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r-fuzzy Z * -Open Sets and r-fuzzy Z* -Continuity In Topological Spaces

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ABSTRACT

The aim of this paper is to introduce and study the notion of r-fuzzy Z*-open sets and r-fuzzy Z*-continuity. Some characterizations of these notions are presented. Also, some fuzzy topological operations such as: r-fuzzy Z*-boundary, r-fuzzy Z*-border, r-fuzzy Z*-exterior, r-fuzzy Z*-limit point are introduced.

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Keyword:. r-fuzzy Z*-open sets , r-fuzzy Z*-boundary, r-fuzzy Z*-border, r-fuzzy Z*-exterior, r-fuzzy Z*-limit point, r-fuzzy Z*-nbd and r-fuzzy Z*-continuity.

INTRODUCTION:

Chang's fuzzy topology [5] has been extended by $\hat{S}$ ostak [23] in fuzzy topology and different level of growths have been made in [12, 13, 20]. Various authors [3, 4, 6, 9, 11] have developed fuzzy continuity between fuzzy topological space in weaker forms also using the idea of fuzzy semi-open sets [3], fuzzy regular open sets [3], fuzzy preopen sets, fuzzy strongly semiopen sets [4], fuzzy γ-open sets [11], fuzzy δ-semiopen sets [2], fuzzy δ-preopen sets [2], fuzzy semi δ-preopen sets [26] and fuzzy e-open sets [22]. In the sense of Chang [5] fts, Ganguly and Saha [10] developed the idea of fuzzy δ-cluster points in fts. In the sense of $\hat{S}$ ostak's fuzzy topological space, Kim and Park [14] developed r-δ-cluster points and δ-closure operators. The weaker forms of fuzzy semi-preopen sets was developed by Park et al. [18] than any other from of fuzzy semi-open or fuzzy preopen sets. In 2008, the formations of e-open sets, e*-open sets and a-open sets in topological spaces are due to Erdal Ekici [[7], [8]]. The same concept developed r-fuzzy topological spaces in [17]. Sobana et al. [24] defined r-fuzzy e-open and r-fuzzy e-closed sets in a fuzzy topological space in the sense of $\hat{S}$ ostak. Vadivel et al. [25] introduced r-fuzzy e*-open and r-fuzzy e*-closed sets in a fuzzy topological space in the sense of $\hat{S}$ ostak.

In this paper we discuss the concept of r-fuzzy Z*-open sets and r-fuzzy Z*-continuity. Some characterizations of these notions are presented. Also, some fuzzy topological operations such as: r-fuzzy Z*-boundary, r-fuzzy Z*-border, r-fuzzy Z*-exterior, r-fuzzy Z*-limit point are introduced.

10.48047/jocaaa.2024.33.07.88

1. PRELIMINARIES:

Throughout this article, we denote nonempty sets by X, Y etc., $I = [0,1]$ and $I_0 = (0,1]$.. For $\alpha \in 1$, $\bar{\alpha}(x) = \alpha$, $\forall x \in X$ A fuzzy point x_t for $t \in I_0$ is an element of I^X such that

$$x_t(y) = \begin{Bmatrix} t & if & y \text{ is equal to } x \\ 0 & if & y \text{ is not equal to } x. \end{Bmatrix}$$

Let $P_t(X)$ denote the set of all fuzzy points in X. A fuzzy point $x_t \in \mu$ iff $t < \mu(x)$. $\mu \in I^X$ is quasi-coincident with ν, denoted by $\mu q \upsilon$, if $\exists x \in X$ such that $\mu(x) + \upsilon(x) > 1$.

If μ is not quasi-coincident with υ, we denoted $\mu \bar{q} \upsilon$. If A is a subset of X, we define the characteristic function χ_A on X by

$$\chi_t(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad if \quad x \notin A. \end{cases}$$ All notations and definitions will be standard in the fuzzy set theory.

Definition 2.2. [23]

A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(2) $\tau(\vee_{i\in\Gamma} v_i) \geq \wedge_{i\in\Gamma} \tau(v_i)$, for any $\{v_i\}_{i\in\Gamma} \subset I^X$,

(3) $\tau(v_1 \wedge v_2) \geq \tau(v_1 \wedge v_2)$, for any $\upsilon_1, \upsilon_2 \in I^X$.

The pair (X, τ) is called a fuzzy topological space or Sˆostak's fuzzy topological space or smooth topological space (for short, fts, sfts, sts).

Remark 2.3. [21]

Let (X, τ) be a fts. Then, for every $r \in I_0$, $\tau = \{\upsilon \in I^X : \tau(\upsilon) \geq r\}$ is a Change's fuzzy topology on X.

10.48047/jocaaa.2024.33.07.88

Theorem 2.4. [21]

Let (X,τ) be a sfts. Then for each $\mu \in I^X$, $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:

$$C_\tau(\mu,r) = \wedge\{\upsilon \in I^X : \mu \le \upsilon, \tau(\underline{1}-\upsilon) \ge r\}..$$

For $\mu,\upsilon \in I^X$ and $r,s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\underline{0},r) = \underline{0},$

(2) $\lambda \le C_\tau(\lambda,r),$

(3) $C_\tau(\lambda,r) \vee C_\tau(\mu,r) = C_\tau(\lambda \vee \mu,r),$

(4) $C_\tau(\lambda,r) \le C_\tau(\mu,s)$ if $r \le s,$

(5) $C_\tau(C_\tau(\lambda,r),r) = C_\tau(\mu,r).$

Theorem 2.5. [21]

Let (X,τ) be a sfts. Then for each $\mu \in I^X$, $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:

$$I_\tau(\mu,r) = \vee\{\upsilon \in I^X : \mu \le \upsilon, \tau(\upsilon) \ge r\}..$$

For $\mu,\upsilon \in I^X$ and $r,s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $I_\tau(\underline{1},r) = \underline{1},$

(2) $\lambda \ge I_\tau(\lambda,r),$

(3) $I_\tau(\lambda,r) \wedge I_\tau(\mu,r) = I_\tau(\lambda \wedge \mu,r),$

(4) $I_\tau(\lambda,r) \le I_\tau(\mu,s)$ if $r \ge s,$

(5) $I_\tau(I_\tau(\lambda,r),r) = I_\tau(\lambda,r),$

10.48047/jocaaa.2024.33.07.88

(6) $I_\tau(\underline{1}-\lambda,r)=\underline{1}-C_\tau(\lambda,r)$ and $C_\tau(\underline{1}-\lambda,r)=\underline{1}-I_\tau(\lambda,r)$.

Definition 2.6. [17]

A point x of X is called δ -cluster point of λ if $I_\tau(C_\tau(U,r),r)\wedge\lambda=\phi,$ for everyopen set U of X containing x. The set of all δ -cluster point of λ is called δ -closure of λ and is denoted $\delta C_\tau(\lambda)$.

Definition 2.6. [17]

A set λ is δ -closed if and only if $\lambda=\delta C_\tau(\lambda,r)$. The complement of a δ -closed set is said to be δ -open [19]. Then δ -interior of a subset λ of X is the union of all δ -open sets of X contained in λ .

Definition 2.8.

