# An Extension of Soft Operations on Generalized Soft Subsets

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**Abstract**

**Existing Literature, Problem and Limitation:** To address problems of fuzzy data in various fields, Molodtsov presented soft set theory, a broad mathematical technique for ambiguity. This theory has been used in a variety of pure and practical mathematical fields. It is evident in this theory that soft subsets and soft equal relations significantly contributed to soft topology, lattices, soft groups, etc. Existing research is limited in that various features, such as associative, distributive, etc., are not confirmed by some current soft subsets for soft product operations. **Purpose:** While studying soft subsets, we observe that several algebraic properties have not yet been investigated on various generalized soft subsets to enhance algebraic structures in soft set theory. So, this article investigates some of these algebraic properties on different generalized soft subsets on different soft operations. **Contribution:** This study demonstrates a few counterexamples that some algebraic properties are unsatisfied by generalized soft subsets. Based on this approach, we present some crucial theorems and results that show these significant features on all soft subsets by employing additional conditions. A universal complement property in soft set theory in relation to soft complements (negation complement ($^c$) and relative complement ($^r$)) is propounded. **Limitation:** The sole restriction of these results is that two generalised soft subsets (soft J-subset and soft L-subset) do not satisfy the union and intersection condition of classical mathematics as described in section 4.

**Keywords:** Soft sets, Soft M-subset, Soft L-subset, Soft F-subset, Soft J-subset, Soft Complements etc.

## 1. Introduction

### *1.1. Problem Statement:*

Most existing techniques for formal reasoning, computing, and modelling have a clear, deterministic, and precise nature. But there are other challenging issues in fields, including economics, engineering, environment etc., that can sometimes involve fuzzy data. Because these challenges contain a variety of forms of uncertainty, we cannot effectively employ classical approaches to solve them. Several theories can be viewed as a framework in mathematics to cope with ambiguities, including the theory of interval mathematics, vague sets, fuzzy sets, and a few others. Molodtsov [1] noted each of these theories includes deficiencies that are inherent to them. To address these issues mathematically, Molodtsov developed the idea of soft set theory. Soft sets could be viewed as a particular type of neighbourhood systems and context-dependent fuzzy sets. The problem of constructing the membership function as well as other related complications are essentially nonexistent in this theory. As a result, it is extremely useful and has potential applications in various areas of mathematics, as shown in [1]. In recent years, many authors [1, 6, 8, 10, 11, 19, 20] worked on operations of soft sets and studied algebraic structures in soft set theory. But we observe that very few works has been done on join($\tilde{\vee}$) and meet($\tilde{\wedge}$) operations of soft sets. Therefore, for extending algebraic structures in soft set theory, we studied these operators on generalized soft subsets, and gives some algebraic properties in this research article.

### *1.2. Previous Research and Limitations:*

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Maji et. al. [11] provided the first detailed explanation of the concept of soft subsets. A complete theoretical examination on soft sets was also written by them, and asserted a few results regarding soft distributive laws with respect to soft products ($\tilde{\wedge}$ and $\tilde{\vee}$) operations of soft sets, but they did not present any supporting data (see 2.6 in [11]). Moreover, according to Ali et al. [6], the results in [10] were inaccurate (see 2.8 in [10]). Therefore, the ideas of generalized soft subsets and soft equal relations were also put forward by Jun and Yang [20]. In an effort to respond to Maji's results (Proposition 2.6 in [11]) and suggested generalized soft distribution laws, they also tried to apply generalized soft equal relations. They started by defining soft J-equal and soft L-equal relations. It is crucial to note that Jun and Yang in [20] and Liu et al. in [19] did not reach the same conclusion on the applicability of distributive laws to all types of soft equal relations. Additionally, Feng and Li [3] thoroughly examined soft product operations, conducted a systematic investigation of five different kinds of soft subsets and developed the free soft algebraic quotient structures linked to soft product operations. But no one study these soft product operations on different types of generalized soft subsets. Therefore, we tried to attempt and explore some operations on various types of generalized soft subsets.

*1.3. Motivation:*

Yadav and Singh [12] first studied El-algebra in soft sets and introduced the concept of soft El-algebra as well as a number of noteworthy algebraic features. But while studying ES structure [13, 18] on soft sets, we observed that the given structure does not make a lattice structure in the sense of soft M-subsets. Therefore, we studied other generalized soft subsets [19, 20] and found that ES structure makes a lattice structure with respect to soft J-subsets. Furthermore, for defining order reversing involution operator on ES structure, we needed complement operation of soft sets. So, we studied two types of complements [6, 11] in soft sets and observed that no one worked on generalized soft sets on these complements. Therefore, in the present article, we derive some algebraic properties of generalized soft subsets on these complement operations.

*1.4. Contribution of the study:*

Soft Set Theory is a mathematical framework that deals with ambiguities and vagueness in real-world scenarios. In this research article, the researchers focused on the algebraic properties of generalized soft subsets. They found that some of these algebraic properties were not satisfied by any of generalized soft subsets, and demonstrated these findings with a few counterexamples.

To overcome this issue, the researchers proposed additional conditions on the elements of parameter set, that would satisfy these significant algebraic features on all soft subsets. They presented some crucial theorems and results supported by real life example that showed how these conditions could be used to achieve these algebraic features. Moreover, the researchers studied the universal complement property on all generalized soft subsets in soft set theory. This property relates to two types of soft complements, which include negation soft complement ($^c$) and relative soft complement ($^r$). The researchers proposed a new approach to achieve this property, which can be used to define soft complements more generally. Overall, this study contributes to the advances of soft set theory by addressing some of the algebraic properties of generalized soft subsets and proposing new conditions to satisfy these properties.

*1.5. Paper Organization:*

This work is divided into six components as: Section 1. provides research problem, previous research, motivation, background etc. Section 2. summarise the fundamental definitions of soft sets and their operations like soft unions ($\tilde{\cup}$), soft intersections ($\tilde{\cap}$), soft products ($\tilde{\vee}$ and $\tilde{\wedge}$) etc. with some basic results. Section 3. gives a brief introduction to four types of soft subsets with an important proposition about their interrelations. Section 4. is devoted to provide some important outcomes on various soft subsets in terms of soft product, soft union and soft intersection operations. In Section 5. we first give a general property of complement on classical sets in mathematics, and then study this property on all generalized soft subsets in the sense of soft negation and soft relative complements [6, 11]. Section 6. provides the conclusion and future work of present study.

*1.6. Background:*

As we know that Molodtsov [1] presented the idea of soft sets as a unique mathematical technique to dealing with ambiguities. The implementation of this theory to a decision-making issue involving rough sets is described by Maji et. al. [10] by utilizing soft sets in the form of a binary information table, and defined first time parameter reduction on soft sets. Further, in [11], they gave a few definitions and operations of soft sets like soft subset ($\tilde{\subseteq}$), soft superset ($\tilde{\supseteq}$), null soft set ($\tilde{\Phi}$), absolute soft sets ($\tilde{A}$), soft complement, soft union ($\tilde{\cup}$), soft intersection ($\tilde{\cap}$), "AND" and "OR" ($\tilde{\wedge}$ and $\tilde{\vee}$) operations. Further, Feng et. al. [4] provided the definition of soft subset in a different way. But, Cagman and Enginoglu [8] built a uni-int decision-making approach by redefining soft sets operations to improve new results. Ali et. al. [6] also gave some new operations on soft sets like extended and restricted intersection, difference and union, relative and negation complements, and proved De-Morgan's law on these operations. Moreover, in [6, 8], they proved that soft products ($\tilde{\vee}$-product and $\tilde{\wedge}$-product) does not hold commutative and associative properties in the sense of soft M-equal relation. For studying these algebraic properties, Jun and Yang [20] gave the definition of generalized soft subset (Soft J-subset in [19]), and proved distributive property called it generalized distributive law, with respect to soft J-equality and generalized interval-valued fuzzy soft equality. After that Liu et. al. [19] combined fuzzy, rough and soft sets to provide four types of generalized soft subsets with some basic properties. They found that the distributive property given in [20] does holds with respect to soft J-equal, and provided a new generalized soft distributive law of soft L-equal. Moreover, they amended the associative property of Maji [11] with respect to soft L-equal and said that this property can be satisfied only by soft L-equality instead of other existing equality.

