GENERALIZATION OF THE ONE-DIMENSIONAL WAVE EQUATION VIA $(p, q)-$ DEFORMATION

SAMI H. ALTOUM

ABSTRACT. In this work, we present a comparative analysis of the q-deformed and (p, q)-deformed formulations of the one-dimensional wave equation within the framework of generalized calculus. The q-deformed wave equation, constructed using the Jackson derivative, introduces a single deformation parameter that encodes discrete-scale effects and leads to modified wave propagation governed by q-d'Alembert-type solutions. The (p, q)-deformed wave equation extends this approach by incorporating two independent deformation parameters, allowing for asymmetric and multi-scale spatial dilations. We show that the q-deformed model is recovered as a special limiting case of the (p, q)-formalism, while both deformations reduce smoothly to the classical wave equation in the undeformed limit. This comparison demonstrates that, although the q-deformation captures essential discretization features, the (p, q)-deformation provides a richer algebraic structure and greater flexibility for modeling wave phenomena in non-uniform and anisotropic media.

1. Introduction

The wave equation is a fundamental second-order linear hyperbolic partial differential equation that describes the propagation of various types of waves, including sound, water, and electromagnetic waves. It appears in numerous fields such as acoustics, fluid dynamics, and electromagnetism. In its simplest one-dimensional form, the wave equation can be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

where x represents the spatial coordinate, t is time, $u = u(x, t)$ denotes the displacement, and c is the wave velocity. Since the square of the velocity, c^2, is independent of its sign, the equation does not determine the direction of wave propagation, allowing solutions that travel both forward $(+x)$ and backward $(-x)$. The one-dimensional wave equation was first formulated by Jean le Rond *D'Alembert* in 1746, and Euler extended it to three dimensions a decade later. This equation effectively describes small oscillations around equilibrium states, where linear approximations, such as *Hook's* law, are valid. Its solutions are essential in areas ranging from fluid mechanics to electromagnetism, optics, gravitational physics, and thermal conduction. Classical studies of vibrating strings, for instance, involved contributions from *D'Alembert*, Euler, Daniel Bernoulli, and Joseph-Louis

2000 *Mathematics Subject Classification.* Primary 60H40; Secondary 46A32, 46F25, 46G20.
Key words and phrases. Wave equation, q-calculus, q-Space Wave Equation.

Lagrange. In modern mathematical physics, the concept of deformation has gained significant attention. The q-calculus, or quantum calculus, extends traditional differentiation and integration using a deformation parameter q. The q-derivative, introduced by Frank Hilton Jackson, provides a q-analogue of the ordinary derivative and has applications in combinatorics, number theory, and statistical mechanics. More generally, (p, q)-deformations introduce two parameters, p and q, offering a richer mathematical framework that bridges classical analysis, quantum theory, and fractional calculus. This approach enables the study of deformed differential and integral operators, q-difference equations, and fractional q-calculus equations [1, 2, 3, 4, 6, 6]. The (p, q)-deformation framework also finds applications in physics. In theories of quantum gravity and deformed quantum mechanics, the deformation parameters can encode phenomena such as a non-zero vacuum energy or curved spacetime. When $p = q = 1$, the classical quantum mechanics framework is recovered. For general (p, q)-values, however, one can formulate deformed wave equations, providing a powerful tool for studying wave propagation in non-classical or curved backgrounds [13, 15, 16]. The study of (p, q)-deformed wave equations is motivated by the need to generalize classical models of wave propagation to incorporate quantum and fractional effects. Such generalizations have potential applications in quantum mechanics, quantum field theory, and materials with non-standard mechanical properties. In this work, we aim to investigate the (p, q)-deformation of the one-dimensional wave equation and explore its analytical solutions under various conditions. More precisely, we are interested by the following problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 D_{q,x}^2 u \tag{1.2}$$

where $D_{q,x}$ is defined using the the q-derivative (see the Preliminaries).