Let (X,τ) be a sfts. Then for each $\lambda\in I^X$, $r\in I_0$, λ is called

1. r-fuzzy regular open (resp. r-fuzzy regular closed) [16] set if $\lambda=I_\tau(C_\tau(\lambda,r),r)$, $(C_\tau(I_\tau(\lambda,r),r)=\lambda)$,
2. r-fuzzy a-open (resp. r-fuzzy a-closed) [23] set if $\lambda\le I_\tau(C_\tau(\delta I_\tau(\lambda,r),r),r)$, $(C_\tau(I_\tau(\delta C_\tau(\lambda,r),r),r)\le\lambda)$,
3. r-fuzzy α -open (resp. r-fuzzy α -closed) [20] set if $\lambda\le I_\tau(C_\tau(I_\tau(\lambda,r),r),r)$, $(C_\tau(I_\tau(C_\tau(\lambda,r),r),r)\le\lambda)$,
4. r-fuzzy preopen (resp. r-fuzzy preclosed) [20] set if $\lambda\le I_\tau(C_\tau(\lambda,r),r)$, $(C_\tau(I_\tau(\lambda,r),r)\le\lambda)$,
5. r-fuzzy δ -preopen (resp. r-fuzzy δ -preclosed) [24] set if $\lambda\le I_\tau(\delta C_\tau(\lambda,r),r)$, $(C_\tau(\delta I_\tau(\lambda,r),r)\le\lambda)$,
6. r-fuzzy δ -semiopen (resp. r-fuzzy δ -preclosed) [26] set if $\lambda\le C_\tau(\delta I_\tau(\lambda,r),r)$,

10.48047/jocaaa.2024.33.07.88

$(I_\tau(\delta C_\tau(\lambda,r),r) \le \lambda)$,

7. r-fuzzy Z-open (resp. r-fuzzy Z-closed) [17] set if
$\lambda \le C_\tau(\delta I_\tau(\lambda,r),r) \vee I_\tau(C_\tau(\lambda,r),r)$

$(C_\tau(\delta I_\tau(\lambda,r),r) \wedge I_\tau(C_\tau(\lambda,r),r) \le \lambda)$,

8. r-fuzzy γ-open (resp. r-fuzzy γ-closed) [24] set if
$\lambda = C_\tau(I_\tau(\lambda,r),r) \vee I_\tau(C_\tau(\lambda,r),r)$

$(C_\tau(I_\tau(\lambda,r),r) \wedge I_\tau(C_\tau(\lambda,r),r) \le \lambda)$,

9. r-fuzzy e-open (resp. r-fuzzy e-closed)[24] set if
$\lambda \le C_\tau(\delta I_\tau(\lambda,r),r) \vee I_\tau(\delta C_\tau(\lambda,r),r)$

$(C_\tau(\delta I_\tau(\lambda,r),r) \wedge I_\tau(\delta C_\tau(\lambda,r),r) \le \lambda)$,

10. r-fuzzy $Z*$-open (resp. r-fuzzy $Z*$-closed) [17] set if
$\lambda \le C_\tau(I_\tau(\lambda,r),r) \vee I_\tau(\delta C_\tau(\lambda,r),r)$

$(C_\tau(I_\tau(\lambda,r),r) \wedge I_\tau(\delta C_\tau(\lambda,r),r) \le \lambda)$,

11. r-fuzzy β-open (resp. r-fuzzy β-closed) [1] set if $\lambda \le C_\tau(I_\tau(C_\tau(\lambda,r),r),r)$

$(I_\tau(C_\tau(C_\tau(\lambda,r),r),r) \le \lambda)$,

12. r-fuzzy $e*$-open (resp. r-fuzzy $e*$-closed) [19] set if $\lambda \le C_\tau(I_\tau(C_\tau(\lambda,r),r),r)$

$(I_\tau(C_\tau(C_\tau(\lambda,r),r),r) \le \lambda)$.

The family of all r-fuzzy regular open (resp. r-fuzzy a-open, r-fuzzy α-open, r-fuzzy preopen , r-fuzzy δ-preopen, r-fuzzy δ-semiopen, r-fuzzy Z-open, r-fuzzy γ-open, r-fuzzy e-open, r-fuzzy β-open, r-fuzzy $e*$-open) is denoted by $RO(X)$ (resp. $aO(X)$, $\alpha O(X)$, $PO(X)$, $\delta PO(X)$, $\delta SO(X)$, $ZO(X)$, $\gamma O(X)$, $eO(X)$, $\beta O(X)$, $e*O(X)$.

Lemma: 2.1 [19]

Let λ,μ be two subsets of (X,τ) Then:

(1) λ is r-fuzzy δ-open if and only if $\lambda = I_\tau(\lambda,r)$,

(2) $X - \delta I_\tau(\lambda,r) = \delta C_\tau(X \setminus (\lambda,r))$ and $\delta I_\tau(X \setminus (\lambda,r)) = X - \delta I_\tau(\lambda,r)$,

(3) $I_\tau(\lambda,r) \le \delta C_\tau(\lambda,r)$ (resp. $\delta I_\tau(\lambda,r) \le I_\tau(\lambda,r)$)), for any subset λ of X,

10.48047/jocaaa.2024.33.07.88

(4) $\delta\text{-}\mathrm{C}_\tau(\lambda \vee \mu, r) = \delta\text{-}\mathrm{C}_\tau(\lambda, r) \vee \delta\text{-}\mathrm{C}_\tau(\mu, r)$,
$\delta\text{-}\mathrm{I}_\tau(\lambda \vee \mu, r) = \delta\text{-}\mathrm{I}_\tau(\lambda, r) \vee \delta\text{-}\mathrm{I}_\tau(\mu, r)$,

Definition: 2.1

A function $f:(X,\tau) \to (Y,\sigma)$ is called r-fuzzy α-continuous [12] (resp. r-fuzzy pre-continuous [11], r-fuzzy γ -continuous [7], r-fuzzy e-continuous [6], r-fuzzy $e*$ -continuous [19]) if $f^{-1}(\lambda)$ is r-fuzzy α-open (resp. r-fuzzy pre open, r-fuzzy γ -, r-fuzzy e-open, r-fuzzy $e*$ -open) in I^X for each λ).

3. r-fuzzy Z*-open sets and r-fuzzy Z*-closed

Definition: 3.1

A subset λ of a topological space (X,τ) is said to be:

(1) r-fuzzy Z*-open if $\lambda \le C_\tau(I_\tau(\lambda,r)) \vee I_\tau(\delta C_\tau(\lambda,r))$,

(2) r-fuzzy Z*-closed if $I_\tau(C_\tau(\lambda,r)) \wedge C_\tau(\delta I_\tau(\lambda,r)) \le \lambda$.

The family of all r-fuzzy Z*-open (resp. r-fuzzy Z*-closed) fuzzy subsets of a fuzzy topological space (X,τ) will be as always denoted by Z*O(X) (resp. Z*C(X)).

Remark: 3.1

The following holds for a fuzzy topological space (X,τ) .

(1) Every r-fuzzy γ -open (resp. e-open) set is r-fuzzy Z*-open,

(2) Every r-fuzzy Z*-open set is r-fuzzy e*-open.

But the converse of the a bovine are not necessarily true in general as shown by the following examples.

Example: 3.1

Let $X = \{a,b,c\}$ and $\mu_1, \mu_2, \upsilon \in I^X$ defined as follows:

$\mu_1(a) = 0.3, \mu_1(b) = 0.2, \mu_1(c) = 0.7,$

$\mu_2(a) = 0.8, \mu_2(b) = 0.8, \mu_2(c) = 0.4,$

10.48047/jocaaa.2024.33.07.88

$\upsilon(a) = 0.8, \upsilon(b) = 0.7, \upsilon(c) = 0.6.$

Define fuzzy topology $\tau = I^X \to I$ as follows

$$\tau(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \lambda = \mu_1, \\ \frac{1}{3} if \lambda = \mu_2, \\ \frac{2}{3} if \lambda = \mu_1 \vee \mu_2, \\ \frac{2}{3} if \lambda = \mu_1 \wedge \mu_2, \\ 0 if \cdot otherwise \end{cases}$$

(1) In (X, τ) fuzzy topological spaces with $0 < r \leq \frac{1}{3}$, υ is r-fuzzy Z*-open set form

$\lambda \leq C_\tau(I_\tau(\lambda, r)) \vee I_\tau(\delta C_\tau(\lambda, r)) = \underline{1}$

For $0 < r \leq \frac{1}{3}$, υ is not r-fuzzy e-open set form $\lambda \neq C_\tau(I_\tau(\delta C_\tau(\lambda, r))$.