In literature, some authors [2, 7, 9] have explored above properties to topological spaces in soft set theory and presented different kinds of soft topological spaces. A full and exhaustive overview of the researches done in soft set theory and the advancements of topological spaces in soft sets was provided by Yadav and Singh [14]. According to Bentley [5], topological spaces can be derived from nearness spaces. Furthermore, Singh with others [15, 16, 17] studied the concepts of soft d-proximity, soft binary heminearness spaces, and nearness of finite order $\mathrm{S}_n$-merotopy respectively.

## 2. Preliminaries:

Some fundamental definitions of soft sets and associated operators are provided in this section. Throughout the whole article, U and E are non-empty finite universal sets of objects and all possible parameters/attributes respectively. The touple (U, E) or $\mathrm{U}_E$ is referred to as a soft universe.

**Definition 2.01([1]):** Let P(U) be the power set of U and A $\subseteq$ E. Then a couple (F, A) is said to be *soft set* over U, if F is a representation defined as:

$$\mathrm{F} : \mathrm{A} \longrightarrow \mathrm{P(U)}.$$

Here, we writes a soft set (F, A) by $\mathrm{F}_A$, where $\mathrm{F}_A = \{(\alpha, \mathrm{F}(\alpha)) \mid \alpha \in \mathrm{A}, \mathrm{F}(\alpha) \in \mathrm{P(U)}\}$. Set F($\alpha$) can be selected at random. Soft set is not a classical set. Then, a significant quantity of information was provided in [1].

**Definition 2.02([11]):** 1. A *null soft set* $\tilde{\Phi}$, is a soft set $\mathrm{F}_A$ on U, if $\forall\ \alpha \in \mathrm{A}$, $\mathrm{F}(\alpha) = \phi$ (null set). 2. An *absolute soft set* $\tilde{A}$, is a soft set $\mathrm{F}_A$ on U, if $\forall\ \alpha \in \mathrm{A}$, $\mathrm{F}(\alpha) = \mathrm{U}$.

**Definition 2.03([11]):** Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are two soft sets on U. Then *Intersection* of $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ over U is defined as: $\mathrm{F}^1_{A_1}\ \tilde{\cap}\ \mathrm{F}^2_{A_2} = \mathrm{F}^3_{A_3}$, where $\mathrm{A}_3 = \mathrm{A}_1 \cap \mathrm{A}_2$, and $\mathrm{F}^3(\alpha) = \mathrm{F}^2(\alpha)$ or $\mathrm{F}^1(\alpha)$, (as both have similar approximation), $\forall\ \alpha \in \mathrm{A}_3$.

**Definition 2.04([6]):** The *extended intersection* of $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ over U is written as $\mathrm{F}^1_{A_1}\ \tilde{\sqcap}_{\mathscr{E}}\ \mathrm{F}^2_{A_2}$ and defined as: $\mathrm{F}^1_{A_1}\ \tilde{\sqcap}_{\mathscr{E}}\ \mathrm{F}^2_{A_2} = \mathrm{F}^3_{A_3}$, where $\mathrm{A}_3 = \mathrm{A}_1 \cup \mathrm{A}_2$, and $\forall\ \alpha \in \mathrm{A}_3$

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$$F^3(\alpha) = \begin{cases} F^1(\alpha) & , \alpha \in A_1 - A_2 \\ F^2(\alpha) & , \alpha \in A_2 - A_1 \\ F^1(\alpha) \cap F^2(\alpha) & , \alpha \in A_1 \cap A_2. \end{cases}$$

**Definition 2.05([6]):** Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets on U such as $A_1 \cap A_2 \neq \phi$. Then, the *restricted intersection* of $F^1_{A_1}$ and $F^2_{A_2}$ is written as $F^1_{A_1} \tilde{\Cap} F^2_{A_2}$, described as $F^1_{A_1} \tilde{\Cap} F^2_{A_2} = F^3_{A_3}$, where $A_3 = A_1 \cap A_2$, and $\forall\ \alpha \in A_3$, $F^3(\alpha) = F^1(\alpha) \cap F^2(\alpha)$.

**Result 2.06:** By the definitions 2.03, 2.04 and 2.05 we can conclude that *intersection and extended intersection* of two non-null soft sets is a non-null soft set, either their approximations are similar or not at the same attribute. However, their *restricted intersection* may be a null soft set, as shown by below example.

**Example 2.07:** Consider U = $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ be the set of five canditates for an interview in a company, and E = $\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the set of five types of jobs, where $\varrho_1$ stands for Network Administrator (NA), $\varrho_2$ stands for User Experience Designer (UED), $\varrho_3$ stands for System Analyst (SA), $\varrho_4$ stands for Database Administrator (DA) and $\varrho_5$ stands for Development Operations Engineer (DOE). let $F^1_{A_1}$ and $F^2_{A_2}$ are two members in selection board, which provides the names of capable canditates for respective jobs in sets $A_1$ and $A_2$ respectively. Here, we consider $F^1_{A_1}$ and $F^2_{A_2}$ are two soft sets over universe set U, defined as:

$F^1_{A_1} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_2, \{\mu_4, \mu_5\})\}$,

$F^2_{A_2} = \{(\varrho_1, \{\mu_3\}), (\varrho_2, \{\mu_1, \mu_2\}), (\varrho_3, \{\mu_4, \mu_5\})\}$.

By definition 2.05, $F^1_{A_1} \tilde{\Cap} F^2_{A_2} = F^3_{A_3}$, where $A_3 = A_1 \cap A_2$. Now, $A_3 = \{\varrho_1, \varrho_2\}$ and hence $F^3_{A_3} = \{(\varrho_1, \phi), (\varrho_2, \phi)\}$. Here, $F^1_{A_1} \tilde{\Cap} F^2_{A_2}$ provides the most suitable canditates for common jobs in respect to the opinions of $F^1_{A_1}$ and $F^2_{A_2}$.

**Definition 2.08([11]):** Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets on U, then the *soft union* is provided as a soft set $F_A$, that satisfies the following criteria:

(i) $A = A_1 \cup A_2$,

(ii) $\forall\ \alpha \in A$,

$$F(\alpha) = \begin{cases} F^1(\alpha) & , \alpha \in A_1 - A_2 \\ F^2(\alpha) & , \alpha \in A_2 - A_1 \\ F^1(\alpha) \cup F^2(\alpha) & , \alpha \in A_1 \cap A_2. \end{cases}$$

**Definition 2.09([11]):** Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets described on U, where $A_1, A_2 \subseteq E$. Then "*AND*" can be defined as: $F^1_{A_1} \tilde{\wedge} F^2_{A_2} = F_{A_1 \times A_2}$, where $\forall\ (\alpha, \beta) \in A_1 \times A_2$, $F(\alpha, \beta) = F^1(\alpha) \cap F^2(\beta)$.

**Definition 2.10([11]):** Let $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets described on U, where $A_1, A_2 \subseteq E$. Then "*OR*" can be defined as: $F^1_{A_1} \tilde{\vee} F^2_{A_2} = F_{A_1 \times A_2}$, where $\forall\ (\alpha, \beta) \in A_1 \times A_2$, $F(\alpha, \beta) = F^1(\alpha) \cup F^2(\beta)$.

