Solutions for Higher Order Cauchy Difference Equation in Free Monoid

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ABSTRACT
Let h: P → Q is a Function, where P is a Group with respect to multiplication and Q be an Abelian Group with respect to addition. In this article, the m^{th} Order Cauchy Difference Equation

$$h.^{(m)}h(p_1, p_2, p_3, \dots, p_{m-1}) = h(C_{m+1}(\prod_{j=1}^{m+1} p_j))$$
$$h(C(\prod_{j=1}^{m+1} p_j)) + \cdots + (-1)^m h(C_1(\prod_{j=1}^{m+1} p_j)), \forall p_1, p_2, \dots, p_{m+1} \in P$$

is discussed, where $h(C_r(\prod_{j=1}^{m} p_j))$ as function of combination r at a time from m objects. we find solutions of $D^{(m)}h = 0$ in Free Monoid.

Keywords: Cauchy Difference Equation; Free Monoid, Abelian Group

INTRODUCTION

From [1] we konw that Jenson's Equation

$$h(p + q) + h(p - q) = 2h(p) \quad (1.1)$$

along h(0) = 0, equal to Cauchy's (Functional) Difference Equation h(p + q) = h(p) + h(q) in R. Let e ∈ P and 0 ∈ Q are identity.

For h: P→Q, Cauchy Difference $D^{(n)}h$, defined by

$$D^{(0)}h = h, \quad (1.2)$$
$$D^{(1)}h(b_1, b_2) = h(b_1b_2) - h(b_1) - h(b_2) \quad (1.3)$$
$$D^{(n+1)}h(b_1, b_2, \dots, b_{n+2}) = D^{(n)}h(b_1, b_2, b_3 \dots, b_{n+2})$$
$$-D^{(n)}h(b_1, b_3, \dots, b_{n+2}) - D^{(n)}h(b_2, b_3, \dots, b_{n+2}) \quad (1.4)$$

where $D^{(1)}h$ denoted as Dh. In [20, 21, 10], General Solution of 2nd and 3rdorder CDE discussed in Free Groups.

In this article, Consider (functional) Cauchy Difference Equation: $h(C_{m+1}(\prod_{j=1}^{m+1} p_j))$

$$h(C_m(\prod_{j=1}^{m+1} p_j)) + \cdots + (-1)h(C_1(\prod_{j=1}^{m+1} p_j)) = 0 \quad (1.5)$$

⇒ from (1.4) that (1.5) equals $D^{(m)}h = 0$.The solution of equation (1.5) define

$$KerD^{(m)}(P, Q) = \{h : P \to Q | h \text{ satisfies } (1.5)\} \quad (1.6)$$

Remark 1

1. $KerD^{(m)}(P, Q)$ is Abelian Group under Addition;
2. $Hom(P, Q) \leq KerD^{(m)}(P, Q)$

2. Properties of Solution for mth Order CDE

Lemma 1 Suppose $h \in KerD^{(m)}(P, Q)$. Then

$$h(e) = 0, \quad (2.1)$$
$$Dh(b_1, b_2) = 0, \quad \text{when } b_1 = e \text{ or } b_2 = e \quad (2.2)$$
$$D^{(2)}h(b_1, b_2, b_3) = 0, \quad \text{when } b_1 = e \text{ or } b_2 = e \text{ or } b_3 = e \quad (2.3)$$
$$\dots\dots\dots$$
$$D^{(m-1)}h(b_1, \dots, b_m) = 0, \quad \text{when } b_1 = e \text{ or } b_2 = e \text{ or} \dots \text{or } b_m = e \quad (2.4)$$
$$D^{(m-1)}h \text{ is Homomorphism w.r.t every variable} \quad (2.5)$$

$h(b^m) = mC_1 h(b) + mC_2 Dh(b,b) + mC_3 D^{(2)}h(p,p,p)\dots + mC_m$
$D^{(m-1)}h(b,\dots,b(m\ \text{times}))$ (2.6)
for every $b_1,\dots,b_m \in P$ and $m \in Z$.
Proof Put $b_1 = e$ in (1.5) $\Rightarrow$ (2.1). Then,
From (2.1) $\Rightarrow$ (2.2)-(2.4)
$Dh(b_1,e) = h(b_1e) - h(b_1) - h(e)$
$= h(b_1) - h(b_1)$
$= 0$
In the same way we get
$Dh(e,b_2) = 0$,
$D^{(2)}h(e,b_2,b_3) = h(eb_2b_3) - h(eb_2) - h(eb_3) - h(b_2b_3) + h(e) + h(b_2) + h(b_3) = 0$
, In the same way we get
$D^{(2)}h(b_1,e,b_3) = 0$,
$D^{(2)}h(b_1,b_2,e) = 0$,
....
$D^{(m-1)}h(e,\dots,b_m) = 0$,
$D^{(m-1)}h(b_1,e,\dots,b_m) = 0$,
$D^{(m-1)}h(b_1,\dots,e) = 0$.
Also, by definition of $D^{(m-1)}h$, we have