(2) In (X, τ) fuzzy topological spaces with $0 < r \leq \frac{1}{3}$, υ is r-fuzzy Z*-open set form

$\lambda \leq C_\tau(I_\tau(\lambda, r)) \vee I_\tau(\delta C_\tau(\lambda, r)) = \underline{1}$

For $0 < r \leq \frac{1}{3}$, υ is not r-fuzzy γ-open set form $\lambda \neq C_\tau(I_\tau(\lambda, r), r) \vee I_\tau(C_\tau(\lambda, r), r)$.

Example: 3.2

Let $X = \{a, b, c\}$ and $\mu_1, \mu_2, \gamma \in I^X$ defined as follows:

$\mu_1(a) = 0.3, \mu_1(b) = 0.4, \mu_1(c) = 0.5,$

$\mu_2(a) = 0.6, \mu_2(b) = 0.5, \mu_2(c) = 0.5,$

$\mu_3(a) = 0.6, \mu_3(b) = 0.4, \mu_3(c) = 0.4,$

$\gamma(a) = 0.6, \gamma(b) = 0.4, \gamma(c) = 0.4.$

10.48047/jocaaa.2024.33.07.88

Define fuzzy topology $\tau = I^X \to I$ as follows

$$\tau(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda = \mu_1, \\ \frac{1}{2} if \lambda = \mu_2, \\ \frac{1}{2} if \lambda = \mu_3, \\ \frac{1}{2} if \lambda = \upsilon, \\ 0 if \cdot otherwise \end{cases}$$

In (X,τ) fuzzy topological spaces with $0 < r \le \frac{1}{2}$, γ is r-fuzzy e*-open set form $\lambda \le C_\tau(I_\tau(\delta C_\tau(\lambda,r)) = \underline{1}$

For $0 < r \le \frac{1}{3}$, γ is not r-fuzzy Z*-open set form $\lambda \ne C_\tau(I_\tau(\lambda,r)) \vee I_\tau(\delta C_\tau(\lambda,r))$.

Remark: 3.2

The following diagram holds for a subset of a space X:

$$\begin{tikzcd}
\text{Open} \arrow[r] & \alpha\text{-open} \arrow[r] & \text{semiopen} \arrow[r] & \gamma\text{-open} \arrow[r] \arrow[d] & \beta\text{-open} \arrow[dd] \\
 & & & \text{Z*-open} \arrow[dr] & \\
\delta\text{-open} \arrow[uu] \arrow[r] & \text{a-open} \arrow[uu] \arrow[r] & \delta\text{-preopen} \arrow[r] & \text{e-open} \arrow[u] \arrow[r] & \text{e*-open}
\end{tikzcd}$$

Proposition: 3.1

Let (X,τ) be a fuzzy topological space .Then the r-fuzzy δ-closure of a r-fuzzy Z*-open fuzzy subset of (X,τ) is r-fuzzy δ-semiopen.

10.48047/jocaaa.2024.33.07.88

Proof: Let $\lambda \leq Z^{*}$O(X). Then
$\delta C_{\tau}(\lambda,r) \leq (C_{\tau}(I_{\tau}(\lambda,r),r)) \vee I_{\tau}(\delta C_{\tau}(\lambda,r),r) \leq \delta C_{\tau}(C_{\tau}(I_{\tau}(\lambda,r),r)) \vee \delta C_{\tau}(I_{\tau}(\delta C_{\tau}(\lambda,r),r))$
$\leq \delta C_{\tau}(I_{\tau}(\lambda,r),r)) \vee \delta C_{\tau}(I_{\tau}(\delta C_{\tau}(\lambda,r),r),r) = \delta C_{\tau}(I_{\tau}(\delta C_{\tau}(\lambda,r),r)) = \delta C_{\tau}(\delta I_{\tau}(\delta C_{\tau}(\lambda,r),r),r) =$
$C_{\tau}(\delta I_{\tau}(\delta C_{\tau}(\lambda,r),r),r)$. Therefore $\delta C_{\tau}(\lambda,r) \in \delta$ SO(X).

Lemma: 3.1

Let (X,τ) be a topological space .Then the following statements are hold.

(1) The union of arbitrary r-fuzzy Z*-open sets is r-fuzzy Z*-open,

(2) The intersection of arbitrary r-fuzzy Z*-closed sets is r-fuzzy Z*-closed.

Proof: (1) It is clear.

Remark: 3.2

By the following we show that the intersection of any two r-fuzzy Z*-open sets is not r-fuzzy Z*-open.

Example: 3.3

Let $X = \{a,b\}$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \gamma_1, \gamma_2 \in I^X$ defined as follows:

$\mu_1(a) = 0.6, \mu_1(b) = 0.4,$

$\mu_2(a) = 0.3, \mu_2(b) = 0.6,$

$\mu_3(a) = 0.6, \mu 3(b) = 0.8,$

$\mu_4(a) = 0.4, \mu_4(b) = 0.6,$

$\mu_5(a) = 0.4, \mu_5(b) = 0.2,$

$\gamma_1(a) = 0.4, \gamma_2(b) = 0.8,$

$\gamma_1(a) = 0.6, \gamma_2(b) = 0.6.$

Define fuzzy topology $\tau = I^X \to I$ as follows

10.48047/jocaaa.2024.33.07.88

$$\tau(\lambda)=\begin{cases} 1 if \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda = \{\mu_1, \mu_1 \vee \mu_2\}, \\ \frac{1}{2} if \lambda = \{\mu_1, \mu_1 \wedge \mu_2, \underline{1} - \mu_1 \wedge \mu_2\} \\ 0 if \cdot otherwise. \end{cases}$$

Since $\gamma_1 and \gamma_2$ are $\frac{1}{2}$-fuzzy Z*-open sets in (X,τ). But $\gamma_1 \wedge \gamma_2 = \mu_4$ it is not $\frac{1}{2}$-fuzzy Z*-open sets.

Definition: 3.2

Let (X,λ) be a topological space. Then:

(1) The union of all r-fuzzy Z*-open sets of X contained in λ is called the r-fuzzy Z*-interior of λ and is denoted by Z*- $I_\tau(\lambda, r)$,

(2) The intersection of all r-fuzzy Z*-closed sets of X containing λ is called the r-fuzzy Z*-closure of λ and is denoted by Z*- $C_\tau(\lambda,r)$.

Theorem: 3.1

Let λ, μ be two subsets of a fuzzy topological space (X,τ). Then the following are hold:

1) Z*-$C_\tau(X-(\lambda,r))$,= $X - Z*I_\tau(\lambda,r)$,
2) Z*-$I_\tau(X-(\lambda,r))$,= $X - Z*C_\tau(\lambda,r)$,
3) If $\lambda \le \mu$, then Z*-$C_\tau(\lambda,r) \le Z^*\text{-}C_\tau(\mu,r)$ and Z*-$I_\tau(\lambda,r) \le Z^*\text{-}I_\tau(\mu,r)$,
4) x $\in$Z*- $C_\tau(\lambda,r)$ if and only if for each a r-fuzzy Z*-open set U contains x, $U \wedge A \ne \phi$,
5) x $\in$Z*-$I_\tau(\lambda,r)$ if and only if there exist a r-fuzzy Z*-open set W such that x $x \in W \le \lambda$,
6) λ is r-fuzzy Z*-open set if and only if $\lambda = Z^*\text{-}I_\tau(\lambda,r)$,
7) λ is r-fuzzy Z*-closed set if and only if $\lambda = Z^*\text{-}C_\tau(\lambda,r)$,
8) $Z^*\text{-}C_\tau(Z^*\text{-}C_\tau(\lambda,r),r) = Z^*\text{-}C_\tau(\lambda,r)$ and $Z^*\text{-}I_\tau(Z^*\text{-}I_\tau(\lambda,r),r) = Z^*\text{-}I_\tau(\lambda,r)$,
9) $Z^*\text{-}C_\tau(\lambda,r) \vee Z^*\text{-}C_\tau(\mu,r) \le Z^*\text{-}C_\tau(\lambda \vee \mu,r)$ and $Z^*\text{-}I_\tau(\lambda,r) \vee Z^*\text{-}I_\tau(\mu,r) \le Z^*\text{-}I_\tau(\lambda \vee \mu,r)$,
10) $Z^*\text{-}I_\tau(\lambda \wedge \mu,r) \le Z^*\text{-}I_\tau(\lambda,r) \wedge Z^*\text{-}I_\tau(\mu,r)$ and $Z^*\text{-}C_\tau(\lambda \wedge \mu,r) \le Z^*\text{-}C_\tau(\lambda,r) \wedge Z^*\text{-}C_\tau(\mu,r)$.