**Result 2.11:** Let $F^1_{A_1} \neq \tilde{\Phi}$, $F^2_{A_2} \neq \tilde{\Phi}$. Then, "AND" of $F^1_{A_1}$ and $F^2_{A_2}$ can be a null soft set i.e. $F^1_{A_1} \tilde{\wedge} F^2_{A_2} = \tilde{\Phi}$ (see example 2.12).

**Example 2.12:** Let U and E are universal sets as given in example 2.07, $F^1_{A_1}$ and $F^2_{A_2}$ are soft sets defined as:

$F^1_{A_1} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_2, \{\mu_5\})\}$,

$F^2_{A_2} = \{(\varrho_1, \{\mu_3\}), (\varrho_3, \{\mu_5\})\}$.

Now, $A_3 = A_1 \times A_2 = \{(\varrho_1, \varrho_1), (\varrho_2, \varrho_1), (\varrho_1, \varrho_3), (\varrho_2, \varrho_3)\}$. Thus, $F^1_{A_1} \tilde{\wedge} F^2_{A_2} = F^3_{A_3} = \{((\varrho_1, \varrho_1), \phi), ((\varrho_2, \varrho_1), \phi), ((\varrho_1, \varrho_3), \phi), ((\varrho_2, \varrho_3), \phi)\}$, where $F^1_{A_1} \tilde{\wedge} F^2_{A_2}$ provides most suitable candidates for one or two perticular jobs at a time with respect to the opinion of $F^1_{A_1}$ and $F^2_{A_2}$.

## 3. Generalized Soft Subsets:

Maji et. al. [11] and Feng et. al. [4] gave two kinds of soft subsets and soft equal relations. Liu et. al. [19] called it soft M-subset, soft F-subset and soft M-equal relation, soft F-equal relation. A generalization of soft subsets was examined by Jun and Yang [20]. Furthermore, Liu et. al. [19] gave the notions of soft J-subset and soft L-subset and demonstrated that soft M-equal and soft F-equal relations are correlate with one another, while others are distinct in general. Here, we provide an overview of four different soft subsets as:

**Definition 3.01([11]):** Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are soft sets defined on U. Then $\mathrm{F}^1_{A_1}$ is a *soft subset* (renamed it a soft M-subset in [19]) of $\mathrm{F}^2_{A_2}$ if:

(i) $\mathrm{A}_1 \subseteq \mathrm{A}_2$,

(ii) For each $\mathrm{a}_1 \in \mathrm{A}_1$, $\mathrm{F}^1(\mathrm{a}_1)$ and $\mathrm{F}^2(\mathrm{a}_1)$ are approximations that are similar.

Soft subset is represented by $\mathrm{F}^1_{A_1} \tilde{\subseteq} \mathrm{F}^2_{A_2}$ or $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2}$. Also $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft M-equal or soft equal, written as $\mathrm{F}^1_{A_1} =_M \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_M \mathrm{F}^1_{A_1}$.

**Definition 3.02([4]):** Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are soft sets on U. Then $\mathrm{F}^1_{A_1}$ is *soft subset* (renamed it a soft F-subset in [19]) of $\mathrm{F}^2_{A_2}$, written as $\mathrm{F}^1_{A_1} \tilde{\subseteq}_F \mathrm{F}^2_{A_2}$, iff $\mathrm{A}_1 \subseteq \mathrm{A}_2$ and $\mathrm{F}^1(\mathrm{a}_1) \subseteq \mathrm{F}^2(\mathrm{a}_1)\ \forall\ \mathrm{a}_1 \in \mathrm{A}_1$. Also, $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft F-equal, denoted as $\mathrm{F}^1_{A_1} =_F \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_F \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_F \mathrm{F}^1_{A_1}$.

**Definition 3.03([19]):** Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are non-empty soft sets. Then $\mathrm{F}^1_{A_1}$ is called *soft J-subset* of $\mathrm{F}^2_{A_2}$ or $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$ if and only if for any $\mathrm{a}_1 \in \mathrm{A}_1$, $\exists\ \mathrm{a}_2 \in \mathrm{A}_2$ such that $\mathrm{F}^1(\mathrm{a}_1) \subseteq \mathrm{F}^2(\mathrm{a}_2)$ (see example 3.04). Also, $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft J-equal, denoted as $\mathrm{F}^1_{A_1} =_J \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_J \mathrm{F}^1_{A_1}$.

**Example 3.04:** Consider U and E are universal sets of candidates and jobs respectively, as given in example 2.07. Let,

$\mathrm{F}^1_{A_1} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_2, \{\mu_3\}), (\varrho_3, \{\mu_2, \mu_3\})\}$,

$\mathrm{F}^2_{A_2} = \{(\varrho_1, \{\mu_1, \mu_2\}), (\varrho_3, \{\mu_2, \mu_3\}), (\varrho_4, \{\mu_2\})\}$ .

Since $\mathrm{A}_1 \neq \mathrm{A}_2$, so $\mathrm{F}^1_{A_1} \neq_M \mathrm{F}^2_{A_2}$. But we can see that, $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_J \mathrm{F}^1_{A_1}$. Hence, $\mathrm{F}^1_{A_1} \tilde{=}_J \mathrm{F}^2_{A_2}$.

Here, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2}$, then from example 2.07 it indicates that both members ($\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$) of selection board selected same candidates for every job in $\mathrm{A}_1$. Similarly, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$, then it indicates that for every job in $\mathrm{A}_1$, the members selected by $\mathrm{F}^1_{A_1}$ are also selected by $\mathrm{F}^2_{A_2}$ for the same or different job in $\mathrm{A}_2$.

**Definition 3.05([19]):** Let $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are non-empty soft sets on U. Then, $\mathrm{F}^1_{A_1}$ is *soft L-subset* of $\mathrm{F}^2_{A_2}$ or $\mathrm{F}^1_{A_1} \tilde{\subseteq}_L \mathrm{F}^2_{A_2}$ if and only if for any $\mathrm{a}_1 \in \mathrm{A}_1$, $\exists\ \mathrm{a}_2 \in \mathrm{A}_2$ such that $\mathrm{F}^1(\mathrm{a}_1) = \mathrm{F}^2(\mathrm{a}_2)$. Soft sets $\mathrm{F}^1_{A_1}$ and $\mathrm{F}^2_{A_2}$ are called soft L-equal, denoted as $\mathrm{F}^1_{A_1} =_L \mathrm{F}^2_{A_2}$, if $\mathrm{F}^1_{A_1} \tilde{\subseteq}_L \mathrm{F}^2_{A_2}$ and $\mathrm{F}^2_{A_2} \tilde{\subseteq}_L \mathrm{F}^1_{A_1}$.

**Proposition 3.06([19]):** Let $\mathrm{F}^1_{A_1} \neq \tilde{\Phi}$ and $\mathrm{F}^2_{A_2} \neq \tilde{\Phi}$. Then,

(1) $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_F \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$,

(2) $\mathrm{F}^1_{A_1} \tilde{\subseteq}_M \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_L \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} \tilde{\subseteq}_J \mathrm{F}^2_{A_2}$,

(3) $\mathrm{F}^1_{A_1} =_M \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} =_L \mathrm{F}^2_{A_2} \implies \mathrm{F}^1_{A_1} =_J \mathrm{F}^2_{A_2}$.

But generally, the converse of the aforementioned arguments does not exist (see examples 2.6, 2.9, 3.3 in [19]).

## 4. Generalized Soft Subsets on Soft Operations:

This section presents some characterizations of above given different types of soft subsets with respect to two properties given below. We can see that all soft subsets satisfy only property 4.01 (1); property 4.01 (2) could be satisfied by soft F-subset and soft M-subset instead of soft J-subset and soft L-subset.

**Property 4.01:** Let P, Q, X and Y are four crisp subsets of the universe set U such as P $\subseteq$ X and Q $\subseteq$ Y. Then we have,
(1). P $\vee$ Q $\subseteq$ X $\vee$ Y, and P $\wedge$ Q $\subseteq$ X $\wedge$ Y,
(2). P $\cap$ Q $\subseteq$ X $\cap$ Y, and P $\cup$ Q $\subseteq$ X $\cup$ Y.