$$D^{(m-1)}h(b_1,b_2,b_3,\dots,b_{m+1}) = h(b_1\dots b_{m+1}) - h(b_1\dots b_m) - h(b_1\dots b_{m-2}b_{m+1}) - \dots - h(b_1b_4\dots b_{m+1}) - h(b_2b_3\dots b_{m+1}) + \dots\dots + (-1)^m[h(b_1) + h(b_2b_3) + h(b_4) + \dots + h(b_{m+1})]$$

and

$$D^{(m-1)}h(b_1,b_2,b_4,\dots,b_{m+1}) + D^{(m-1)}h(b_1,b_3,b_4,\dots,b_{m+1})$$
$$= h(b_1b_2b_4\dots b_{m+1}) - h(b_1b_2b_4\dots b_m) - h(b_1b_2b_4\dots b_{m-1}b_{m+1}) - h(b_1b_4\dots b_{m+1}) - \dots - h(b_2b_4\dots b_{m+1}) + \dots\dots\dots + (-1)^m[h(b_1) + h(b_2) + h(b_4)\dots + h(b_{m+1})]$$
$$+ h(b_1b_3b_4\dots b_{m+1}) - h(b_1b_3b_4\dots b_m) - h(b_1b_3b_4\dots b_m) - h(b_1b_3b_4\dots b_{m-1}b_{m+1}) - \dots - h(b_1b_4\dots b_{m+1}) - h(b_3b_4\dots b_{m+1}) + \dots\dots\dots + (-1)^m[h(b_1) + h(b_3) + h(b_4) + \dots + h(b_{m+1})]$$

From easy simplification,

$$D^{(m-1)}h(b_1,b_2,b_3,\dots,b_{m+1}) - D^{(m-1)}h(b_1,b_2,b_4,\dots,b_{m+1}) - D^{(m-1)}h(b_1,b_3,b_4,\dots,b_{m+1}) = D^{(m)}h(b_1,b_2,b_3,\dots,b_{m+1}) = 0$$

$\Longrightarrow$ $D^{(m-1)}h(.,b_2,\dots,b_m)$ is Homomorphism.
Similarly, we can prove $D^{(m-1)}h(b_1,.,b_3,\dots,b_m)$,..., $D^{(m-1)}h(b_1,\dots,.,b_m)$ and $D^{(m-1)}h(b_1,\dots,b_{m-1},.)$ are homomorphism.
Hence, which is proved (2.5).
Now Take (2.6).
(2.6) is true for n = 0,1,2 from (2.1) and from Dh Definition.
Assume that (2.6) true for k=m-1≥2, k ∈ N then

$$h(b^m) = h(bbb\dots b(m\ \text{times}))$$
$$= m\ \ h(bb\dots b(\text{m-1 times})) - mC_{m-2}\ \ h(bb\dots b(\text{m-2 times})) + mD_{m-3}\,h(bb\dots b(\text{m-3 times})) - \dots + (-1)^m m\ \ h(b) + D^{(m-1)}h(b,b,\dots,b(m\ \text{times}))$$
$$= m\ \ h(b^{m-1}) - mD_{m-2}\ \ h(b^{m-2}) + mD_{m-3}\ \ h(b^{m-3}) - \dots + (-1)^m m\ \ h(b) + D^{(m-1)}h(b,b,\dots,b(m\ \text{times}))$$
$$= m\,[(m-1)\ \ h(b) + (m-1)D_2\ \ Dh(b,b) + (m-1)D_3\ D^{(2)}h(b,b,b) + \dots + (m-1)D_{m-1}\ \ D^{m-2}h(b,b,\dots,b(\text{m-1 times})]$$
$$- mD_{m-2}\,[(m-2)\ \ h(b) + (m-2)D_2\ \ Dh(b,b) + (m-2)D_3\ \ D^{(2)}h(b,b,b) + \dots + (m-2)D_{m-2}\ \ D^{m-3}h(b,b,\dots,b(\text{m-2 times})]$$
$$+ mD_{m-3}\,[(m-3)\ \ h(b) + (m-3)D_2\ \ Dh(b,b) + (m-3)D_3\ \ D^{(2)}h(b,b,b) + \dots + (m-3)D_{m-3}\ \ D^{m-4}h(b,b,\dots,b(\text{m-3 times})]$$
$$- \dots + (-1)^m m\ \ h(b) + D^{(m-1)}h(b,b,\dots,b(m\ \text{times}))$$

