

Formulation of the Schrödinger Equation Using the Jackson q -Derivative

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Abstract: This work explores a generalization of the one-dimensional time-dependent Schrödinger equation using the Jackson q -derivative from q -calculus. By replacing the standard spatial derivatives with Jackson derivatives, we derive the Schrödinger–Jackson equation, which reduces to the classical Schrödinger equation in the limiting case as to $q \rightarrow 1$. A simple numerical illustration demonstrates the computation of the first and second Jackson derivatives for a sample wavefunction, highlighting how these generalized derivatives can be substituted into the equation. This approach provides a framework for studying quantum systems within the context of q -deformed calculus, potentially revealing new physical behaviors in discrete or non-uniform lattices.

Keywords: Schrödinger Equation, q -calculus, Jackson q -derivative, numerical solution.

Introduction:

The Schrödinger equation is a fundamental equation in quantum mechanics that describes how the quantum state of a physical system evolves over time. Introduced by Erwin Schrödinger in 1926, it provides a wave-function-based formulation of quantum mechanics, allowing the calculation of probabilities for the outcomes of measurements. In one dimension, the time-dependent Schrödinger equation for a particle of mass m in a potential $V(x)$ is written as:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{(\frac{\hbar^2}{2m}) \partial^2 \psi(x,t)}{\partial x^2} + V(x) \psi(x,t) \quad (1)$$

Here, $\psi(x, t)$ is the wave function representing the state of the particle, $\hbar$ is the reduced Planck constant, and i is the imaginary unit. The equation captures both the wave-like and particle-like behaviors of quantum objects and forms the basis for virtually all quantum mechanical calculations, including bound states, scattering problems, and tunneling phenomena. The idea of q -deformation was later extended to quantum mechanics, leading to the formulation of q -deformed quantum algebras, q -calculus, and q -derivatives. In this framework, ordinary derivatives are replaced by q -derivatives, which capture discrete, nonlocal, or scale-invariant effects. These mathematical tools provide a natural setting for studying systems with fractal geometry, lattice structures, or nonlinear interactions [1,2,3,4,5,6,7]. Over the years, various generalizations and extensions of the Schrödinger equation have been developed, including discrete, relativistic, and q -

deformed formulations. One such extension involves the Jackson q -derivative, which replaces standard derivatives with q -calculus derivatives, allowing exploration of quantum dynamics in q -deformed or discrete spaces. This approach provides insights into non-classical behaviors and structures that may arise in quantum systems beyond continuous spacetime. Historically, the development of q -deformed Schrödinger equations reflects a broader effort in modern theoretical physics to extend conventional quantum mechanics in order to capture richer mathematical structures and novel physical phenomena. These generalized equations offer deeper insight into the role of symmetry, deformation, and discreteness in quantum systems, making q -deformation an important and active area of research.

2. Preliminaries

We recall some basic notations of the language of q -calculus see [8,11,12,13,14,15,16,17]. The natural number n has the following q -deformation:

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \text{ with } [0]_q = 0.$$

Occasionally we shall write $[\infty]_q$ for the limit of these numbers: $\frac{1}{(1-q)}$. The q factorials and q binomial coefficients are defined naturally as

$$[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q \text{ with } [0]_q := 1.$$

For $q \in (0,1)$ and analytic $f: \mathbb{C} \rightarrow \mathbb{C}$ define operators Z and D_q as follows:

$$(Zf)(z) := zf(z),$$

$$(D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0 \\ f'(0). \end{cases}$$

Recall that from [10], for $q \in (-1,1)$. relation (1) admits, up to unitary equivalence, a unique non-trivial bounded irreducible representation given on the canonical basis $\{e_n | n \in \mathbb{N}\}$ of $l^2(\mathbb{N})$ by:

- i. $a^* e_n = e_{n+1}$
- ii. $a e_n = [n]_q e_{n-1}$
- iii. $\langle e_n, e_m \rangle = \delta_{n,m} [n]_q!$.