2. Preliminaries

We recall some basic notations of the language of p, q-calculus (see [12, 10, 16]). The (p, q)−derivative of a function f with respect to x is defined by

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x} \quad (x \neq 0)$$

and $(D_{p,q} f)(0) = f'(0)$, provided that f is differentiable at 0, and $D_{p,q} = D_{q,p}$. We recall some basic notations of the language of p, q−calculus. The natural number n has the following (p, q)− deformation for $(0 < q < p < 1)$ where

$$[n]_{p,q} := p^{n-1} + p^{n-2} q + p^{n-3} q^2 \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}.$$

It happens clearly that $D_{p,q} x^n = [n]_{p,q} x^{n-1}$. Note also that for $p = 1$, the (p, q) derivative reduces to the Hahn derivative given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad (x \neq 0).$$

As with ordinary derivative, the action of the (p, q) −derivative of a function is a linear operator. More precisely, for any constants a and b,

$$D_{p,q}\ af(x) + bg(x) = aD_{p,q}f(x)bD_{p,q}g(x).$$

The twin-basic number is a natural generalization of the q−number, that is

$$\lim_{p\to 1}[n]_{p,q} = [n]_q \frac{1-q^n}{1-q^n},\ \ q \neq 1.$$

The linear (p, q) −derivative operator satisfies the following properties:

(i): $lim_{p,q\uparrow 1}(D_{p,q}f)(x) = f'(x)$,
(ii): $D_{p,q}(x^n) = [n]_{p,q}x^{n-1}$,
(iii): $D_{p,q}(f(x)g(x)) = D_{p,q}f(x)g(qx) + f(px)D_{p,q}g(x)$,
(iv): $D_{p,q}\ \frac{f(x)}{g(x)} = \frac{D_{p,q}f(x)g(px)-f(px)D_{p,q}g(x)}{g(px)g(qx)}$.

3. q−Space Wave Equation

In the reminder of this paper we take $q \in (0, 1)$. In this section,we introduce the following equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 D^2_{q,x}u. \tag{3.1}$$

Eq. (3.1) will be called q−space wave equation.

Theorem 3.1. *For* $0 < q < 1$, *we have*

(1) *u given by*

$$u(t, x) = \left(Ae^{-t\sqrt{c^2L}} + Be^{t\sqrt{c^2L}}\right) e_q(x\sqrt{L})$$

is solution of Eq. (3.1), for any constants A and B, where $L > 0$.

(2) *u given by*

$$u(t, x) = \left(A\cos(t\sqrt{-c^2L}) + B\sin(t\sqrt{-c^2L})\right) e_q(jx\sqrt{-L})$$

is solution of Eq. (3.1,) for any constants A and B, where $L < 0$ *and* $j^2 = -1$.

Proof. For $0 < q < 1$, we get

$$D_{q,x}u(t, x) = \frac{u(t, x) - u(t, qx)}{x(1 - q)} := h(t, x) \tag{3.2}$$

and

$$D^2_{q,x}u(t, x) = \frac{h(t, x) - h(t, qx)}{x(1 - q)}. \tag{3.3}$$

Putting

$$u(t, x) = R(t)S(x).$$

we obtain

$$S(x)R''(t) = c^2R(t)D^2_{q,x}S(x)$$

then,

$$\frac{D^2_{q,x}S(x)}{S(x)} = \frac{R''(t)}{c^2R(t)} = L,$$

for a constant L. This implies

$$R''(t) = Lc^2R(t) \tag{3.4}$$

$$D^2_{q,x}S(x) = LS(x). \tag{3.5}$$

Using Eq. (3.3) in Eq. (3.5), we obtain

$$\frac{qS(x) - (1+q)S(qx) + S(q^2x)}{qx^2(1-q)^2} = LS(x)$$

$$qS(x) - (1+q)S(qx) + S(q^2x) = qx^2(1-q)^2LS(x).$$

This gives

$$q\left(1 - x^2(1-q)^2L\right)S(x) = (1+q)S(qx) - S(q^2x). \tag{3.6}$$

From which, we obtain

$$q\left(1 - q^2x^2(1-q)^2L\right)S(qx) = (1-q)S(q^2x) - S(q^3x)$$

$$\vdots$$

$$q\left(1 - q^{2n}x^2(1-q)^2L\right)S(q^nx) = (1+q)S(q^{n+1}x) - S(q^{n+2}x)$$

$$q\left(1 - q^{2n+2}x^2(1-q)^2L\right)S(q^{n+1}x) = (1+q)S(q^{n+2}x) - S(q^{n+3}x).$$