**Proposition 4.02:** Let $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets defined on U. Then,
(1). If $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.
(2). If $F^1_{A_1} \tilde{\subseteq}_M F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_M F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.
(3). If $F^1_{A_1} \tilde{\subseteq}_J F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_J F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\wedge} F^4_{A_4}$,
(4). If $F^1_{A_1} \tilde{\subseteq}_L F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_L F^4_{A_4}$, then $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_L F^2_{A_2} \tilde{\vee} F^4_{A_4}$ and $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_L F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.

**Proof:** We simply demonstrate the correctness of (1) and (3); same method can be used to obtain (2) and (4).

(1). Let $F^1_{A_1} \tilde{\vee} F^3_{A_3} = F_{A_1 \times A_3}$ and $F^2_{A_2} \tilde{\vee} F^4_{A_4} = G_{A_2 \times A_4}$, where $F(\alpha, \gamma) = F^1(\alpha) \cup F^3(\gamma)$, $\forall$ $(\alpha, \gamma) \in A_1 \times A_3$ and $G(\beta, \delta) = F^2(\beta) \cup F^4(\delta)$, $\forall$ $(\beta, \delta) \in A_2 \times A_4$. Since, $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$. So, we have $A_1 \subseteq A_2$, $A_3 \subseteq A_4$, $F^1(\alpha) \subseteq F^2(\alpha)$ $\forall$ $\alpha \in A_1$ and $F^3(\gamma) \subseteq F^4(\gamma)$ $\forall$ $\gamma \in A_3$. It implies that, $A_1 \times A_3 \subseteq A_2 \times A_4$ and $F^1(\alpha) \cup F^3(\gamma) \subseteq F^2(\alpha) \cup F^4(\gamma)$, $\forall$ $(\alpha, \gamma) \in A_1 \times A_3$. Hence, $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\vee} F^4_{A_4}$. Similarly, we can prove that $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.

(3). Since, $F^1_{A_1} \tilde{\subseteq}_J F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_J F^4_{A_4}$, so for every $\alpha \in A_1$, $\exists$ $\beta \in A_2$ such that $F^1(\alpha) \subseteq F^2(\beta)$. Similarly, for every $\gamma \in A_3$, $\exists$ $\delta \in A_4$ such that $F^3(\gamma) \subseteq F^4(\delta)$. Now, let $F^1_{A_1} \tilde{\vee} F^3_{A_3} = F_{A_1 \times A_3}$ and $F^2_{A_2} \tilde{\vee} F^4_{A_4} = G_{A_2 \times A_4}$. Then $F(\alpha, \gamma) = F^1(\alpha) \cup F^3(\gamma)$, $\forall$ $(\alpha, \gamma) \in A_1 \times A_3$ and $G(\beta, \delta) = F^2(\beta) \cup F^4(\delta)$, $\forall$ $(\beta, \delta) \in A_2 \times A_4$. But $F^1(\alpha) \cup F^3(\gamma) \subseteq F^2(\beta) \cup F^4(\delta)$. This implies that, for any $(\alpha, \gamma) \in A_1 \times A_3$, $\exists$ $(\beta, \delta) \in A_2 \times A_4$ such that $F^1(\alpha) \cup F^3(\gamma) \subseteq F^2(\beta) \cup F^4(\delta)$. Hence from definition 3.03, $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\vee} F^4_{A_4}$. Similarly, we can prove that $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_J F^2_{A_2} \tilde{\wedge} F^4_{A_4}$.

**Proposition 4.03:** Let $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets over U. Then,
(1). $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$, implies $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cup} F^4_{A_4}$ and $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cap} F^4_{A_4}$,
(2). If $F^1_{A_1} \tilde{\subseteq}_M F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_M F^4_{A_4}$, implies $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\cup} F^4_{A_4}$ and $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

**Proof:** We just varify the validity of (1); subsequent work could be use identical methods to establish (2). To this end, Let $F^1_{A_1} \tilde{\cup} F^3_{A_3} = J_{\dot{D}}$, where $\dot{D} = A_1 \cup A_3$, and $F^2_{A_2} \tilde{\cup} F^4_{A_4} = J^{'}_{D^{'}}$, where $D^{'} = A_2 \cup A_4$. By definition 2.08, we have

$$J(\dot{d}) = \begin{cases} F^1(\dot{d}) & ,\dot{d} \in A_1 - A_3 \\ F^3(\dot{d}) & ,\dot{d} \in A_3 - A_1 \\ F^1(\dot{d}) \cup F^3(\dot{d}) & ,\dot{d} \in A_1 \cap A_3, \end{cases} \quad , \forall\ \dot{d} \in \dot{D},$$

$$J^{'}(d^{'}) = \begin{cases} F^2(d^{'}) & ,d^{'} \in A_2 - A_4 \\ F^4(d^{'}) & ,d^{'} \in A_4 - A_2 \\ F^2(d^{'}) \cup F^4(d^{'}) & ,d^{'} \in A_2 \cap A_4. \end{cases} \quad , \forall\ \mathrm{d}^{'} \in \mathrm{D}^{'}$$

To prove $J_{\dot{D}} \tilde{\subseteq}_F J^{'}_{D^{'}}$, we show that $\dot{D} \subseteq D^{'}$ and $J(\alpha) \subseteq J^{'}(\alpha)$, $\forall$ $\alpha \in \dot{D}$. Since $F^1_{A_1} \tilde{\subseteq}_F F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_F F^4_{A_4}$, so $A_1 \subseteq A_2$, $F^1(\alpha) \subseteq F^2(\alpha)$, $\forall$ $\alpha \in A_1$, and $A_3 \subseteq A_4$, $F^3(\alpha) \subseteq F^4(\alpha)$, $\forall$ $\alpha \in A_3$. It implies that $\dot{D} = A_1 \cup A_3 \subseteq A_2 \cup A_4 = D^{'}$ and $\dot{D} \cap D^{'} = \dot{D}$. Now,

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Case 1. If $\alpha \in$ $A_1$ - $A_3$, then $J(\alpha) = F^1(\alpha) \subseteq F^2(\alpha) = J^{'}(\alpha)$.
Case 2. If $\alpha \in$ $A_3$ - $A_1$ then $J(\alpha) = F^3(\alpha) \subseteq F^4(\alpha) = J^{'}(\alpha)$.
Case 3. If $\alpha \in A_1 \cap A_3$, then $J(\alpha) = F^1(\alpha) \cup F^3(\alpha) \subseteq F^2(\alpha) \cup F^4(\alpha) = J^{'}(\alpha)$.
Hence, $\forall\, \alpha \in \dot{D}$, $J(\alpha) \subseteq J^{'}(\alpha)$ and thus we finally conclude that $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cup} F^4_{A_4}$. Similarly, we can prove that $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\subseteq}_F F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

The following examples provides an explanation to how the property 4.01 (2) need not be satisfied by soft J-subsets and soft L-subsets, as we discussed above.