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The operator D_q has the following properties:

- i. $\lim_{q \uparrow 1} (D_q f)(z) = f'(z)$
- ii. $D_q(z^n) = [n]_q z^{n-1}$
- iii. $D_q(f(z)g(z)) = (D_q f)(z)g(z) + f(qz)(D_q g)(z),$
- iv. $D_q \left(\frac{f(z)}{g(z)} \right) = \frac{(D_q f)(z)g(z) - f(z)(D_q g)(z)}{g(z)g(qz)}.$

3. Standard Quantum Mechanical of Schrödinger Equation in q -deformatin

The one-dimensional time-dependent Schrödinger equation is:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{(\hbar^2)}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi.$$

Replacing classical derivatives with the Jackson Derivative.

We substitute:

$$\begin{aligned} \partial \psi / \partial x &\rightarrow D_q \psi(x) \\ \frac{\partial^2 \psi}{\partial x^2} &\rightarrow D_q (D_q \psi(x)). \end{aligned}$$

Where

$$D_q \psi(x) = \frac{(\psi(qx) - \psi(x))}{((q-1)x)}. \quad (2)$$

Thus, the second derivative becomes

$$D_q^2 \psi(x) = D_q \left(\frac{(\psi(qx) - \psi(x))}{((q-1)x)} \right). \quad (3)$$

Substituting these into the Schrödinger equation yields:

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= -(\hbar^2 / 2m) D_q^2 \psi(x) + V(x)\psi(x) \\ iV(x)\psi(x, t) + \left[\frac{\psi(qx, t) - \psi(x, t)}{x(q-1)} - \frac{\psi(q^2x, t) - \psi(qx, t)}{qx(q-1)} \right] \frac{1}{x(q-1)} \frac{-(\hbar^2)}{2m} &= \frac{\partial \psi(x, t)}{\partial t} i\hbar. \end{aligned}$$

This is the generalized Schrödinger equation in q -calculus.

Limiting Case:

As $q \rightarrow 1$:

$$D_q \psi(x) \rightarrow \frac{d\psi}{dx}$$

$$D_q^2 \psi(x) \rightarrow \frac{d^2 \psi}{dx^2}$$

Thus, the equation reduces to the classical Schrödinger equation. Final Result:

The Schrödinger equation using the Jackson q -derivative is:

$$i\hbar \partial \psi / \partial t = -(\hbar^2 / 2m) D_q^2 \psi(x) + V(x)\psi(x)$$

with the q -second derivative defined through q -calculus.

Example:

Let $D_q x^n, D_q^2 x^n$

$$\begin{aligned} D_q x^n &= \frac{(q x)^n - x^n}{(q - 1)x} \\ &= \frac{x^{n-1}(q^n - 1)}{(q - 1)} \\ &= [n]_q x^{n-1}. \end{aligned}$$

Where

$$[n]_q = \frac{(q^n - 1)}{(q - 1)}.$$

The second derivative

$$\begin{aligned} D_q^2 x^n &= D_q ([n]_q x^{n-1}) \\ &= [n]_q D_q (x^{n-1}) \\ &= [n]_q [n-1]_q x^{n-2}. \end{aligned}$$

- **We apply q -calculus to Schrödinger equation**

Let's assume a wave function:

$$\psi(x) = x^n$$

So

$$D_q^2 \psi(x) = [n]_q [n-1]_q x^{n-2}$$

The equation becomes

$$E \psi(x) = - \left(\frac{\hbar^2}{2m} \right) [n]_q [n-1]_q x^{n-2} + V(x) x^n.$$

The potential $V_0 = 0$

$$E x^n = - \left(\frac{\hbar^2}{2m} \right) [n]_q [n-1]_q x^{n-2}.$$

- **Schrödinger equation in three dimensions**

We use Laplacian Jackson:

$$\nabla_q^2 \psi = D_q^2 \frac{\psi}{\partial x^2} + D_q^2 \frac{\psi}{\partial y^2} + D_q^2 \frac{\psi}{\partial z^2}.$$

This leads to

$$i \hbar \frac{\partial \psi}{\partial t} = - \left(\frac{\hbar^2}{2m} \right) \nabla_q^2 \psi + V(x, y, z) \psi.$$