$= m \quad h(b) + mD2 \quad Dh(b, b) + mD3 \quad D^{(2)}h(b, b, b)$
$+ \ldots + mDm \quad D^{(m-1)}h(b, \ldots, b(m \text{ times}))$
when we used the definition of $D^{(m-1)}h$ and (2.5) ⇒ (2.6) for all m≥0.
From (1.4) and (2.1), for any fixed integer m>0, we have
$h(b^{-m}) = -m \quad h(b) + -mD2 \quad Dh(b, b) + -mD3 \quad D^{(2)}h(b, b, b)$
$+ \ldots + -mDm \quad D^{(m-1)}h(b, \ldots, b(m \text{ times}))$ from (2.5) and the above result for m>0. (2.6) is true for m<0.

Remark 2
The following are pairwise equivalent for h:P→ Q:
(i) $h \in KerD^{(m)}(P, Q)$;
(ii) $D^{(m-1)}h(., b2, \ldots, bm)$ is Homomorphism;
(iii) $D^{(m-1)}h(b1, ., b3, \ldots, bm)$ is Homomorphism;
....
(iv) $D^{(m-1)}h(b1, \ldots, ., bm)$ is Homomorphism;
(v) $D^{(m-1)}h(b1, \ldots, bn-1, .)$ is Homomorphism;

Proposition 1 Suppose that $h \in KerD^{(m)}(P, Q)$. Then

$$\begin{aligned} h(b_1^{m_1} b_2^{m_2} \ldots b_t^{m_t}) = \sum_{1 \le j \le t} [\, m_j h(b_j) + m_j D_2\, Dh(b_j, b_j) + \\ m_j D_3\, D^{(2)} h(b_j, b_j, b_j) + \cdots + m_j D_m\, D^{(m-1)} h\left(b_j, b_j, \ldots, b_j (m \text{ times})\right)] \\ + \sum_{1 \le j_1 < j_2 \le t} Dh(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}) + \sum_{1 \le j_1 < j_2 < j_3 \le t} m_{j_1} m_{j_2} m_{j_3} D^{(2)} h(b_{j_1}, b_{j_2}, b_{j_3}) \\ + \cdots + \sum_{1 \le j_1 < j_2 < \cdots < j_m \le t} m_{j_1} m_{j_2} \ldots m_{j_m} D^{(m-1)} h(b_{j_1}, b_{j_2}, \ldots, b_{j_m}) \qquad (2.7) \end{aligned}$$

for $m_j \in Z$ and every $p_j \in P, j = 1,2, \ldots t$ such that $b_i \neq b_{i+1}, i = 1,2, \ldots, t-1$

Proof To prove the Proposition the following lemma used, which was proved in [20]

Lemma 2 The following identity is valid for function h:P→Q and t ∈ N;

$$h(b_1 b_2 \ldots b_t) = \sum_{m \le t} \sum_{1 \le j_1 < j_2 < \cdots < j_m \le t} D^{(m-1)} h(b_{j_1}, b_{j_2}, \ldots, b_{j_m}) \qquad (2.8)$$

Replace b_j in (2.8) by $b_j^{m_j}$, we have

$$h(b_1^{m_1} b_2^{m_2} \ldots b_t^{m_t}) = \sum_{m \le t} \sum_{1 \le j_1 < j_2 < \cdots < j_n \le t} D^{(n-1)} h\left(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}, \ldots, b_{j_n}^{m_{j_n}}\right)$$