**Example 4.04:** Let U = {1, 2, 3, 4, 5} and E = {a, b, c, d, e} are the universe set and the essential set of parameters. Consider, $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets defined over U as:
$F^1_{A_1}$ = {(a, {2, 3}), (b, {1, 3, 4})},
$F^2_{A_2}$ = {(c, {2, 3}), (d, {1, 3, 4, 5})},
$F^3_{A_3}$ = {(a, {3, 5})},
$F^4_{A_4}$ = {(c, {4}), (e, {1, 3, 5})}.
Then, it is clear that $F^1_{A_1} \tilde{\subseteq}_J F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_J F^4_{A_4}$, since $F^1(a) \subseteq F^2(c)$, $F^1(b) \subseteq F^2(d)$ and $F^3(a) \subseteq F^4(e)$ respectively. Now, let us write $F^1_{A_1} \tilde{\cup} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cup} F^4_{A_4} = J^{'}_{D'}$, where D = {a, b} and $D^{'}$ = {c, d, e} such that $J_D$ = {(a, {2, 3, 5}), (b, {1, 3, 4})} and $J^{'}_{D'}$ = {(c, {2, 3, 4}), (d, {1, 3, 4, 5}), (e, {1, 3, 5})}. Then, clearly we can see that $J(b) \subseteq J^{'}(d)$ but for $a \in D$ there does not exists any $\alpha^{'} \in D^{'}$ such as $J(a) \subseteq J^{'}(\alpha^{'})$ which gives $J_D \tilde{\nsubseteq}_J J^{'}_{D'}$ or $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\nsubseteq}_J F^2_{A_2} \tilde{\cup} F^4_{A_4}$. Also by definition 2.05, let us write $F^1_{A_1} \tilde{\cap} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cap} F^4_{A_4} = J^{'}_{D'}$, where D = {a} and $D^{'}$ = {c} such that $J_D$ = {(a, {3})} and $J^{'}_{D'}$ = {(c, $\phi$)}. Therefore, we have $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\nsubseteq}_J F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

**Example 4.05:** Let U and E are universal sets as given in example 4.04. Consider, $F^1_{A_1}$, $F^2_{A_2}$, $F^3_{A_3}$ and $F^4_{A_4}$ are four soft sets defined over U as:
$F^1_{A_1}$ = {(a, {2}), (b, {1, 3, 4})},
$F^2_{A_2}$ = {(c, {2}), (d, {1, 3, 4}), (e, {5})},
$F^3_{A_3}$ = {(a, {3, 5})},
$F^4_{A_4}$ = {(c, {2, 4}), (e, {3, 5})}.
Then, clearly $F^1_{A_1} \tilde{\subseteq}_L F^2_{A_2}$ and $F^3_{A_3} \tilde{\subseteq}_L F^4_{A_4}$, since $F^1(a) = F^2(c)$, $F^1(b) = F^2(d)$ and $F^3(a) = F^4(e)$ respectively. Now, let us write $F^1_{A_1} \tilde{\cup} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cup} F^4_{A_4} = J^{'}_{D'}$, where D = {a, b} and $D^{'}$ = {c, d, e} such that $J_D$ = {(a, {2, 3, 5}), (b, {1, 3, 4})} and $J^{'}_{D'}$ = {(c, {2, 4}), (d, {1, 3, 4}), (e, {3, 5})}. Therefore, we have $J(b) = J^{'}(d)$ but for $a \in R \nexists\, \alpha^{'} \in D^{'}$ such as $J(a) = J^{'}(\alpha^{'})$. Thus, $J_D \tilde{\nsubseteq}_L J^{'}_{D'}$ or $F^1_{A_1} \tilde{\cup} F^3_{A_3} \tilde{\nsubseteq}_L F^2_{A_2} \tilde{\cup} F^4_{A_4}$. Also by definition 2.05, let us write $F^1_{A_1} \tilde{\cap} F^3_{A_3} = J_D$ and $F^2_{A_2} \tilde{\cap} F^4_{A_4} = J^{'}_{D'}$, where D = {a} and $D^{'}$ = {c, e} such that $J_D$ = {(a, $\phi$)} and $J^{'}_{D'}$ = {(c, {2}), (e, {5})}. Therefore, we have $F^1_{A_1} \tilde{\cap} F^3_{A_3} \tilde{\nsubseteq}_L F^2_{A_2} \tilde{\cap} F^4_{A_4}$.

## 5. Complement Property and Generalized Soft Subsets:

In this section, first we define a universal complement property on classical subsets and soft complements (negation $^c$ and relative $^r$ complement) on soft sets. In soft set theory, it is obvious that none of the soft subsets presented in section 4 satisfy the complement property 5.01. However, by applying a restrictions on their parameter sets, we show the validity of the specified complement property on all soft subsets.

**Definition 5.01(Universal Complement Property):** Let the universe set be X. If $U \subseteq X$ and $V \subseteq X$ such as $U \subseteq V$, then $V^{'} \subseteq U^{'}$, where "$'$" is called complement operator defined as $U^{'}$ = X - U.

**Definition 5.02([11]):** The soft set $(F_A)^c$ is the complement of soft set $F_A$, described as $(F_A)^c = F^c_{\rceil A}$, where $F^c$ is a mapping as: $F^c : \rceil A \longrightarrow P(U)$, such that $\forall\, \alpha \in \rceil A$, $F^c(\alpha) = U - F(\neg\alpha)$. Here, $\rceil A$

is read as "NOT set of a set A"; $\rceil$A = {$\neg\alpha_1$, $\neg\alpha_2$,....., $\neg\alpha_n$}, where $\neg\alpha_i$ = not $\alpha_i$, $\forall$ i. (It should be observed that the operators $\rceil$ and $\neg$ are distinct). This type of soft complement is called *"negation complement (neg-complement or pseudo-complement [6])"*.

**Definition 5.03([6]):** The soft set $(F_A)^r$ is the complement of soft set $F_A$, defined as $(F_A)^r = F^r_A$, where $F^r$ is a mapping: $F^r : A \longrightarrow P(U)$, such as $\forall\ \alpha \in A$, $F^r(\alpha)$ = U - F($\alpha$). This type of soft complement is called *"Relative Complement"*.

Clearly, $((F_A)^c)^c = F_A$ and $((F_A)^r)^r = F_A$. However, it is noted that the parameter set in relative complement $(F_A)^r$ is still the original set A of parameters, instead of $\rceil$A in negation complement $(F_A)^c$. The following theorem provides an important result that, if $F_{A_1} \tilde{\subseteq} G_{A_2}$, then $(F_{A_1})^c \tilde{\subseteq} (G_{A_2})^c$ and $(F_{A_1})^r \tilde{\subseteq} (G_{A_2})^r$ with respect to soft M-subset and soft L-subset.

**Theorem 5.04:** Let $F_{A_1}$ and $G_{A_2}$ are soft sets defined on U. Then,
(1). $F_{A_1} \tilde{\subseteq}_M G_{A_2} \iff F^c_{\rceil A_1} \tilde{\subseteq}_M G^c_{\rceil A_2}$,
(2). $F_{A_1} \tilde{\subseteq}_M G_{A_2} \iff F^r_{A_1} \tilde{\subseteq}_M G^r_{A_2}$,
(3). $F_{A_1} \tilde{\subseteq}_L G_{A_2} \iff F^c_{\rceil A_1} \tilde{\subseteq}_L G^c_{\rceil A_2}$,
(4). $F_{A_1} \tilde{\subseteq}_L G_{A_2} \iff F^r_{A_1} \tilde{\subseteq}_L G^r_{A_2}$.

**Proof:** We only varify the correctness of (1) and (3); using similar techniques we can give the proof of (2) and (4).

(1). Let $F_{A_1} \tilde{\subseteq}_M G_{A_2}$. Then we have $A_1 \subseteq A_2$ and for all $\alpha \in A_1$, F($\alpha$) = G($\alpha$). Now by defintion 5.02, we write $(F_{A_1})^c = F^c_{\rceil A_1}$, where $F^c(\neg\alpha)$ = U - F($\alpha$), $\forall\ \neg\alpha \in \rceil A_1$ or $\forall\ \alpha \in A_1$. Since F($\alpha$) = G($\alpha$), and F($\alpha$), G($\alpha$) are crisp subsets of universe set U, so we can find that U - F($\alpha$) = U - G($\alpha$), for all $\alpha \in A_1$. Also, we know that $A_1 \subseteq A_2$ iff $\rceil A_1 \subseteq \rceil A_2$. It implies that, for any $\neg\alpha \in \rceil A_1$, $F^c(\neg\alpha) = G^c(\neg\alpha)$. This shows that $F^c_{\rceil A_1} \tilde{\subseteq}_M G^c_{\rceil A_2}$.