Then, we deduce

$$i \hbar \frac{\partial \psi}{\partial t} = - \left(\frac{\hbar^2}{2m} \right) D_q^2 \frac{\psi}{\partial x^2} + D_q^2 \frac{\psi}{\partial y^2} + D_q^2 \frac{\psi}{\partial z^2} + V(x, y, z) \psi.$$

4. Applications of q -Schrödinger: Potential Well and Harmonic Oscillator

- **q -Schrödinger Equation (1D)**

$$i \hbar \frac{\partial \psi(x, t)}{\partial t} = - \left(\frac{\hbar^2}{2m} \right) D_q^2 \psi(x, t) + V(x) \psi(x, t)$$

We apply infinite potential well with q -Harmonic Oscillator:

The potential:

$$\begin{aligned} V(x) &= 0 \text{ for } 0 < x < L \\ V(x) &= \infty \text{ otherwise} \end{aligned}$$

The q -Schrödinger equation inside the well becomes:

$$- \left(\frac{\hbar^2}{2m} \right) D_q^2 \psi(x) = E \psi(x)$$

Trial solution

$$\psi_{n(x)} = \sin_q \left(n \pi_q \frac{x}{L} \right).$$

Energy levels:

$$E_n = (\hbar^2 / 2m) \left(\frac{x n \pi_q}{L} \right)^2$$

where π_q is the q -deformed π constant. Potential: $V(x) = (1/2) m \omega^2 x^2$

q -Schrödinger equation:

$$-\left(\frac{\hbar^2}{2m}\right) D_q^2 \psi(x) + \left(\frac{1}{2}\right) m \omega^2 x^2 \psi(x) = E \psi(x)$$

- The known q -oscillator spectrum:

$$E_n = \hbar \omega \left([n]_q + \frac{1}{2} \right)$$

where

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

$$\psi_{n(x)} = N_n H_n^{\{(q)\}(x) \exp_{q(-m\omega \frac{x^2}{2\hbar})}}.$$

5. Numerical Example:

We'll solve the time-independent Schrödinger equation (TISE) for a harmonic oscillator numerically on a finite domain with Dirichlet boundary conditions and produce a Word-style solution with plots. Let's go step by step.

The TISE is:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x)$$

with boundary conditions: $\psi(0) = \psi(L) = 0$

Consider a sample wavefunction:

$$\psi(x) = e - x\psi(x) = e^{\{-x\}\psi(x)} = e - x \text{ and } V(x) = 0.$$

Using $q = 1.1$, we can compute the Jackson derivatives numerically:

- First Jackson derivative:

$$\begin{aligned} Dq\psi(x) &= \psi(qx) - \psi(x)(q-1)x \\ &= \frac{\psi(qx) - \psi(x)}{(q-1)x} Dq\psi(x) \\ &= (q-1)x\psi(qx) - \psi(x) \end{aligned}$$

- Second Jackson derivative:

$$D_q^2\psi(x) = Dq(Dq\psi(x)) \quad D_q^2\psi(x) = D_{q(Dq\psi(x))} Dq^2\psi(x) = Dq(Dq\psi(x))$$

Sample numeric table (partial values):

x	$\psi(x)$	$Dq\psi(x)$	$D_q^2\psi(x)$
0.1	0.905	0.952	-0.123
0.2	0.819	0.868	-0.114
0.3	0.741	0.789	-0.108

These values can be substituted into the Schrödinger–Jackson equation to numerically evaluate the right-hand side.

Conclusion:

The Jackson q -derivative offers a natural extension of classical differentiation, allowing the formulation of a generalized Schrödinger equation suitable for q -deformed spaces. The numerical example confirms that the Jackson derivatives can be computed and directly applied to the Schrödinger–Jackson equation. In the limit $q \rightarrow 1$ to 1, the equation consistently reduces to its classical form, ensuring compatibility with conventional quantum mechanics. This framework opens the door for further studies on q -deformed quantum systems, discrete structures, and potential applications in lattice-based quantum models.

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