Then, we deduce that

$$\begin{aligned}
& q(1-\alpha_x)S(x) + qS(qx) - q^3\alpha_xS(qx) \\
& \vdots \\
+ \;& qS(q^nx) - q^{2n+1}\alpha_xS(q^nx) \\
+ \;& qS(q^{n+1}x) - q^{2n+3}\alpha_xS(q^{n+1}x) \\
= \;& S(qx) + qS(qx) - S(q^2x) \\
+ \;& S(q^2x) + qS(q^2x) - S(q^3x) \\
& \vdots \\
+ \;& S(q^{n+1}x) + qS(q^{n+1}x) - S(q^{n+2}x) \\
+ \;& S(q^{n+2}x) + qS(q^{n+2}x) - S(q^{n+3}x)
\end{aligned}$$

$$q(1-\alpha_x)S(x) - \alpha_x(q^3S(qx)) + \cdots + q^{2n+3}S(q^{n+1}x) \tag{3.7}$$
$$= S(qx) + qS(q^{n+2}x) - S(q^{n+3}x)$$

where

$$\alpha_x = x^2(1-q)^2L.$$

For S given by

$$S(x) = \sum_{i=0}^{\infty} a_ix^i,$$

we get

$$\alpha_x\left(q^3\sum_{i=0}^{\infty} a_i\,q^ix^i + \cdots + q^{2n+3}\sum a_i\,q^{(n+1)i}x^i\right)$$

$$= q^3(1-q)^2L\sum_{i=0}^{\infty} a_i q^i \left(1 + q^i + \cdots + (q^{2+i})^n\right) x^{i+2}$$

$$= q^3(1-q)^2L\sum_{i=0}^{\infty} a_i q^i \left(\frac{1-(q^{2+i})^{n+1}}{1-q^{2+i}}\right) x^{i+2} := \beta$$

as $n \to \infty$, the last term gives

$$\beta \to q^3(1-q)^2L\sum_{i=0}^{\infty} a_{i-2}\frac{q^i}{1-q^i}x^i. \tag{3.8}$$

Then, Eq. (4.1) and Eq. (3.8) we get

$$q\left(1-x^2(1-q)^2L\sum_{i=0}^{\infty} a_i x^i - \beta\right) = \left(\sum_{i=0}^{\infty} a_i q^i x^i\right) + (q-1)a_0$$

Therefore, we get

$$\left(\sum_{i=0}^{\infty} qa_i x^i\right) - q(1-q)^L\sum_{i=0}^{\infty} a_i x^{i+2} - \beta = \left(\sum_{i=0}^{\infty} a_i q^i x^i\right) + (q-1)a_0$$

which gives

$$qa_0 + qa_1x + \left(\sum_{i=0}^{\infty} qa_i x^i\right) - q(1-q)^2L\left(\sum_{i=0}^{\infty} a_{i-2}x^i\right) - \beta$$