The vanish of $D^{(n-1)}h$ for $n \ge (m+1)$ yields

$$\begin{aligned} h(b_1^{m_1} b_2^{m_2} \ldots b_t^{m_t}) = \sum_{1 \le j \le t} h\left(b_j^{m_j}\right) + \sum_{1 \le j_1 < j_2 \le t} Dh\left(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}\right) \\ + \sum_{1 \le j_1 < j_2 < j_3 \le t} D^{(2)} h\left(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}, b_{j_3}^{m_{j_3}}\right) + \cdots + \\ \sum_{1 \le j_1 < j_2 < \cdots < j_m \le t} D^{(m-1)} h\left(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}, \ldots, b_{j_m}^{m_{j_m}}\right) \end{aligned}$$

Therefore, From (2.6) and (2.5), we have

$h(b_j^{m_j}) = m_j \quad h(b_j) + m_j D_2 \quad Dh(b_j, b_j) + \ldots + m_j D_m \quad D^{(m-1)}h(b_j, b_j, \ldots, b_j(m \text{ times}))$

$$D^{(2)} h\left(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}, b_{j_3}^{m_{j_3}}\right) = m_{j_1} m_{j_2} m_{j_3} D^{(2)} h(b_{j_1}, b_{j_2}, b_{j_3})$$

... ...

$$D^{(m-1)} h\left(b_{j_1}^{m_{j_1}}, b_{j_2}^{m_{j_2}}, \ldots, b_{j_m}^{m_{j_m}}\right) = m_{j_1} m_{j_2} \ldots m_{j_m} D^{(m-1)} h\left(b_{j_1}, b_{j_2}, \ldots, b_{j_m}\right)$$

This is (2.7). Hence the proof.

Solution for m^{th} order CDE in a Free Monoid

By [18, 24], from the embedding system of Free Monoid into Free Group In this part, Initially, we find solution of (1.5) for P which is Free Monoid on a onlyone character p.

Theorem 1 Suppose that define Free Monoid P on a single character p. Then $h \in KerD^{(m)}(P, Q)$ iff

$$h(p^m) = m\ h(p) + mD2\, Dh(p,p) + mD3\ D^{(2)}h(p,p,p) + \ldots + mDm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))\quad \forall m \in W \quad (3.1)$$

Proof ⇒.

Assume that $h \in KerD^{(m)}(P, Q)$

From (2.6) ⟹

$$h(p^m) = m\ h(p) + mD2\, Dh(p,p) + mD3\ D^{(2)}h(p,p,p) + \ldots + mDm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))\quad \forall m \in W$$

⇐.Take $h(p^m) = m\ h(p) + mD2\ Dh(p,p) + mD3\ D^{(2)}h(p,p,p) + \ldots + mDm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))\quad \forall m \in W$

as the dfn., of h on $P = \langle p \rangle$.

To Prove: $h \in KerD^{(m)}(P, Q)$

i.e., we want to prove that $D^{(m-1)}h$ is Homomorphism w.r.t. every variable.

Let

$$y_1 = p^{n1}, y_2 = p^{n2}, \ldots, y_m = p^{nm}$$

be m elements of P.

From (1.4) and (3.1) ⇒

$$D^{(m-1)}h(y_1, y_2, \ldots, y_m) = D^{(m-1)}h(p^{n1}, p^{n2}, \ldots, p^{nm})$$
$$= h(p^{n_1+n_2+\ldots+n_m}) - h(p^{n_1+n_2+\ldots+n_{m-1}}) - h(p^{n_1+\ldots+n_{m-2}+n_m})$$
$$- \ldots - h(p^{n2+\ldots+nm}) + \ldots\ldots$$
$$+(-1)^m [h(p^{n1}) + h(p^{n2}) + \ldots + h(p_n{}^m)]$$
$$= M1\ h(p) + M1D2\ Dh(p,p) + \ldots + M1Dm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$
$$-M2\ h(p) - M2D2\ Dh(p,p) - \ldots - M2Dm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$
$$-M3\ h(p) - M3D2\ Dh(p,p) - \ldots - M3Dm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$
$$-M4\ h(u) - M4D2\ Dh(p,p) - \ldots - M4Dm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$
$$- \ldots\ldots$$
$$+(-1)^m n_1\ h(p) + n_1D_2\ Dh(p,p) + \ldots + n_1D_m\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$
$$+ n_2\ h(p) + n_2D_2\ Dh(p,p) + \ldots + n_mD_m\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$
$$+ \ldots\ldots$$
$$+ n_m\ h(p) + nmD2\ Dh(p,p) + \ldots + nmDm\ D^{(m-1)}h(p,p,\ldots,p(m\ \text{times}))$$

Where $M1 = n1 + n2 + \ldots + nm$, $M2 = n1 + n2 + \ldots + nm-1$, $M3 = n1 + \ldots + nm-2 + nm$, $M4 = n2 + \ldots + nm$, …….