10.48047/jocaaa.2024.33.07.88

Remark: 3.3

By the following example we show that the inclusion relation in parts (9) and (10) of the above theorem cannot be replaced by equality.

Example: 3.4

Let $X = \{a,b\}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \in I^X$ defined as follows:

$\lambda_1(a) = 0.5, \lambda_1(b) = 0.4,$

$\lambda_2(a) = 0.3, \lambda_2(b) = 0.5,$

$\lambda_3(a) = 0.5, \lambda_2(b) = 0.7,$

$\lambda_4(a) = 0.4, \lambda_4(b) = 0.5,$

$\lambda_5(a) = 0.4, \lambda_5(b) = 0.1,$

$\mu_1(a) = 0.4, \mu_1(b) = 0.7,$

$\mu_2(a) = 0.5, \mu_2(b) = 0.5,$

$\mu_3(a) = 0.3, \mu_3(b) = 0.1,$

$\mu_4(a) = 0.1, \mu_4(b) = 0.7.$

Define fuzzy topology $\tau = I^X \to I$ as follows

$$\tau(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \lambda = \{\mu_1, \mu_1 \vee \mu_2\}, \\ \frac{1}{2} if \lambda = \{\mu_1, \mu_1 \wedge \mu_2, \underline{1} - \mu_1 \wedge \mu_2\} \\ 0 if \cdot otherwise. \end{cases}$$

1. If $\mu_1 and \mu_2$ are $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1, \frac{1}{2}) = \mu_1$, $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_2, \frac{1}{2}) = \mu_2$ and
$\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1 \vee \mu_2, \frac{1}{2}) = \underline{1}$, $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1 \vee \mu_2, \frac{1}{2}) \neq$ $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1, \frac{1}{2}) \vee$ $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_2, \frac{1}{2})$,

10.48047/jocaaa.2024.33.07.88

2. If $\mu_1 and \mu_2$ are $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1,\frac{1}{2})=\mu_1$, $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_2,\frac{1}{2})=\mu_2$ and $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1\wedge\mu_2,\frac{1}{2})=\lambda_4$, $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1\wedge\mu_2,\frac{1}{2})\neq$ $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_1,\frac{1}{2})\wedge$ $\frac{1}{2}$-fuzzy $Z^*\text{-}C_\tau(\mu_2,\frac{1}{2})$,
3. If $\mu_3 and \mu_4$ then $\mu_3\vee\mu_4=\mu(a)=0.3,\mu(b)=0.7$ and hence $\frac{1}{2}$-fuzzy $Z^*\text{-}I_\tau(\mu_3,\frac{1}{2})=\phi$, $\frac{1}{2}$-fuzzy $Z^*\text{-}I_\tau(\mu_4,\frac{1}{2})=\mu_4$ and $\frac{1}{2}$-fuzzy $Z^*\text{-}I_\tau(\mu_1\vee\mu_2,\frac{1}{2})\neq$ $\frac{1}{2}$-fuzzy $Z^*\text{-}I_\tau(\mu_3,\frac{1}{2})\vee$ $\frac{1}{2}$-fuzzy $Z^*\text{-}I_\tau(\mu_4,\frac{1}{2})$.

Theorem: 3.2

Let λ , μ be two fuzzy subsets of a fuzzy topological space (X,τ). Then the following are hold:

(1) $Z^*\text{-}C_\tau(C_\tau(\lambda\vee\mu,r),r)=C_\tau(\lambda,r)\vee Z^*\text{-}C_\tau(\lambda,r)$,
(2) $Z^*\text{-}I_\tau(I_\tau(\lambda\wedge\mu,r),r)=I_\tau(\lambda,r)\wedge Z^*\text{-}I_\tau(\lambda,r)$,

Proof: (1)
$Z^*\text{-}C_\tau(C_\tau(\lambda,r),r)\vee\mu\geq Z^*\text{-}C_\tau(C_\tau(\lambda,r),r)\vee Z^*\text{-}C_\tau(\mu,r)\geq C_\tau(\lambda,r)\vee Z^*\text{-}C_\tau(\mu,r)$.
The other inclusion, $C_\tau(\lambda,r)\vee\mu\leq C_\tau(\lambda,r)\vee Z^*\text{-}C_\tau(\mu,r)$ which is r-fuzzy Z*-closed.
Hence, $Z^*\text{-}C_\tau(C_\tau(\lambda,r),r)\vee\mu\leq C_\tau(\lambda,r)\vee Z^*\text{-}C_\tau(\mu,r)$. Therefore,
$Z^*\text{-}C_\tau((C_\tau(\lambda,r),r)\vee\mu)=C_\tau(\lambda,r)\vee Z^*\text{-}C_\tau(\mu,r)$, (2) It is follows from (1).

Theorem: 3.3

Let (X,τ) be a fuzzy topological space and $\lambda\leq X$. Then λ is a r-fuzzy Z*-open set if and only if $\lambda=SI_\tau(\lambda,r)\vee\delta PC_\tau(\lambda,r)$.

Proof: It is clear.

Proposition: 3.2

Let (X,τ) be a topological space and $\lambda\leq X$. Then λ is a r-fuzzy Z*-closed set if and only if $\lambda=SC_\tau(\lambda,r)\wedge\delta PI_\tau(\lambda,r)$..

10.48047/jocaaa.2024.33.07.88

Proof: It follows from Theorem 3.3.

Proposition: 3.3 Let λ be a fuzzy subset of a fuzzy topological space (X,λ). Then:

(1) Z*- $C_\tau(\lambda,r) =$ $SC_\tau(\lambda,r) \wedge \delta PI_\tau(\lambda,r)$,

(2) $PC_\tau(\delta PI_\tau(\lambda,r) =$ $SI_\tau(\lambda,r) \wedge \delta PC_\tau(\lambda,r)$.

Lemma: 3.2

Let λ be a subset of a topological space (X,λ). Then the following are hold:

(1) $PC_\tau(\delta PI_\tau(\lambda,r) =$ $\delta PI_\tau(\lambda,r) \vee C_\tau(I_\tau(\lambda,r),r)$,

(2) $PI_\tau(\delta PC_\tau(\lambda,r) =$ $\delta PC_\tau(\lambda,r) \wedge I_\tau(C_\tau(\lambda,r),r)$.

Proof: (1) By Lemma 2.2 and Proposition 2.1, $PC_\tau(\delta PI_\tau(\lambda,r),r) =$
$\delta PI_\tau(\lambda,r) \vee C_\tau(I_\tau(\delta PI_\tau(\lambda,r),r),r) = \delta PI_\tau(\lambda,r) \vee (C_\tau(I_\tau(\lambda,r),r) \wedge \delta C_\tau(I_\tau(\lambda,r),r)) =$

$\delta PI_\tau(\lambda,r) \vee C_\tau(I_\tau(\lambda,r),r)$.