$$= a_0 + a_1qx + \left(\sum_{i=0}^{\infty} a_i q^i x^i\right) + (q-1)a_0.$$

This implies that

$$qa_i - q(1-q)^2La_{i-2} - q(1-q)^2L\frac{q^i}{1-q^i}a_{i-2} = a_iq^i.$$

Hence, we obtain

$$q(1-q^{i-1})a_i = q(1-q)^2L\left(1+\frac{q^i}{1-q^i}\right)a_{i-2}.$$

Then

$$q[i-1]_qa_i = q(1-q)L\left(\frac{1-q^i+q^i}{1-q^i}\right)a_{i-2}.$$

This implies that

$$[i-1]_qa_i = L\frac{(1-q)}{1-q^i}a_{i-2}.$$

Which gives

$$[i-1]_qa_i = L\frac{1}{[i]_q}a_{i-2}.$$

Therefore, we get

$$a_i = \frac{L}{[i]_q[i-1]_q}a_{i-2} \quad i \geq 2. \tag{3.9}$$

If $L > 0$, then

$$a_i = \frac{(\sqrt{L})^i}{[i]_q!}.$$

Therefore, we obtain

$$\begin{aligned} a_{i+2} &= \frac{(\sqrt{L})^{i+2}}{[i+2]_q!} \\ &= \frac{(\sqrt{L})^i}{[i]_q!} \cdot \frac{(\sqrt{L})^2}{[i+2]_q[i+1]_q} \\ &= a_i \frac{L}{[i+2]_q[i+1]_q}. \end{aligned}$$

This verifies Eq. (3.9). Then, we get

$$\begin{aligned} S(x) &= \sum_{i=0}^{\infty} a_i x^i \\ &= \sum_{i=0}^{\infty} \frac{(\sqrt{L}x)^i}{[i]_q!} \\ &= e_q(x\sqrt{L}). \end{aligned}$$

If $L < 0$, we take

$$a_i = \frac{(j\sqrt{-L})^i}{[i]_q!}, \quad j = \sqrt{-1}.$$

Hence, we obtain

$$\begin{aligned} a_{i+2} &= (j)^{i+2}\left(\sqrt{-L}\right)^{i+2} \\ &= \frac{\left(j\sqrt{-L}\right)^i}{[i]_q!} \frac{j^2(-L)}{[i+2]_q[i+1]_q} \\ &= a_i \frac{-(-L)}{[i+2]_q[i+1]_q} \\ &= \frac{L}{[i+2]_q[i+1]_q} a_i. \end{aligned}$$

This verifies Eq. (3.9). Then, we obtain

$$\begin{aligned} S(x) &= \sum_{i=0}^{\infty} a_i x^i \\ &= e_q\left(jx\sqrt{-L}\right). \end{aligned}$$

Hence, we obtain for $(L > 0)$

$$u(t, x) = \left(Ae^{-t\sqrt{c^2L}} + Be^{t\sqrt{c^2L}}\right) e_q(x\sqrt{L}).$$

If $L < 0$, we obtain

$$u(t, x) = \left(A\cos(t\sqrt{-c^2L}) + B\sin(t\sqrt{-c^2L})\right) e_q(jx\sqrt{-L})$$

where A and B are constants. □

4. (p, q)– Space Wave Equation

In this section, we introduce the (p, q)– Space Wave Equation defined as follows:

$$\frac{\partial^2 u}{\partial t^2} = c^2 D^2_{p,q,x} u. \tag{4.1}$$

Eq. (4.1) will be called (p, q)–space wave equation.

Theorem 4.1. *For $0 < p, q < 1$, Eq. (4.1) has a solution given by*

$$u(x, t) = \sum_{n=1}^{\infty} [A_n \cos(ck_n t) + B_n \sin(ck_n t)] \sin_{p,q}(k_n x) \tag{4.2}$$

The coefficients A_n and B_n are determined from the initial conditions

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x),$$

Proof. For $0 < p, q < 1$, we get The standard one-dimensional wave equation is given by

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x \in \mathbb{R}, \ t > 0,$$

where c is the wave speed. For a sufficiently smooth function $f(x)$, the (p, q)-derivative is defined as