From very lengthy calculation, we have

$$D^{(m-1)}h(p^{n1}, p^{n2}, \ldots, p^{nm}) = n_1n_2\ldots n_m D^{(m-1)}h(p,p,\ldots\ldots,p(m\ \text{times}))$$

⟹ $D^{(m-1)}h$ is Homomorphism w.r.t any variable

Finally, for the Free Monoid P on an alphabet ⟨ P⟩ with $|P| \geq 2$,gives solution of (1.5).

For $p \in P$

$$p = p_1^{m_1} p_2^{m_2} \ldots p_l^{m_l}, \quad \text{where}\ \ p_j \in P, m_j \in W \qquad (3.2)$$

Define the functions T, T2, T3, for every fixed $u \in P$ and fixed pair of distinct $u, v \in P$:

$$T(p;u) = \textstyle\sum_{p_i=u} m_i \qquad (3.3)$$
$$T2(p;u,v) = \textstyle\sum_{j<k, p_i=u, p_j=v} m_j m_k \qquad (3.4)$$
$$T3(p;u,v) = \textstyle\sum_{j>k, p_i=u, p_j=v} m_j m_k \qquad (3.5)$$

along (3.2). By [20, 21], the functions T , T2, T3 are well defined. Also, T , T2, T3 verified the following:

$T(pq; u) = T(p; u) + T(q; v)$ (3.6)

$T_2(p; u, v) = T_3(p; v, u)$ (3.7)

Proposition 2 The following assertions true, for any fixed u ∈ P and fixedpair of distinct u,v in P,

(i) $T(\cdot; u) \in \mathrm{KerD}^{(m)}(P, W)$;

(ii) $T2(\cdot; u, v) \in \mathrm{KerD}^{(m)}(P, W)$;

(iii) $T3(\cdot; u, v) \in \mathrm{KerD}^{(m)}(P, W)$;

Proof From (3.6), (i) is obviously true.

For (ii). Take $p_1, p_2, \ldots, p_{m+1}$ in the Free Monoid P is of the form

$$p_1 = b_{11}^{s_{11}} b_{12}^{s_{12}} \ldots b_{1l}^{s_{1l}}$$

$$p_2 = b_{21}^{s_{21}} b_{22}^{s_{22}} \ldots b_{2l}^{s_{2l}}$$

$$p_3 = b_{31}^{s_{31}} b_{32}^{s_{32}} \ldots b_{3l}^{s_{3l}}$$

....

$$p_{m-1} = b_{(m-1)1}^{s_{(m-1)1}} b_{(m-1)2}^{s_{(m-1)2}} \ldots b_{(m-1)l}^{s_{(m-1)l}}$$

$$p_m = b_{(m)1}^{s_{(m)1}} b_{(m)2}^{s_{(m)2}} \ldots b_{(m)l}^{s_{(m)l}}$$

$$p_{m+1} = b_{(m+1)1}^{s_{(m+1)1}} b_{(m+1)2}^{s_{(m+1)2}} \ldots b_{(m+1)l}^{s_{(m+1)l}}$$

Let $S_1 = \sum_{1i<1j,\ b1i=u,\ b1j=v} S_{1i}S_{1j}$; $S_2 = \sum_{2i<2j,\ b2i=u,\ b2j=v} S_{2i}S_{2j}$; $S_3 = \sum_{3i<3j,\ b3i=u,\ b3j=v} S_{3i}S_{3j}$

....

$S_{(m-1)} = \sum_{(m-1)i<(m-1)j,\ b(m-1)i=u,\ b(m-1)j=v} S_{(m-1)i}S_{(m-1)j}$; $S_m = \sum_{mi<mj,\ bmi=u,\ bmj=v} S_{mi}S_{mj}$;

$S_{(m+1)} = \sum_{(m+1)i<(m+1)j,\ b(m+1)i=u,\ b(m+1)j=v} S_{(m+1)i}S_{(m+1)j}$ $S_{12} = \sum_{\ b1i=u,\ b2j=v} S_{1i}S_{2j}$;

$S_{13} = \sum_{\ b1i=u,\ b3j=v} S_{1i}S_{3j}$ .... $S_{1(m-1)} = \sum_{\ b1i=u,\ b(m-1)j=v} S_{1i}S_{(m-1)j}$;