(3) It follows from (1).

Proposition: 3.4

Let λ be a fuzzy subset of a fuzzy topological space (X,τ). Then:

(1) Z*- $C_\tau(\lambda,r) =$ $\lambda \vee PI_\tau(\delta PC_\tau(\lambda,r),r)$,

(2) Z*- $I_\tau(\lambda,r) =$ $\lambda \wedge PC_\tau(\delta PI_\tau(\lambda,r),r)$.

Proof: (1) By Lemma 3.2, $\lambda \vee PI_\tau(\delta PC_\tau(\lambda,r),r) = \lambda \vee ((\delta PC_\tau(\lambda,r)) \wedge I_\tau(C_\tau(\lambda,r),r))$

$= (\lambda \vee (\delta PC_\tau(\lambda,r))) \wedge (\lambda \vee (I_\tau(C_\tau(\lambda,r),r))) = \delta PC_\tau(\lambda,r) \wedge SC_\tau(\lambda,r) =$ Z*- $C_\tau(\lambda,r)$.

(2) It follows from (1).

Theorem: 3.4

Let λ be a fuzzy subset of a fuzzy topological space (X,λ). Then the following are equivalent:

(1) λ is a r-fuzzy Z*-open set,

(2) $\lambda \leq PC_\tau(\delta PI_\tau(\lambda,r),r)$,

10.48047/jocaaa.2024.33.07.88

(3) there exists U $\in \delta$ PO(X) such that $U \leq \lambda \leq PC_{\tau}(U,r)$,

(4) $PC_{\tau}(\lambda,r) = PC_{\tau}(\delta PI_{\tau}(\lambda,r),r)$.

Proof: (1) $\to$(2). Let λ be a r-fuzzy Z*-open set. Then, $\lambda = Z^*\text{-}\mathrm{I}_{\tau}(\lambda,r)$ and by Proposition 3.4, $\lambda = \lambda \wedge PC_{\tau}(\delta PI_{\tau}(\lambda,r),r)$ and hence , $\lambda \leq PC_{\tau}(\delta PI_{\tau}(\lambda,r),r)$.

(3) $\to$ (1). Let $\lambda \leq PC_{\tau}(\delta PI_{\tau}(\lambda,r),r)$. Then by Proposition 3.4,
$\lambda \leq \lambda \wedge PC_{\tau}(\delta PI_{\tau}(\lambda,r),r) = Z^*\text{-}\mathrm{I}_{\tau}(\lambda,r)$ and hence $\lambda = Z^*\text{-}\mathrm{I}_{\tau}(\lambda,r)$. Thus λ is r-fuzzy Z*-open,

(2) $\to$ (3). It follows from putting $U = \delta\mathrm{PI}_{\tau}(\lambda,r)$,

(3) $\to$ (2). Let there exists $U \in \delta\mathrm{PO}(X)$ such that $U \leq PC_{\tau}(U,r)$. Since $U \leq \lambda$, then $PC_{\tau}(U,r) \leq PC_{\tau}(\delta PI_{\tau}(\lambda,r),r)$ therefore $\lambda \leq PC_{\tau}(U,r) \leq PC_{\tau}(\delta PI_{\tau}(\lambda,r),r)$,

(4) $\leftrightarrow$(4). It is clear.

Theorem: 3.5

Let λ be a fuzzy subset of a fuzzy topological space (X,τ). Then the following are equivalent:

(1) λ is a r-fuzzy Z*-closed set,

(2) $\delta PI_{\tau}(PC_{\tau}(\lambda,r),r) \leq \lambda$,

(3) there exists $U \in \delta PC(X,r)$ such that $PI_{\tau}(U,r) \leq \lambda \leq U$,

(4) $PI_{\tau}(\lambda,r) = PI_{\tau}(\delta PC_{\tau}(\lambda,r),r)$.

Proof: It follows from Theorem 3.4.

Proposition: 3.5 If λ is a r-fuzzy Z*-open fuzzy subset of a fuzzy topological space (X,λ) such that $\lambda \leq \mu \leq PC_{\tau}(\lambda,r)$, then μ is r-fuzzy Z*-open.

4. SOME TOPOLOGICAL OPERATIONS.

Definition: 4.1

Let (X,τ). be a fuzzy topological space and $\lambda \leq X$. Then the r-fuzzy Z*-boundary of λ (briefly, $Z^*\text{-b}(\lambda,r)$ is defined by $Z^*\text{-b}(\lambda,r) = Z^*\text{-}C_{\tau}(\lambda,r) \wedge Z^*\text{-}C_{\tau}(X-(\lambda,r))$.

10.48047/jocaaa.2024.33.07.88

Theorem: 4.1

If λ is a fuzzy subsets of a fuzzy topological space (X,τ), then the following statement are hold:

(1) Z^*-b(λ,r) is r-fuzzy Z*-closed,

(2) Z^*-b$(\lambda,r) = Z^*\text{-}C_\tau(X-(\lambda,r))$,

(3) Z^*-b$(\lambda,r) = Z^*\text{-}C_\tau(\lambda,r) - Z^*\text{-}I_\tau(\lambda,r)$,

(4) Z^*-b$(\lambda,r) \wedge Z^*\text{-}I_\tau(\lambda,r) = \phi$,

(5) Z^*-b$(\lambda,r) \wedge Z^*\text{-}I_\tau(\lambda,r) = Z^*\text{-}C_\tau(\lambda,r)$,

(6) Z^*-b$(Z^*\text{-b}(\lambda,r),r) \leq Z^*\text{-}b(\lambda,r)$,

(7) Z^*-b$(Z^*\text{-I}_\tau(\lambda,r),r) \leq Z^*\text{-}b(\lambda,r)$,

(8) Z^*-b$(Z^*\text{-C}_\tau(\lambda,r),r) \leq Z^*\text{-}b(\lambda,r)$,

(9) $Z^*\text{-I}_\tau(\lambda,r),r) = \lambda - Z^*\text{-}b(\lambda,r)$,

(10) Z^*-b$((\lambda,r)\wedge(\mu,r)) \leq Z^*\text{-}b(\lambda,r) \vee Z^*\text{-b}(\mu,\text{r})$.

Proof: (1) It is clear.

Theorem: 4.2

If λ is a fuzzy subset of a fuzzy topological space X, then the following statement are hold:

(1) λ is a r-fuzzy Z*-open set if and only if $\lambda \wedge Z^*\text{-b}(\lambda,\text{r}) = \phi$,

(2) λ is a r-fuzzy Z*-closed set if and only if $Z^*\text{-b}(\lambda,\text{r}) \leq \lambda$,,

(3) λ is a r-fuzzy Z*-clopen set if and only if $Z^*\text{-b}(\lambda,\text{r}) = \phi$, .

Proof: (1) It follows from Theorem 4.1.

Definition: 4.2

$Z^*\text{-Bd}(\lambda,\text{r}) = \lambda - Z^*\text{-I}_\tau(\lambda,\text{r})$, is said to be Z*-border of λ.

Theorem: 4.2

10.48047/jocaaa.2024.33.07.88

For a fuzzy subset λ of a space X, the following statements hold:

(1) $Z^*\text{-Bd}(\lambda,\text{r}) \leq \lambda,$ for any $\lambda \leq X$,

(2) $\lambda = Z^*\text{-I}_\tau(\lambda,r) \vee Z^*\text{-}Bd(\lambda,r)$,

(3) $Z^*\text{-I}_\tau(\lambda,r) \wedge Z^*\text{-}Bd(\lambda,r) = \phi$,

(4) λ is r-fuzzy Z*-open if and only if $Z^*\text{-}Bd(\lambda,r) = \phi$,

(5) $Z^*\text{-}Bd(Z^*\text{-I}_\tau(\lambda,r),r) = \phi$,

(6) $Z^*\text{-}I_\tau(Z^*\text{-Bd}(\lambda,r),r) = \phi$,

(7) $Z^*\text{-}Bd(Z^*\text{-Bd}(\lambda,r),r) = Z^*\text{-Bd}(\lambda,\text{r})$,

(8) $Z^*\text{-Bd}(\lambda,r) = \lambda \wedge Z^*\text{-C}_\tau(\text{X - }(\lambda,\text{r}))$..