Conversely, let us take $F^c_{\rceil A_1} \tilde{\subseteq}_M G^c_{\rceil A_2}$. It implies that $\rceil A_1 \subseteq \rceil A_2 \implies A_1 \subseteq A_2$, and $\forall\ \neg\alpha \in \rceil A_1$, $F^c(\neg\alpha) = G^c(\neg\alpha)$. Thus, U - F($\alpha$) = U - G($\alpha$) which gives F($\alpha$) = G($\alpha$). So we have $\forall\ \alpha \in A_1$, F($\alpha$) = G($\alpha$). Hence $F_{A_1} \tilde{\subseteq}_M G_{A_2}$.

(3). Let $(F_{A_1})^c = F^c_{\rceil A_1}$, where $F^c(\neg\alpha)$ = U - F($\alpha$), $\forall\ \neg\alpha \in \rceil A_1$ or $\forall\ \alpha \in A_1$. Since $F_{A_1} \tilde{\subseteq}_L G_{A_2}$. Therefore, for any $\alpha \in A_1$, $\exists\ \beta \in A_2$ such that F($\alpha$) = G($\beta$). Consequently U - F($\alpha$) = U - G($\beta$). We also know that for any $\alpha \in A_1$, $\neg\alpha \in \rceil A_1$. So, we can find that for any $\neg\alpha \in \rceil A_1$ $\exists\ \neg\beta \in \rceil A_2$ such that $F^c(\neg\alpha)$ = U - F($\alpha$) = U - G($\beta$) = $G^c(\neg\beta)$. Hence, we have $F^c_{\rceil A_1} \tilde{\subseteq}_L G^c_{\rceil A_2}$.

Conversely, let $F^c_{\rceil A_1} \tilde{\subseteq}_L G^c_{\rceil A_2}$. Then for any $\neg\alpha \in \rceil A_1$, $\exists\ \neg\beta \in \rceil A_2$ such that $F^c(\neg\alpha) = G^c(\neg\beta)$. Thus, U - F($\alpha$) = U - G($\beta$) which implies F($\alpha$) = G($\beta$). Also, $\neg\alpha \in \rceil A_1$ if and only if $\alpha \in A_1$. Thus, for any $\alpha \in A_1$ $\exists\ \beta \in A_2$ such as F($\alpha$) = G($\beta$). Therefore, $F_{A_1} \tilde{\subseteq}_L G_{A_2}$.

**Remark 5.05:** From the theorem 5.04, we conclude that the property 5.01 is not satisfied by soft L-subset. Whenever soft M-subset satisfies 5.01, the attribute sets should be equal. That is, if $A_1 = A_2$ and $F_{A_1} \tilde{\subseteq}_M G_{A_2}$, then $G^r_{A_2} \tilde{\subseteq}_M F^r_{A_1}$ and $G^c_{\rceil A_2} \tilde{\subseteq}_M F^c_{\rceil A_1}$ (see theorem 5.09). We give the following example only for soft L-subset. One can also see it for soft M-subsets, when $A_1 \subset A_2$.

**Example 5.06:** Let U and E are universal sets as given in example 4.04, $F_{A_1} \neq \tilde{\Phi}$ and $G_{A_2} \neq \tilde{\Phi}$ are defined as: $F_{A_1}$ = {(a, {2, 3}), (b, {1, 4, 5})}, $G_{A_2}$ = {(c, {2, 3}), (d, {3, 4}), (e, {1, 4, 5})}. So, we can see that for a $\in A_1$, $\exists$ c $\in A_2$ such as F(a) = G(c), and for b $\in A_1$, $\exists$ e $\in A_2$ such that F(b) = G(e). Therefore, $F_{A_1} \tilde{\subseteq}_L G_{A_2}$. Now by defintions 5.02 and 5.03, we have $F^c_{\rceil A_1}$ = {($\neg$a, {1, 4, 5}), ($\neg$b, {2, 3})}, $G^c_{\rceil A_2}$ = {($\neg$c, {1, 4, 5}), ($\neg$d, {1, 2, 5}), ($\neg$e, {2, 3})}, $F^r_{A_1}$ = {(a, {1, 4, 5}), (b, {2, 3})} and $G^r_{A_2}$ = {(c, {1, 4, 5}), (d, {1, 2, 5}), (e, {2, 3})}. So we can see that $A_2 \nsubseteq A_1$ and for $\neg$e $\in \rceil A_2$ there does not exists any element $\neg e' \in \rceil A_1$ such that $G^c(\neg e) = F^c(\neg e')$. It implies that $G^c_{\rceil A_2} \tilde{\nsubseteq}_L F^c_{\rceil A_1}$. Similarly, we can find that $G^r_{A_2} \tilde{\nsubseteq}_L F^r_{A_1}$.

**Remark 5.07:** The above properties given in the theorem 5.04 and definition 5.01 are not satisfied

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by soft F-subset and soft J-subset. Here, we give an example only for soft J-subset. Similarly, one can give an example for soft F-subset.

**Example 5.08:** Let U and E are universal sets as given in example 4.04, $F_{A_1}$ and $G_{A_2}$ are soft sets over U, defined as: $F_{A_1} = \{(a, \{2, 3\}), (b, \{1, 3, 4\})\}$, $G_{A_2} = \{(c, \{1, 2, 3\}), (d, \{1, 3, 4, 5\}), (e, \{2, 4\})\}$. Clearly, we can see that $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Now, we have $F^r_{A_1} = \{(a, \{1, 4, 5\}), (b, \{2, 5\})\}$ and $G^r_{A_2} = \{(c, \{4, 5\}), (d, \{2\}), (e, \{1, 3, 5\})\}$. It implies that neither $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$ nor $F^r_{A_1} \tilde{\subseteq}_J G^r_{A_2}$. Similarly, we can find that neither $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$ nor $F^c_{\rceil A_1} \tilde{\subseteq}_J G^c_{\rceil A_2}$.

The following theorems proves the validity of the stated complement property 5.01 on all generalized soft subsets by taking an onto mapping between sets of parameters/attributes.

**Theorem 5.09:** Let $F_{A_1}$ and $G_{A_2}$ are soft sets on U and $A_1 = A_2$. Then,
(1). $F_{A_1} \tilde{\subseteq}_M G_{A_2} \iff G^r_{A_2} \tilde{\subseteq}_M F^r_{A_1}$,
(2). $F_{A_1} \tilde{\subseteq}_M G_{A_2} \iff G^c_{\rceil A_2} \tilde{\subseteq}_M F^c_{\rceil A_1}$,
(3). $F_{A_1} \tilde{\subseteq}_F G_{A_2} \iff G^r_{A_2} \tilde{\subseteq}_F F^r_{A_1}$,
(4). $F_{A_1} \tilde{\subseteq}_F G_{A_2} \iff G^c_{\rceil A_2} \tilde{\subseteq}_F F^c_{\rceil A_1}$.

**Proof:** We give only the validity of argument (4); the proof of other statements (1), (2) and (3) can be obtained by the same method. Let us take $F_{A_1} \tilde{\subseteq}_F G_{A_2}$. Then $A_1 \subseteq A_2$ and for all $\alpha \in A_1$, $F(\alpha) \subseteq G(\alpha)$ which gives U - $G(\alpha)$ U - $F(\alpha)$. That is, $G^c(\neg\alpha) \subseteq F^c(\neg\alpha)$. Since, $A_1 = A_2$ iff $\rceil A_1 = \rceil A_2$, so we have $\forall\ \neg\alpha \in \rceil A_1$, $G^c(\neg\alpha) \subseteq F^c(\neg\alpha)$. Hence $G^c_{\rceil A_2} \tilde{\subseteq}_F F^c_{\rceil A_1}$.