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0, \quad p \neq q.$$

Higher-order (p, q)-derivatives are defined iteratively:

$$D^2_{p,q} f(x) = D_{p,q} \left(D_{p,q} f(x)\right).$$

When, $p, q \to 1$, the (p, q)-derivative reduces to the classical derivative:

$$\lim_{p,q \to 1} D_{p,q} f(x) = \frac{df(x)}{dx}.$$

We define the (p, q)-deformed wave equation as

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 D^2_{p,q} u(x, t)$$

Now, this equation models wave propagation on a nonlocal or quantum-deformed spatial structure. Assume a separable solution of the form

$$u(x, t) = X(x)T(t). \tag{4.3}$$

Substituting into the (p, q)-wave equation gives

$$X(x)T''(t) = c^2 T(t) D^2_{p,q} X(x).$$

Dividing both sides by $c^2 X(x)T(t)$ yields

$$\frac{T''(t)}{c^2 T(t)} = \frac{D^2_{p,q} X(x)}{X(x)} = -\lambda,$$

where $\lambda > 0$ is a separation constant. The temporal part satisfies

$$T''(t) + c^2\lambda T(t) = 0.$$

Hence, we deduce the general solution is

$$T(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = c\sqrt{\lambda}.$$

Now, we get the spatial (p, q)-eigenvalue problem which takes the form

$$D_{p,q}^2 X(x) + \lambda X(x) = 0. \tag{4.4}$$

This equation represents the (p, q)-analogue of the Helmholtz equation. We consider a finite domain $x \in (0, L)$ with boundary conditions

$$u(0, t) = u(L, t) = 0.$$

These conditions imply

$$X(0) = 0 \Rightarrow D = 0,$$

and

$$\sin_{p,q}(kL) = 0.$$

Thus, the eigenvalues are discrete:

$$k_n = \frac{n\pi_{p,q}}{L}, \quad n = 1, 2, 3, \ldots$$

where $\pi_{p,q}$ denotes the first positive zero of $\sin_{p,q}(x)$. Finally, we obtain the full solution of the (p, q)-wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} [A_n \cos(ck_n t) + B_n \sin(ck_n t)] \sin_{p,q}(k_n x). \tag{4.5}$$

The coefficients A_n and B_n are determined from the initial conditions

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x),$$

using (p, q)-Fourier sine expansions. Which complete the proof. In the limit $p, q \to 1$, we recover

$$D_{p,q} \to \frac{d}{dx}, \quad \sin_{p,q}(x) \to \sin(x),$$

and the solution reduces to the classical wave equation solution. Which completes the proof. □

5. (p, q)-Exponential and Trigonometric Functions

The (p, q)-exponential function is defined by the series

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!},$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Using the (p, q)-exponential, we define the (p, q)-trigonometric functions as

$$\sin_{p,q}(x) = \frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i},$$
$$\cos_{p,q}(x) = \frac{e_{p,q}(ix) + e_{p,q}(-ix)}{2}.$$

Lemma 5.1. *For $k \in \mathbb{R}$, the following identity holds:*

$$D^2_{p,q} \sin_{p,q}(kx) = -k^2 \sin_{p,q}(kx).$$

Proof. Using the known properties of the (p, q)-exponential,

$$D_{p,q}e_{p,q}(kx) = ke_{p,q}(kx), \quad D^2_{p,q} e_{p,q}(kx) = k^2 e_{p,q}(kx),$$

we compute

$$\begin{aligned} D^2_{p,q} \sin_{p,q}(kx) &= \frac{1}{2i}\left(D^2_{p,q} e_{p,q}(ikx) - D^2_{p,q} e_{p,q}(-ikx)\right) \\ &= \frac{1}{2i}\left(-k^2 e_{p,q}(ikx) + k^2 e_{p,q}(-ikx)\right) \\ &= -k^2 \sin_{p,q}(kx). \end{aligned}$$

From the eigenvalue equation, we obtain

$$\lambda = k^2.$$

Which completes the proof. □

From Lemma 5.1 we get solution of the (p, q)-deformed wave equation using (p, q)-trigonometric functions

Theorem 5.2. *For $0 < q < 1$, Eq. (4.1) has solution given by*

(1) *has the spatial solution*

$$X(x) = C \cos_{p,q}(\lambda x) + D \sin_{p,q}(\lambda x). \tag{5.1}$$

(2) *has a general solution*

$$u(x, t) = \left(C \cos_{p,q}(\lambda x) + D \sin_{p,q}(\lambda x)\right)\left(A \cos(c\lambda t) + B \sin(c\lambda t)\right). \tag{5.2}$$

Boundary conditions determine the allowed values of λ and the constants.