Proof: (6) Let $x \in Z^*\text{-}I_\tau(Z^*\text{-Bd}(\lambda,r),r)$. Then $x \in Z^*\text{-Bd}(\lambda,r)$. Since, $Z^*\text{-Bd}(\lambda,r) \leq \lambda$, then $x \in Z^*\text{-}I_\tau(Z^*\text{-Bd}(\lambda,r),r) \leq Z^*\text{-I}_\tau(\lambda,r)$. Hence, $x \in Z^*\text{-}I_\tau(\lambda,r) \wedge Z^*\text{-Bd}(\lambda,r)$, which contradicts (3). Thus, $Z^*\text{-}I_\tau(Z^*\text{-Bd}(\lambda,r),r) = \phi$,
(8) $Z^*\text{-Bd}(\lambda,r) = \lambda - Z^*\text{-I}_\tau(\lambda,r) = \lambda - (X - (Z^*\text{-C}_\tau(X - \lambda))) = \lambda \wedge Z^*\text{-C}_\tau(X - \lambda)$.

Definition: 4.3

Let (X,τ) be a topological space and $\lambda \leq X$. Then the set $X - (Z^*\text{-C}_\tau(X - \lambda))$ is called the Z*-exterior of λ and is denoted by Z*-ext(λ, r). A point $p \in X$ is called a Z*- exterior point of λ, if it is a Z*-interior point of $X - \lambda$.

Theorem: 4.3

If λ and μ are two fuzzy subsets of a fuzzy topological space (X,τ), then the following statement are hold:

(1) $Z^*\text{-E}_\tau(\lambda,r)$ is r-fuzzy Z*-open,

(2) $Z^*\text{-E}_\tau(\lambda,r) = Z^*\text{-I}_\tau(X - \lambda)$,

(3) $Z^*\text{-E}_\tau(Z^*\text{-E}_\tau(\lambda,r),r) = Z^*\text{-I}_\tau(Z^*\text{-C}_\tau(\lambda,r),r)$,

(4) $Z^*\text{-E}_\tau(\text{X - }(Z^*\text{-E}_\tau(\lambda,r))) = Z^*\text{-E}_\tau(\lambda,r)$,

(5) $Z^*\text{-I}_\tau(\lambda,\text{r}) \leq Z^*\text{-E}_\tau(Z^*\text{-E}_\tau(\lambda,r),r)$,

(6) $Z^*\text{-E}_\tau(\lambda,\text{r}) \wedge Z^*\text{-b}(\lambda,r) = \phi$,

10.48047/jocaaa.2024.33.07.88

(7) $Z^*\text{-E}_\tau(\lambda,\mathrm{r}) \vee Z^*\text{-b}(\lambda,r) = Z^*\text{-C}_\tau(X-(\lambda,r))$,

(8) $\{Z^*\text{-I}_\tau(\lambda,\mathrm{r}), Z^*\text{-b}(\lambda,r)$ and $Z^*\text{-E}_\tau(\lambda,r)\}$ form a partition of X,

(9) If $\lambda \le \mu$, then $Z^*\text{-E}_\tau(\mu,\mathrm{r}) \le Z^*\text{-E}_\tau(\lambda,r)$,

(10) $Z^*\text{-E}_\tau(\phi) = X$ and $Z^*\text{-E}_\tau(X) = \phi$,

(11) $Z^*\text{-E}_\tau(\lambda \vee \mu,r) \le Z^*\text{-E}_\tau(\lambda,r) \vee Z^*\text{-E}_\tau(\mu,r)$,

(12) $Z^*\text{-E}_\tau(\lambda \wedge \mu,r) \ge Z^*\text{-E}_\tau(\lambda,r) \wedge Z^*\text{-E}_\tau(\mu,r)$,

(13) $Z^*\text{-E}_\tau(\lambda \vee \mu,r) \le Z^*\text{-E}_\tau(\lambda \wedge \mu,r)$.

Proof: It follows from Theorems 3.1 and 4.1.

Remark: 4.1

The inclusion relation in parts (11) and (12) of the above theorem cannot be replaced by equality as is shown by the following example.

Example: 4.1

Let $X = \{a,b\}$ and $\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in I^X$ defined as follows:

$\lambda_1(a) = 0.3, \lambda_1(b) = 0.4, \lambda_1(c) = 0.5,$

$\lambda_2(a) = 0.6, \lambda_2(b) = 0.5, \lambda_2(c) = 0.5,$

$\lambda_3(a) = 0.6, \lambda_3(b) = 0.5, \lambda_3(c) = 0.4,$

$\lambda_4(a) = 0.3, \lambda_4(b) = 0.4, \lambda_4(c) = 0.4,$

$\mu_1(a) = 0.4, \mu_1(b) = 0.7, \mu_1(c) = 0.5,$

$\mu_2(a) = 0.4, \mu_2(b) = 0.1, \mu_2(c) = 0.5.$

Define fuzzy topology $\tau = I^X \to I$ as follows

$$\tau(\lambda) = \begin{Bmatrix} 1 if \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda = \{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}, \\ 0 if \cdot otherwise. \end{Bmatrix}$$

10.48047/jocaaa.2024.33.07.88

1. If μ_1 and μ_2 are $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1,\frac{1}{2})=\mu_1$, $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_2,\frac{1}{2})=\mu_2$ and $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1\vee\mu_2,\frac{1}{2})=\underline{0}$, $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1\vee\mu_2,\frac{1}{2})\neq$ $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1,\frac{1}{2})\vee$ $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_2,\frac{1}{2})$,
2. If $\mu_1 and \mu_2$ are $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1,\frac{1}{2})=\mu_1$, $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_2,\frac{1}{2})=\mu_2$ and $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1\wedge\mu_2,\frac{1}{2})=\lambda_4$, $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1\wedge\mu_2,\frac{1}{2})\neq$ $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_1,\frac{1}{2})\wedge$ $\frac{1}{2}$-fuzzy Z^*-$E_\tau(\mu_2,\frac{1}{2})$.

Definition: 4.4

Let λ be a fuzzy subset of a fuzzy topological space (X,τ). Then a point $p\in X$ is called a r-fuzzy Z*-limit point of a set $\lambda\le X$ if every Z*-open set $G\le X$ containing p contains a point of λ other than p. The set of all r-fuzzy Z*-limit points of λ is called a Z*-derived set of λ and is denoted by Z^*-$D_\tau(\lambda,r)$.

Theorem: 4.4

If λ and μ are two fuzzy subsets of a fuzzy topological space X, then the following statement are hold:

(1) If $\lambda\le\mu$, then Z^*-$D_\tau(\lambda,r)\le Z^*$-$D_\tau(\mu,r)$,

(2) Z^*-$D_\tau(\lambda,r)\vee Z^*$-$D_\tau(\mu,r)\le Z^*$-$D_\tau(\lambda\vee\mu,r)$,

(3) Z^*-$D_\tau(\lambda\wedge\mu,r)\le Z^*$-$D_\tau(\lambda,r)\wedge Z^*$-$D_\tau(\mu,r)$,

(4) λ is a r-fuzzy Z*-closed set if and only if it contains each of its r-fuzzy Z*-limit points,

(5) Z^*-$C_\tau(\lambda,r)=\lambda\vee Z^*$-$D_\tau(\lambda,r)$.

Proof: It is clear.