Conversely, let $G^c_{\rceil A_2} \tilde{\subseteq}_F F^c_{\rceil A_1}$. Then $\rceil A_2 \subseteq \rceil A_1$ and $\forall\ \neg\beta \in \rceil A_2$, $G^c(\neg\beta) \subseteq F^c(\neg\beta)$. Therefore, U - $G(\beta) \subseteq$ U - $F(\beta)$ which implies $F(\beta) \subseteq G(\beta)$. Since, $A_1 = A_2$ iff $\rceil A_1 = \rceil A_2$, hence $\forall\ \alpha \in A_1$, $F(\alpha) \subseteq G(\alpha)$.

**Theorem 5.10:** Let $F_{A_1}$ and $G_{A_2}$ are soft sets on U. If there exists a surjective or onto mapping f : $A_1 \longrightarrow A_2$ as; for any $\alpha \in A_1$, $f(\alpha) = \beta$ where $\beta \in A_2$, such that $F(\alpha) \subseteq G(f(\alpha))$, then $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Hence, $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$ and $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$.

**Proof:** Since f is a mapping as, for any $\alpha \in A_1$, $f(\alpha) = \beta$, such as $F(\alpha) \subseteq G(f(\alpha))$. So for any $\alpha \in A_1$, $\exists\ \beta = f(\alpha) \in A_1$ such as $F(\alpha) \subseteq G(\beta)$. So we have $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Now, for any $\alpha \in A_1$, $\exists\ \beta \in A_2$ such as $F(\alpha) \subseteq G(\beta)$. That is, U - $G(\beta) \subseteq$ U - $F(\alpha)$. Consequently $G^c(\neg\beta) \subseteq F^c(\neg\alpha)$. We also know that f is onto mapping, so for every $\beta \in A_2$, $\exists\ \alpha \in A_1$ such as $\beta = f(\alpha)$. Thus, $\neg\beta = f(\neg\alpha)$. Hence, for any $\neg\beta \in \rceil A_2$, $\exists\ \neg\alpha \in \rceil A_1$ such as $G^c(\neg\beta) \subseteq F^c(\neg\alpha)$. Therefore, we have $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$. Similarly, we can prove $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$.

**Theorem 5.11:** Let $F_{A_1}$ and $G_{A_2}$ are soft sets over U. If there exists a surjective or onto mapping f : $A_1 \longrightarrow A_2$ as; for any $\alpha \in A_1$, $f(\alpha) = \beta$ where $\beta \in A_2$, such as $F(\alpha) = G(f(\alpha))$, then $F_{A_1} \tilde{\subseteq}_L G_{A_2}$. Hence, $G^c_{\rceil A_2} \tilde{\subseteq}_L F^c_{\rceil A_1}$ and $G^r_{A_2} \tilde{\subseteq}_L F^r_{A_1}$.

**Proof:** Due to similar proof to that of the Theorem 5.10, the proof is excluded.

The following real world example first makes two soft sets as soft J-subset using a mapping between two different parameter sets and then show above given complement property (theorem 5.10) on them. Here, noted that these parameter sets may share common elements.

**Example 5.12:** Let U = $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ be the universal set of five canditates for an interview in a company, and E = $\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \tilde{\varrho_1}, \tilde{\varrho_2}, \tilde{\varrho_3}, \tilde{\varrho_4}\}$ be the universal set of all attributes/parameters defined on U, where all $\varrho_i$'s (i $\in \{1, 2, 3, 4, 5\}$) represents the name of jobs as given in example 2.07, and all $\tilde{\varrho_j}$'s (j $\in \{1, 2, 3, 4\}$) represents the required qualifications for respective jobs such as; $\tilde{\varrho_1}$ indicates bachelor in information technology (Bach. I.T.), $\tilde{\varrho_2}$ indicates bachelor in computer science (Bach. C.S.), $\tilde{\varrho_3}$ indicates bachelor in information technology with course work in design and web developer (Bach. I.T. + C.W. in Dgn and Web devl.), and $\tilde{\varrho_4}$ indicates bachelor in computer science with course work in business administration (Bach. C.S. + C.W. in B.A.). Below table provides the connection between jobs and required qualifications for corresponding jobs.

Table 1: Data table for jobs and required qualifications

| **Attributes representation of jobs** | **Name of jobs** | **Required qualifications of corresponding jobs** | **Representation of required qualifications in attribute form** |
|---|---|---|---|
| $\varrho_1$ | Network Administrator (NA) | Bach. I.T. or Bach. C.S. | $\tilde{\varrho_1}$ or $\tilde{\varrho_2}$ |
| $\varrho_2$ | User Experience Designer (UED) | Bach. I.T. + C.W. in Dgn and Web devl. | $\tilde{\varrho_3}$ |
| $\varrho_3$ | System Analyst (SA) | Bach. C.S. or related fields + C.W. in B.A. or Management or Finance | $\tilde{\varrho_4}$ |
| $\varrho_4$ | Database Administrator (DA) | Bach. C.S. | $\tilde{\varrho_2}$ |
| $\varrho_5$ | Development Operations Engineer (DOE) | Bach. I.T. or Bach. C.S. | $\tilde{\varrho_1}$ or $\tilde{\varrho_2}$ |

Now, consider two soft sets $F_{A_1}$ and $G_{A_2}$, where $F_{A_1}$ provides the canditates for corresponding jobs in $A_1 = \{\varrho_1, \varrho_2, \varrho_3\}$ and $G_{A_2}$ provides the canditates according to the qualifications in $A_2 = \{\tilde{\varrho_1}, \tilde{\varrho_2}\}$, defined as:

$F_{A_1} = \{(\varrho_1, \{\mu_1, \mu_3\}), (\varrho_2, \{\mu_2\}), (\varrho_3, \{\mu_3, \mu_5\})\}$,
$G_{A_2} = \{(\tilde{\varrho_1}, \{\mu_2, \mu_4\}), (\tilde{\varrho_2}, \{\mu_1, \mu_3, \mu_5\})\}$.

From given table 1 and soft sets $F_{A_1}$ and $G_{A_2}$, we can see that there exist an onto mapping f defined as:
$f : A_1 \longrightarrow A_2$, where $f(\varrho_1) = \tilde{\varrho_2}$, $f(\varrho_2) = \tilde{\varrho_1}$, and $f(\varrho_3) = \tilde{\varrho_2}$, such as for every $\varrho_i \in A_1 \; \exists \; \tilde{\varrho_j} \in A_2$, $F(\varrho_i) \tilde{\subseteq} G(f(\varrho_i))$. So, clearly we have $F_{A_1} \tilde{\subseteq}_J G_{A_2}$. Now,
$F^c_{\rceil A_1} = \{(\neg\varrho_1, \{\mu_2, \mu_4, \mu_5\}), (\neg\varrho_2, \{\mu_1, \mu_3, \mu_4, \mu_5\}), (\neg\varrho_3, \{\mu_1, \mu_2, \mu_4\})\}$,
$G^c_{\rceil A_2} = \{(\neg\tilde{\varrho_1}, \{\mu_1, \mu_3, \mu_5\}), (\neg\tilde{\varrho_2}, \{\mu_2, \mu_4\})\}$.
Therefore, $G^c_{\rceil A_2} \tilde{\subseteq}_J F^c_{\rceil A_1}$. Similarly, we can see that $G^r_{A_2} \tilde{\subseteq}_J F^r_{A_1}$.

By utilizing above results (5.01 - 5.11) of complements on all generalized soft subsets, we provide here some more results on soft product operators. Proofs will be similar to above results. So, we exclude their proofs.