Proof.

$$\frac{\partial^2 u(x, t)}{\partial t^2} = c^2 D^2_{p,q} u(x, t),$$

where the (p, q)-derivative is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}.$$

Assuming a separable solution of the form

$$u(x, t) = X(x)\, T(t),$$

substitution into the wave equation yields

$$\frac{T''(t)}{c^2 T(t)} = \frac{D^2_{p,q} X(x)}{X(x)} = -\lambda^2,$$

where λ is a separation constant. The time-dependent equation

$$T''(t) + c^2\lambda^2 T(t) = 0$$

has the classical solution

$$T(t) = A\cos(c\lambda t) + B\sin(c\lambda t). \tag{5.3}$$

The spatial equation takes the form

$$D_{p,q}^2 X(x) + \lambda^2 X(x) = 0.$$

Its solutions can be expressed in terms of (p, q)-trigonometric functions, defined through the (p, q)-exponential

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q}!}, \qquad [n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Accordingly, the (p, q)-cosine and (p, q)-sine functions are given by

$$\cos_{p,q}(x) = \frac{e_{p,q}(ix) + e_{p,q}(-ix)}{2}, \qquad \sin_{p,q}(x) = \frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i}.$$

Thus, the spatial solution holds

$$X(x) = C\cos_{p,q}(\lambda x) + D\sin_{p,q}(\lambda x). \tag{5.4}$$

Combining the temporal and spatial parts, the general solution of the (p, q)-deformed wave equation is

$$u(x, t) = \left(C\cos_{p,q}(\lambda x) + D\sin_{p,q}(\lambda x)\right)\left(A\cos(c\lambda t) + B\sin(c\lambda t)\right). \tag{5.5}$$

In the limit $p, q \to 1$, the (p, q)-trigonometric functions reduce to the classical sine and cosine functions, and the standard wave equation solution is recovered. □

6. (p, q)-d'Alembert Solution

In the classical case, the one-dimensional wave equation admits the d'Alembert solution

$$u(x, t) = F(x - ct) + G(x + ct).$$

We now construct its (p, q)-analogue.

6.1. Factorization of the (p, q)-Wave Operator.

The (p, q)– wave equation is.

$$\frac{\partial^2 u}{\partial t^2} - c^2 D_{p,q}^2 u = 0.$$

This operator can be factorized formally as

$$\left(\frac{\partial}{\partial t} - cD_{p,q}\right)\left(\frac{\partial}{\partial t} + cD_{p,q}\right) u(x, t) = 0.$$

Hence, the solution is obtained by solving the first-order equations

$$\left(\frac{\partial}{\partial t} - cD_{p,q}\right) u = 0, \tag{6.1}$$

$$\left(\frac{\partial}{\partial t} + cD_{p,q}\right) u = 0. \tag{6.2}$$

6.2. First-Order (p, q) Transport Equations.

Let.

$$u(x, t) = F(x, t).$$

Then,

$$\frac{\partial F}{\partial t} = cD_{p,q}F.$$

Using the definition of the (p, q)-derivative,

$$\frac{\partial F}{\partial t} = \frac{c}{(p-q)x}\, F(px, t) - F(qx, t) \, .$$

This equation describes a *nonlocal transport process* in space. A similar equation holds for the second factor with opposite sign.

Remark 6.1. The (p, q)-d'Alembert solution provides a natural extension of classical traveling waves to nonlocal and quantum-deformed geometries. Compared to the Jackson q-derivative, the (p, q)-formalism offers greater flexibility, symmetry, and modeling power, making it suitable for applications in quantum groups, discrete lattices, and deformed space-time wave propagation.