Definition: 4.5

A fuzzy subset N of a fuzzy topological space (X,τ) is called a r-fuzzy Z*-neighbourhood (briefly, Z*-nbd) of a point $p\in X$ if there exists a r-fuzzy Z*-open set W such that

10.48047/jocaaa.2024.33.07.88

$p \in W \leq N$. The class of all r-fuzzy Z*-nbds of $p \in X$ is called the r-fuzzy Z*-neighbourhood system of P and denoted by Z*-Np .

Theorem: 4.5

A fuzzy subset G of a fuzzy topological space X is r-fuzzy Z*-open if and only if it is Z*-nbd, for every fuzzy point $p \in G$.

Proof: It is clear.

Theorem: 4.6

In a fuzzy topological space (X,τ). Let $Z^*\text{-}N_p$ be the Z*-nbd System of a point (X,τ). Then the following statement is hold:

(1) $Z^*\text{-}N_p$ is not empty and p belongs to each member of $Z^*\text{-}N_p$,

(2) Each fuzzy superset of members of N_p belongs to $Z^*\text{-}N_p$,

(3) Each member $N \in Z^*\text{-}N_p$ is a fuzzy superset of a member $W \in Z^*\text{-}N_p$, where W is Z*-nbd of each fuzzy point $p \in W$.

Proof: Obvious.

5. r-Fuzzy Z*-continuous mappings.

Definition: 5.1

A function $f:(X,\tau) \to (Y,\sigma)$ is called r-fuzzy Z*-continuous if $f^{-1}(\gamma)$ is r-fuzzy Z*-open in X for each $\gamma \in \sigma$.

Remark: 5.1. Let $f:(X,\tau) \to (Y,\sigma)$ be mapping from a space (X,τ) into a space (Y,σ), The following hold:

$$\begin{tikzcd}
\gamma-\text{continous} \arrow[dr] & & \\
& \text{Z*-continous} \arrow[r] & \text{e*-continuous} \\
\text{e-continuous} \arrow[ur] & &
\end{tikzcd}$$

10.48047/jocaaa.2024.33.07.88

Now , the following examples show that these implication are not reversible.

Example: 5.1

Let $X = Y = \{a,b,c\}$ and $\mu_1, \mu_2 \in I^X$ $\mu_2, \mu_3 \in I^Y$, $\mu_5 \in I^Z$ defined as follows:

$\mu_1(a) = 0.4, \mu_1(b) = 0.5, \mu_1(c) = 0.6,$

$\mu_2(a) = 0.4, \mu_2(b) = 0.5, \mu_2(c) = 0.4,$

$\mu_3(a) = 0.6, \mu_3(b) = 0.5, \mu_3(c) = 0.4,$

$\mu_4(a) = 0.4, \mu_4(b) = 0.5, \mu_4(c) = 0.6,$ $\mu_5(a) = 0.6, \mu_5(b) = 0.5, \mu_5(c) = 0.4,$

Define fuzzy topology $\tau, \sigma, \eta = I^X \to I$ as follows

$$\tau(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \lambda = \mu_1, \\ \frac{1}{2} if \lambda = \mu_2, \\ 0 if \cdot otherwise \end{cases}$$

$$\sigma(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0}, \underline{1}\},, \\ \frac{1}{2} if \lambda = \mu_3, \\ \frac{1}{2} if \lambda = \mu_4, \\ \frac{1}{2} if \lambda = \mu_3 \vee \mu_4, \\ \frac{1}{2} if \lambda = \mu_3 \wedge \mu_4, \\ 0 if \cdot otherwise \end{cases}$$

$$\eta(\lambda) = \begin{cases} 1 if \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \lambda = \mu_5, \\ 0 if \cdot otherwise \end{cases}$$

10.48047/jocaaa.2024.33.07.88

Then the identity mapping $id_X : (X,\tau) \to (Y,\sigma)$ is $\frac{1}{2}$-fuzzy $Z*$- continuous but not $\frac{1}{2}$-fuzzy γ- continuous and $id_X : (X,\tau) \to (Z,\eta)$ is $\frac{1}{2}$-fuzzy $Z*$- continuous but not $\frac{1}{2}$-fuzzy e- continuous

Example: 5.2

Let $X = \{a,b,c\}, Y = \{a,b\}$ and $\mu_1, \mu_2, \mu_3 \in I^X$ $\mu_4, \mu_5, \mu_6 \in I^Y$ defined as follows:

$\mu_1(a) = 0.3, \mu_1(b) = 0.6, \mu_1(c) = 0.6,$

$\mu_2(a) = 0.4, \mu_2(b) = 0.5, \mu_2(c) = 0.6,$

$\mu_3(a) = 0.5, \mu_3(b) = 0.5, \mu_3(c) = 0.5,$

$\mu_4(a) = 0.6, \mu_4(b) = 0.6,$

$\mu_5(a) = 0.5, \mu_5(b) = 0.5,$

$\mu_6(a) = 0.5, \mu_6(b) = 0.4.$

Define fuzzy topology $\tau, \sigma = I^X \to I$ as follows

10.48047/jocaaa.2024.33.07.88

$$\tau(\lambda)=\begin{cases} 1 if \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda = \mu_1, \\ \frac{1}{2} if \lambda = \mu_2, \\ \frac{1}{2} if \lambda = \mu_3, \\ 0 if \cdot otherwise \end{cases}$$

$$\sigma(\lambda)=\begin{cases} 1 if \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \lambda = \mu_4, \\ \frac{1}{2} if \lambda = \mu_5, \\ 0 if \cdot otherwise \end{cases}$$

Then the identity mapping $id_X : (X,\tau) \to (Y,\sigma)$ is $\frac{1}{2}$-fuzzy e^*- continuous but not $\frac{1}{2}$-fuzzy Z^*- continuous .

Theorem: 5.1

Let $f:(X,\tau) \to (Y,\sigma)$ be a mapping. Then the following statements are equivalent:

(1) f is r-fuzzy Z^*-continuous.

(2) For each $x \in X$ and $\gamma \in \sigma$ containing $f(X)$, there exists $U \in Z^*O(X)$ containing x such that $f(\omega) \le \gamma$,

(3) The inverse image of each r-fuzzy closed set in Y is r-fuzzy Z^*-closed in X,

(4) $I_\tau(C_\tau(f^{-1}(\mu,r),r),r) \wedge C_\tau(\delta I_\tau(f^{-1}(\mu,r),r) \le f^{-1}(C_\tau(\mu,r))$, for each $\mu \le Y$,

(5) $f^{-1}(I_\tau(\mu,r) \le C_\tau(I_\tau(f^{-1}(\mu,r),r)) \vee I_\tau(\delta C_\tau(f^{-1}(\mu,r),r)$, for each $\mu \le Y$,

(6) If f is bijective, then $I_\tau(f(\lambda,r)) \le f(C_\tau(I_\tau(\lambda,r),r)) \vee f(I_\tau(\delta C_\tau(\mu,r),r))$, for each $\lambda \le X$,

(7) If f is bijective, then $f(I_\tau(C_\tau(\lambda,r),r)) \wedge f(C_\tau(\delta I_\tau(\lambda,r),r)) \le C_\tau(f(\lambda,r))$, for each $\lambda \le X$.