**Theorem 5.13:**
(1). If $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\vee} F^4_{A_4}$, then $F^{1c}_{\rceil A_1} \tilde{\wedge} F^{3c}_{\rceil A_3} \tilde{\subseteq}_M F^{2c}_{\rceil A_2} \tilde{\wedge} F^{4c}_{\rceil A_4}$. Similarly, if $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\wedge} F^4_{A_4}$, then $F^{1c}_{\rceil A_1} \tilde{\vee} F^{3c}_{\rceil A_3} \tilde{\subseteq}_M F^{2c}_{\rceil A_2} \tilde{\vee} F^{4c}_{\rceil A_4}$. Also, it holds with respect to soft L-subset.
(2). Let $A_1 \times A_3 = A_2 \times A_4$. Then, $F^1_{A_1} \tilde{\vee} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\vee} F^4_{A_4}$, implies that $F^{2c}_{\rceil A_2} \tilde{\wedge} F^{4c}_{\rceil A_4} \tilde{\subseteq}_M F^{1c}_{\rceil A_1} \tilde{\wedge} F^{3c}_{\rceil A_3}$. Similarly, $F^1_{A_1} \tilde{\wedge} F^3_{A_3} \tilde{\subseteq}_M F^2_{A_2} \tilde{\wedge} F^4_{A_4}$, implies that $F^{2c}_{\rceil A_2} \tilde{\vee} F^{4c}_{\rceil A_4} \tilde{\subseteq}_M F^{1c}_{\rceil A_1} \tilde{\vee} F^{3c}_{\rceil A_3}$. Also, it holds with respect to soft F-subset.
Both points are also true for relative complement ($^r$).

**Theorem 5.14:** Let $F^1_{A_1} \tilde{\vee} F^3_{A_3} = F_{A_1 \times A_3}$, $F^2_{A_2} \tilde{\vee} F^4_{A_4} = G_{A_2 \times A_4}$ and f be an onto mapping defined as: f : $A_1 \times A_3 \longrightarrow A_2 \times A_4$; for any $(\alpha_1, \alpha_3) \in A_1 \times A_3$, $f(\alpha_1, \alpha_3) = (\alpha_2, \alpha_4)$ where $(\alpha_2, \alpha_4) \in A_2$, $A_4$, such as:
(1). If $F(\alpha_1, \alpha_3) \subseteq G(f(\alpha_1, \alpha_3))$, then $F_{A_1 \times A_3} \tilde{\subseteq}_J G_{A_2 \times A_4}$. Hence, $G^c_{\rceil A_1 \times \rceil A_3} \tilde{\subseteq}_J G^c_{\rceil A_2 \times \rceil A_4}$ and $G^r_{A_1 \times A_3} \tilde{\subseteq}_J G^r_{A_2 \times A_4}$.
(2). If $F(\alpha_1, \alpha_3) = G(f(\alpha_1, \alpha_3))$, then $F_{A_1 \times A_3} \tilde{\subseteq}_L G_{A_2 \times A_4}$. Hence, $G^c_{\rceil A_1 \times \rceil A_3} \tilde{\subseteq}_L G^c_{\rceil A_2 \times \rceil A_4}$ and $G^r_{A_1 \times A_3} \tilde{\subseteq}_L G^r_{A_2 \times A_4}$.

## 6. CONCLUSION AND FUTURE WORK

Due to the non-availability of complement property on generalized soft subsets in soft set theory, generalized soft subsets can not be used to study various algebraic structures. This research provides a platform in this area. It presents crucial results on soft operations using various generalized soft subsets. It is also shown here that the classical property of intersection and union (Property 4.01(2)) only holds with respect to soft M-subset and soft F-subset but not for soft J-subset and soft L-subset. Further, we provide the complement property 5.01 for given soft subsets, and prove that the property is not satisfied by any generalized soft subsets for which the relevant counterexamples are given. But, this problem is solved in the given study by an onto mapping between the sets of parameters on all generalized soft subsets.

In classical mathematics, subsets, operators and complements are very important concepts when studying algebraic structures such as topology, lattices, and Boolean algebra. In soft set theory, these concepts are also crucial for studying these structures. To achieve this, some researchers have provided soft topological spaces in various forms using soft union, soft intersection, soft M-subsets, soft F-subsets, and soft complement operators.

In addition, some researchers have focused on soft product operations to enhance algebraic properties, and it has been found that these soft product operations can be utilized in soft lattice structures. As such, it is suggested that future research can expand on these findings by investigating various soft subsets in other algebraic properties to make lattice structures on various generalized soft subsets. By studying these structures in soft set theory, researchers can gain a better understanding of how uncertainty and vagueness can affect the algebraic properties of subsets, operators, and complements. The findings from this research can also have practical applications in various fields such as decision-making, data analysis, and artificial intelligence.

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# References

[1] D. Molodtsov, Soft set theory - First results, *Comput. Math. Appl.*, 37, 19-31 (1999).

[2] D.N. Georgiou, A.C. Megaritis and V.I. Petropoulos, On soft topological spaces, *Appl. Math. Inf. Sci.*, 7(5), 1889-1901 (2013).

[3] F. Feng and Y. Li, Soft subsets and soft product operations, *Information Sciences*, DOI: 10.1016/j.ins.2013.01.001.

[4] F. Feng, C.X. Li, B. Davvaz and M.I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, *Soft Computing*, 14, 899-911 (2010).

[5] H.L. Bentley, Nearness spaces and extensions of topological spaces, *Studies in Topology, Academic Press, New York*, 47-66 (1975).

[6] M.I. Ali, F. Feng, X. Liu, W.K. Min and M. Shabir, On some new operations in soft set theory, *Computers and Mathematics with Applications*, 57, 1547-1553 (2009).

[7] M. Shabir and M. Naz, On soft topological spaces, *Comput. Math. Appl.*, 61(7), 1786-1799 (2011).

[8] N. Cagman and S. Enginoglu, Soft set theory and uni-int decision making, *European Journal of Operational Research*, 207, 848–855 (2010).

[9] N. Cagman, S. Karatas and S. Enginoglu, Soft topology, *Comput. Math. Appl.*, 62(1), 351-358 (2011).

[10] P.K. Maji, A.R. Roy and R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, 44, 1077–1083 (2002).

[11] P.K. Maji, R. Biswas and A.R. Roy, Soft set theory, *Comput. Math. Appl.*, 45, 555-562 (2003).

[12] P. Yadav and R. Singh, El-Algebra in Soft Sets, *Journal of Algebraic Statistics*, 13(2), 1455-1462 (2022).

[13] P. Yadav and R. Singh, On Soft Sets based on ES Structure, El-Algebra, *In Proceedings: 5th International Conference on Information Systems and Computer Networks (ISCON 2021) (IEEE Xplore)*, https://doi.org/10.1109/ISCON52037.2021.9702321 (2021).

[14] P. Yadav, R. Singh and K. Khurana, A Review on soft topological spaces, *Psychology and Education*, 57(9), 1430-1442 (2020).

[15] R. Singh and A.K. Umrao, On finite order nearness in soft set theory, *WSEAS Transactions on Mathematics*, 18, 118-122 (2019).

[16] R. Singh and A.K. Umrao, A study on soft d-proximity, *Jour. of Adv. Research in Dynamical and Control System*, 10(05), 1911-1914 (2018).

[17] R. Singh and R. Chauhan, On soft heminearness spaces, *AIP Conference proceeding*, DOI: 10.1063/1.5086638.

[18] X. Chen, P. Yadav, R. Singh and S.M.N. Islam, ES Structure based on soft J-subset, *Mathematics*, 11 (853), https://doi.org/10.3390/math11040853 (2023).

[19] X.Y. Liu, F. Feng and Y.B. Jun, A note on generalized soft equal relations, *Computers and Mathematics with Applications*, 64, 572-578 (2012).

[20] Y.B. Jun and X. Yang, A note on the paper "combination of interval-valued fuzzy set and soft set", *Computers and Mathematics with Applications*, 61, 1468-1470 (2011).