$S_{1m} = \sum_{\ b1i=u,\ bmj=v} S_{1i}S_{mj}$; $S_{1(m+1)} = \sum_{\ b1i=u,\ b(m+1)j=v} S_{1i}S_{(m+1)j}$;

$S_{23} = \sum_{\ b2i=u,\ b3j=v} S_{2i}S_{3j}$ ....$S_{2(m-1)} = \sum_{\ b2i=u,\ b(m-1)j=v} S_{2i}S_{(m-1)j}$;

$S_{2m} = \sum_{\ b2i=u,\ bmj=v} S_{2i}S_{mj}$; $S_{2(m+1)} = \sum_{\ b2i=u,\ b(m+1)j=v} S_{2i}S_{(m+1)j}$;

..................................$S_{(m-1)m} = \sum_{\ b(m-1)i=u,\ bmj=v} S_{(m-1)i}S_{mj}$;

$S_{(m-1)(m+1)} = \sum_{\ b(m-1)i=u,\ b(m+1)j=v} S_{(m-1)i}S_{(m+1)j}$; $S_{m(m+1)} = \sum_{\ bmi=u,\ b(m+1)j=v} S_{mi}S_{(m+1)j}$

Then

$$T_2(p_1p_2\ldots p_{m+1};u,v) = S_1+S_2+S_3+\ldots+S_m+S_{(m+1)}+S_{12}+S_{13}+\ldots+S_{1(m+1)}+$$
$$+S_{23}+\ldots+S_{2(m+1)}+\ldots+S_{(m-1)m}+S_{(m-1)(m+1)}+S_{m(m+1)}$$
$$T_2(p_1p_2\ldots p_m;u,v) = S_1+S_2+S_3+\ldots+S_m+S_{12}+S_{13}+\ldots+S_{1m}+$$
$$+S_{23}+\ldots+S_{2m}+\ldots+S_{(m-1)m}$$

$$T_2(p_1p_2\ldots p_{(m-1)}p_{(m+1)};u,v) = S_1+S_2+S_3+\ldots+S_{(m-1)}+S_{(m+1)}+S_{12}$$
$$+S_{13}+\ldots+S_{1(m-1)}+S_{1(m+1)}+S_{23}+\ldots+S_{2(m+1)}+\ldots+S_{(m-1)(m+1)}$$

$$\ldots\ldots$$

$$T_2(p_1p_2;u,v) = S_1+S_2+S_{12}$$
$$T_2(p_1p_3;u,v) = S_1+S_3+S_{13}$$

$$\ldots\ldots$$

$$T_2(p_1p_{m+1};u,v) = S_1+S_{(m+1)}+S_{1(m+1)}$$
$$T_2(p_2p_3;u,v) = S_2+S_3+S_{23}$$

$$\ldots\ldots$$

$$T_2(p_2p_{m+1};u,v) = S_2+S_{(m+1)}+S_{2(m+1)}$$

$$\ldots\ldots$$

$$T_2(p_mp_{m+1};u,v) = S_m+S_{(m+1)}+S_{m(m+1)}$$
$$T_2(p_1;u,v) = S_1$$
$$T_2(p_2;u,v) = S_2$$

$$\ldots\ldots$$

$$T_2(p_m;u,v) = S_m$$
$$T_2(p_{m+1};u,v) = S_{m+1}$$

Therefore, we have

$$T_2(p_1p_2\ldots p_{m+1};u,v) - T_2(p_1p_2\ldots p_m;u,v) - T_2(p_1\ldots p_{m-1}p_{m+1};u,v)$$
$$-\ldots - T_2(p_2\ldots p_mp_{m+1};u,v)$$
$$+\ldots\ldots$$
$$+(-1)^m\,[T_2(p_1;u,v)+T_2(p_2;u,v)+\ldots+T_2(p_{m-1};u,v)$$
$$+T_2(p_n;u,v)+T_2(p_{m+1};u,v)] = 0$$

Hence (ii) is proved.
Similarly we can prove (iii) or directly from (3.7) we can prove (iii).

CONCLUSION

We studied m[th] Order Cauchy difference equation and solution has been found in Free Monoid generated by single and more than 2 character. We can extend this work to the other type Cauchy functional equations and also we can find the solution for the any functional equations in Free Semi Group and Different type of Groups.

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