Conclusion

The comparative study of the q-deformed and (p, q)-deformed wave equations highlights a clear structural and conceptual relationship between the two deformation schemes. The q-deformed formulation, based on the Jackson derivative, represents the simplest extension of the classical wave equation and is characterized by a single deformation parameter that introduces discrete-scale effects into wave propagation. Its solutions retain a deformed d'Alembert structure and converge to the classical solutions in the limit $q \to 1$. The (p, q)-deformed wave equation generalizes the q-deformation by introducing two independent deformation parameters, thereby allowing asymmetric and multi-scale spatial dilations. As a result, the q-deformed wave equation appears as a special case of the (p, q)-formalism, confirming the unifying nature of the latter. This additional flexibility leads to a richer algebraic framework and broadens the range of physical systems that can be effectively modeled. Consequently, while the q-deformation is well suited for describing single-scale discretized media, the (p, q)-deformation provides a more comprehensive and versatile approach for the analysis of wave dynamics in generalized and non-uniform settings.

Conflict of interest

All authors declare that they have no competing interests.

References

[1] S. H. Altoum, Othman.H.A and Rguigui.H, *Quantum white noise Gaussian kernel operators*, Chaos, Solitons & Fractals, Vol 104: pp. 468-476, DOI: 10.1016/j.chaos.08.039, (2017).

[2] S. H. Altoum, *q-Deformation of Transonic Gas Equation*, Journal of Mathematics and Statistics, Volume 14: pp. 88-93, DOI: 10.3844/jmssp, (2018).

[3] S. H. Altoum, *q-sl2 and Associated Wave and Heat Equations*, American Journal of Applied Sciences, Vol 15: pp. 261-266, DOI: 10.3844/ajassp, (2018).

[4] S. H. Altoum, *q-deformation of the square white noise Lie algebra*, Transactions of Razmadze Mathematical Institute, Vol 172: pp. 133-139, (2018).

[5] S. BERMUDO, P. KORUS and J. E. NAPOLES VALD'ES, *ON q-HERMITE-HADAMARD INEQUALITIES FOR GENERAL CONVEX FUNCTIONS*, Acta Math. Hungar, Vol 162 (1): pp. 364-374, (2020).
[6] R. Bhatia and Tanvi Jain, *Variational Principles for Symplectic Eigenvalues,* Canadian mathematical bulletin, Vol 64(3), pp 1-7, 2020.
[7] G. W. Bluman and Kumei *S*, *Symmetries And Differential Equations,* New York: Springer-Verlag, (1989).
[8] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Birkhauser Springer, Basel, (2012).
[9] G. D. FrattaCyrill, B. Muratov, Filipp Rybakov and Valeriy Slastikov *Variational Principles of Micromagnetics Revisited,* SIAM Journal on Mathematical Analysis, Vol 52(4), pp 3580-3599, (2020).
[10] H. W. Guggenheimer, *Differential Geometry,* MC Grahill Book Campany, (1963).
[11] J. H. He, *Generalized variational principles for buckling analysis of circular cylinders,* Acta Mech, Springer, (2020).
[12] H. F. Jackson, *q-Difference equations*, Am. J. Math, Vol 32: pp. 305-314, (1990).
[13] D. F. Lawden, *Introduction To Tensor Calculus, Relativity And Cosmology,* Third Edition, Dover Publications, (2003).
[14] J. M. Lee, *Introduction To Smooth Manifolds,* Graduate Texts In Mathematics, Springer-Verlag, (2003).
[15] J. A. Medina, Salvador Cruz Garcia, Juan E. Napoles Valdes, Thelma Galeana Moyaho, *Some New Results in the q-Calculus*, Discontinuity, Nonlinearity, and Complexity, Vol: 10(4), pp. 755-763, (2021).
[16] J. E. Valdes, Jorge A. Castillo Medina, Paulo M. Guzman and Luciano Miguel Lugo, *A New Local Fractional Derivative of q-uniform Type*, Discontinuity, Nonlinearity, and Complexity, Vol 8(1): pp. 101-109, (2019).

DEPARTMENT OF MATHEMATICS, MANAR AL JANOUB COLLEGE OF SCIENCE AND TECHNOLOGY, SAUDI ARABIA
Email address: s.altom@mjc.edu.sa