cospectral regular graphs, then they are also cospectral with respect to the Laplacian matrix.
Using the above lemmas we obtain the following Theorem

Theorem 5.1. Let G_i and H_i be r_i regular graphs, i= 1,2, where G_1 may not be distinct to H_1. If any graphs G_1 and H_1 are A-cospectral, and G_2 and H_2 are A- cospectral then $S(G_1)\,\ddot{\square}\,C(G_2)$ (respectively, $S(G_1)\,\bar{\bar{\square}}C(G_2)$, $S(G_1)\bar{\dot{\square}}C(G_2)$ and $S(H_1)\,\ddot{\square}\,C(H_2)$) (respectively, $S(H_1)\,\bar{\bar{\square}}C(H_2)$, $S(H_1)\bar{\dot{\square}}C(H_2)$and $S(H_1)\dot{\bar{\square}}C(H_2)$) are simultaneously A–cospectral and L- cospectral.

REFERENCES

[1] S. Butler, A note about cospectral graphs for the adjacency and normalized Laplacian matrices, Linear and Multilinear Algebra, 58(3)(2010), pp. 387-90.

[2] D.M. Cardoso, D. Freitas, M. A. A.Martins and F. A. Robbino, Spectra of graphs obtained by a generalization of the join graph operation, Discrete Mathematics, 313(5) (2013), pp. 733-741.

[3] D. M. Cvetkovic, P. Rowlinson and S. Simic, An introduction to the theory of graph spectra, Cambridge University Press, (2010).

[4] A. Das and P. Panigrahi, Construction of simultaneous cospectral graphs for Adjacency, Laplacian and normalized Laplacian Matrices, Kragujevac Journal of Mathematics, 47(6) (2023), pp. 947-964.

[5] C. Godsil and G. F. Royale, Algebraic graph theory, Springer Science & Business Media, (2001).

[6] R. A. Horn, R. A. Horn and C.R. Johnson, Topics in matrix analysis, Cambridge university press, (1994)

[7] T. K. Jahfar and A. V. Chithra, Central vertex join and central edge join of two graphs, AIMS Mathematics, 5 (6)(2020), pp-7214–7234.

[8] T. K. Jahfar and A. V. Chithra, Spectra of new graph operations based on central graph, arXiv preprint arXiv:2107.00854 (2021).

[9] X. Liu and P. Lu, Spectra of subdivision-vertex and subdivision-edge neighbourhood coronae, Linear algebra and its applications. 438(8) (2013), pp. 3547-59.

[10] X. Liu, Z. Zhang, Spectra of subdivision-vertex join and subdivision-edge join of two graphs, Bulletin of the Malaysian Mathematical Sciences Society. 42(2019), pp. 15-31.

[11] Z Lu, X. Ma and M Zhang, Spectra of graph operations based on splitting graph, Journal of Applied Analysis and Computation 13(1)(2023) pp. 133-155

[12] C. McLeman C and E. McNicholas, Spectra of coronae, Linear algebra and its applications, 435(5)(2011), pp. 998–1007.

[13] L. Xiaogang and L. Pengli, Spectra of subdivision-vertex and subdivision-edge neighbourhood coronae Linear Algebra and its Applications, 438(8) (2013), pp. 3547-3559

[14] J. V. Vivin, M. M. Akbar and K. Thilagavathi, On harmonious colouring of central graphs, Advances and applications in Discrete Mathematics, 2(1), pp 17–33