10.48047/jocaaa.2024.33.07.88

Proof: (1) $\leftrightarrow$ (2) and (1) $\leftrightarrow$ (3) are obvious,

(3) $\leftrightarrow$ (4). Let $\mu \le Y$, then by (3) $f^{-1}(C_\tau(\mu,r))$ is r-fuzzy Z^*-closed. This means

$$f^{-1}(C_\tau(\mu,r) \ge I_\tau(C_\tau(f^{-1}(\mu,r),r) \wedge C_\tau(\delta I_\tau(f^{-1}(\mu,r),r)$$
$$\ge I_\tau(C_\tau(f^{-1}(\mu,r),r)) \wedge C_\tau(\delta I_\tau(f^{-1}(\mu,r),r)),$$

(4) $\leftrightarrow$ (5). By replacing $Y-\mu$ instead of μ in (4) ,we have

$I_\tau(C_\tau(f^{-1}(Y-\mu),r),r) \wedge C_\tau(\delta I_\tau(f^{-1}(Y-\mu),r),r) \le f^{-1}(C_\tau(Y-\mu),r)$ and therefore $f^{-1}(I_\tau(\mu,r) \le C_\tau(I_\tau(f^{-1}(\mu,r),r)) \vee I_\tau(\delta C_\tau(f^{-1}(\mu,r),r)$,

(5) $\leftrightarrow$ (6). Follows directly by replacing λ instead of $f^{-1}(\lambda)$ in (5) and applying the bijection of f,

(6) $\rightarrow$ (7). By complementation of (6) and applying the bijective of f, we have

$f(I_\tau(C_\tau(X-\lambda),r),r) \wedge f(C_\tau(\delta I_\tau(X-\lambda),r),r) \le C_\tau(f(X-\lambda),r)$ .We obtain the required by replacing λ instead of $X-\lambda$,

(7) $\rightarrow$ (1). Let $\upsilon \in \sigma$. But $\omega = Y-\upsilon$, by (7), we have $f(I_\tau(C_\tau(f^{-1}(\omega,r),r)) \wedge f(C_\tau(\delta I_\tau(f^{-1}(\omega,r),r)) \le C_\tau(ff^{-1}(\omega,r)) \le C_\tau(\omega,r) = \omega$. So $I_\tau(C_\tau(f^{-1}(\omega,r),r))) \wedge C_\tau(\delta I_\tau(f^{-1}(\omega,r),r))) \le f^{-1}(\omega,r)) \Rightarrow f^{-1}(\omega,r)$ is r-fuzzy Z^*-closed and therefore $f^{-1}(\upsilon,r) \in Z^*O(X)$.

Theorem: 5.2.

Let $f:(X,\tau) \to (Y,\sigma)$ be a mapping. Then the following statements are equivalent:

(1) f is Z^*-continuous,

(2) Z^*-$C_\tau(f^{-1}(\mu,r)) \le f^{-1}(C_\tau(\mu,r))$, for each $\mu \le Y$,

(3) $f(Z^*$-$C_\tau(\mu,r)) \le C_\tau(f(\mu,r))$, for each $\lambda \le X$,

(4) Z^*-$Bd_\tau(f^{-1}(\mu,r)) \le f^{-1}(Bd_\tau(\mu,r))$, for each $\mu \le Y$,

(5) $f(Z^*$-$D_\tau(\lambda,r)) \le C_\tau(f(\lambda,r))$, for each $\lambda \le X$,

(6) $f^{-1}(I_\tau(\mu,r)) \le Z^*$-$I_\tau f^{-1}((\mu,r))$, for each $\mu \le Y$,

10.48047/jocaaa.2024.33.07.88

Proof: (1) $\to$ (2). Let $\mu \le Y$, $f^{-1}(C_\tau(\mu,r))$ is Z^*-closed in X. Then
$Z^*\text{-C}_\tau(f^{-1}(\mu,r)) \le Z^*\text{-C}_\tau(f^{-1}(C_\tau(\mu,r))) = f^{-1}(C_\tau(\mu,r))$,

(2) $\to$ (3). Let $\lambda \le X$, then $f(\lambda) \le Y$, by (2),
$f^{-1}(C_\tau(\lambda,r)) \ge Z^*\text{-C}_\tau(f^{-1}(f(\lambda,r))) \ge Z^*\text{-C}_\tau(\lambda,r).$

Therefore, $C_\tau(f(\lambda,r)) \ge \text{f}(f^{-1}(C_\tau(f(\lambda,r))) \ge f(Z^*\text{-C}_\tau(\lambda,r))..$

(3) $\to$ (1). Let $\omega \le Y$ be open set. Then, $F = Y - \omega$ is closed in Y and
$f^{-1}(F) = X - f^{-1}(\omega)$.

Hence, by (3), $f(Z^*\text{-C}_\tau f^{-1}((F,r)) \le C_\tau(f(f^{-1}(F,r)) \le C_\tau(F,r) = (F,r)$ thus,
$Z^*\text{-C}_\tau f^{-1}((F,r)) \le (f^{-1}(F,r)$,

So, $f^{-1}((F,r)) = X - (f^{-1}(\omega,r)) \in Z^*C(X)$ and therefore $f^{-1}(\omega,r) \in Z^*O(X).$,

(4) $\to$ (6). Let $\mu \le Y$. Then by (4),
$Z^*\text{-Bd}_\tau(f^{-1}((\mu,r)) = f^{-1}(\mu,r)) - Z^*\text{-}I_\tau(f^{-1}(\mu,r) \le f^{-1}(Bd_\tau(\mu,r)) =$
$f^{-1}(\mu - I_\tau(\mu,r)) = f^{-1}(\mu,r) - f^{-1}(I_\tau(\mu,r))$

this implies $f^{-1}(I_\tau(\mu,r) \le Z^*\text{-}(f^{-1}(\mu,r))$,

(6) $\to$ (4). Let $\mu \le Y$. Then by (6), $f^{-1}(I_\tau(\mu,r) \le Z^*\text{-}(\text{I}_\tau(f^{-1}(\mu,r)))$, we have
$f^{-1}(\mu,r) - Z^*\text{-}(\text{I}_\tau(f^{-1}(\mu,r)) \le f^{-1}(\mu,r) - f^{-1}(I_\tau(\mu,r))$,
$\Rightarrow Z^*\text{-Bd}_\tau(f^{-1}(\mu,r)) \le f^{-1}(Bd_\tau(\mu,r))$,

(1) $\to$ (5). It is obvious, since f is Z*-continuous and by (3),
$f(Z^*\text{-}(C_\tau(\mu,r)) \le C_\tau(f(\lambda,r))$ for each $\lambda \le X$. So,
$f(Z^*\text{-}(D_\tau(\mu,r)) \le f(Z^*\text{-}C_\tau(\lambda,r) \le C_\tau(f(\lambda,r))$.

(5) $\to$ (1). Let $\upsilon \le Y$ be open set. Then, $F = Y - \upsilon$ is closed in Y and
$f^{-1}(F) = X - f^{-1}(\upsilon)$.

Hence, by (5), $f(Z^*\text{-}(D_\tau(f^{-1}(F,r)) \le C_\tau(f(f^{-1}(F,r)) \le C_\tau(F,r) = F$. Hence,
$Z^*\text{-}(\text{D}_\tau(f^{-1}(F,r)) \le f^{-1}(F)$,. By Theorem 4.4, $f^{-1}(F) = X - f^{-1}(\upsilon)$ is Z*-closed in X. Therefore, $f^{-1}(\upsilon)$ is Z*-open in X,

10.48047/jocaaa.2024.33.07.88

(1) $\to$ (6). Let $\mu \le Y$. Then $f^{-1}(I_\tau(\mu,r))$ is Z*-open in X. Thus,

$f^{-1}(I_\tau(\mu,r)) = Z^*\text{-}\mathrm{I}_\tau(f^{-1}(I_\tau(\mu,r)) \le Z^*\text{-}\mathrm{I}_\tau(f^{-1}(\mu,r))$. Therefore,

$f^{-1}(I_\tau(\mu,r)) \le Z^*\text{-}\mathrm{I}_\tau(f^{-1}((\mu,r))$,

(5) $\to$ (1). Let $\mu \le Y$ be an open set. Then $f^{-1}(\upsilon,r) = f^{-1}(I_\tau(\upsilon,r)) \le f^{-1}(Z^*\text{-}I_\tau(\upsilon,r))$.

Hence, $f^{-1}(\upsilon)$ is Z^*-open in X . Therefore, f is Z*-continuous.

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