A Typical Openness and Continuity in ξ-Nano Topological Spaces Via Ideals

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ABSTRACT
In this work, we present ξ-Nano ideal topological spaces by the context of idealization in a general topological spaces and examine the connection between ξ-Nano topological space and ξ-Nano ideal topological spaces. Also we established a class of functions known as ξ- Nano ideal continuous functions and explored their properties interms of ξI –open sets and ξI -closed sets. Furthermore, Also we strive to establish the definitions of ξI -open maps and ξI -closed maps.

Keywords: Nano-topology, ξ-Nano topology, ξ-Nano Ideal topology, ξI -open and ξI -closed sets.

1. INTRODUCTION

The inclusion of ideals in the general topological spaces were originated by Vaidyanatha Swamy [16] and Kuratowski [9]. Also, it has been studied by P. Samuels [13]. Furthermore, detailed investigation were carried out on the ideal and new topology is determined by D. Jankovic and J.R. Hamlet [7]. A collection of non-empty subsets I of a set $\boldsymbol{X}$ is said to be an ideal which is closed under the subsets operation.

1. H ∈ I, S ⊂ H imply S ∈ I (Heridity).
2. H, S ∈ I imply H ∪ S ∈I (Finite-Additive)

If $(\boldsymbol{X}, \tau)$ be a topological space along with an ideal I $\subseteq \boldsymbol{X}$ then (**X**, τ, I)-is called an ideal topological space. Let a subset $\mathcal{U} \subseteq X$. we denote $\mathcal{U}^*(I)=\{x\in X : \mathcal{U} \cap \mathcal{Y} \notin I$, for every neighborhood $\mathcal{Y}$ of x $\}$ and $cl^*(\mathcal{U}) = \mathcal{U} \cup \mathcal{U}^*(I)$ defines a Kuratowski-closure operator cl*(.) from $\mathcal{P}(\boldsymbol{X}) \to \mathcal{P}(\boldsymbol{X})$, for every subset $\mathcal{U}$ of $\boldsymbol{X}$, which defines a new topology

τ^*_ξ (I , τ_ξ) = {V ⊆ X : cl* (X − V) = $\boldsymbol{X}$ − V } , simply $\tau^*(I)$ denoted as τ^* .

The members belonging to τ^* are referred as * - openset and its complements are refereed as * - closedset. Recently, the nano topology concept is established by Lellis Thivagar and C.Richard [14],[15] . It contains a maximum of just five open sets and it is the smallest topology as compared to many topological space. Moreover, Jenavee [8], provided the extension of nano topology, with the help of ξ--opensets, which was ξ-nano topology. In the context of ideal topological spaces from the general topological space, we present the ξ-nano ideal topological spaces by including the topological ideal in the ξ-Nano topology space.

2. PRELIMINARIES

Here, we review the certain definitions that will come in handy in the follow-up. Through- out this paper, we consider $\mathcal{H}$ as universe and establish $\mathcal{R}$ as an equivalency-relation defined on $\mathcal{H}$.

Definition 2.1. [15] On the Universe $\mathcal{H}$, an equivalence relation $\mathcal{R}$ is known as the indiscernibility relation. It is believed that the members who belongs to the same equivalence class are indistinguishable from one another. The approximation space is refereed as the Pair $(\mathcal{H},\mathcal{R})$.

In a approximation space $(\mathcal{H},\mathcal{R})$ and X is a subset of the universe $\mathcal{H}$, then the lower, upper-approximations and boundary-region are characterized as follows:

1. The lower-approximations

$$\mathcal{L}_{\mathcal{R}}(\boldsymbol{X}) = \bigcup_{x\in \mathbf{H}} \{ \mathcal{R}(\mathbf{x}) : \mathcal{R}(\mathbf{x}) \subseteq \boldsymbol{X} \}$$

2. The upper-approximations

$\mathcal{U}_{\mathcal{R}}(\boldsymbol{X}) = \bigcup_{x\in\mathbf{H}} \{\mathcal{R}(\mathbf{x}) : \mathcal{R}(\mathbf{x}) \cap \boldsymbol{X} \neq \emptyset\}$

where $\mathcal{R}(\mathbf{x})$ represents the equivalency class indicated by ***x*.**

3. The Boundary region $\mathcal{B}_{\mathcal{R}}(\boldsymbol{X})$ is characterized by the difference between the upper and lower-approximations referred as

$\mathcal{B}_{\mathcal{R}}(\boldsymbol{X}) = \mathcal{U}_{\mathcal{R}}(\boldsymbol{X}) - \mathcal{L}_{\mathcal{R}}(\boldsymbol{X})$

Definition 2.2. [15] In the Universe $\mathcal{H}$**,** if $\mathcal{R}$ is an equivalency relation, then $\mathfrak{I}_{\mathcal{R}}(X)$ = {$\mathcal{H}$, ∅, $\mathcal{L}_{\mathcal{R}}$ (**X**), $\mathcal{U}_{\mathcal{R}}$ (**X**), $\mathcal{B}_{\mathcal{R}}$ (**X**)} where **X** ⊆ $\mathcal{H}$ follows specific axioms.

1. $\mathcal{H}$, ∅ ∈ $\mathfrak{I}_{\mathcal{R}}(X)$

2. Any sub-collection of members of $\mathfrak{I}_{\mathcal{R}}(X)$ and whose union is also i n $\mathfrak{I}_{\mathcal{R}}(X)$

3. The sub-collection of finite members of $\mathfrak{I}_{\mathcal{R}}(X)$ and whose intersection is also in $\mathfrak{I}_{\mathcal{R}}(X)$. Then, $\mathfrak{I}_{\mathcal{R}}(X)$ or simply $\mathfrak{I}_{\mathcal{N}}$ is topology on $\mathcal{H}$ referred as the nano-topology on $\mathcal{H}$ so that ($\mathcal{H}$, -) is a nono $\mathfrak{I}_{\mathcal{N}}$ is open) and its complement -openset (n -are referred to nano $\mathfrak{I}_{\mathcal{N}}$ topological space. The members in .nano-closedset (n-closed)

Remark 2.1. $\mathfrak{I}_{\mathcal{N}}$ basis is represented as $\beta_{\mathfrak{I}_{\mathcal{N}}}(X)$ = {$\mathcal{H}$, $\mathcal{L}_{\mathcal{R}}$ ($\boldsymbol{X}$), }.$\mathcal{B}_{\mathcal{R}}$ ($\boldsymbol{X}$)

Definition 2.3. [8] A ξ -nano-open set is a subset J of $\mathcal{H}$ and there is a non empty n-open set Z of for $\mathfrak{I}_{\mathcal{N}}$ which

1. Z ≠ ∅, $\mathcal{H}$
2. J ⊆ N_{int} (J) ∪ Z.

A ξ -nano topological space is referred as a collection every ξ-open set, including ∅ and $\mathcal{H}$**,** that satisfies the topological definitions and it is denoted as $(\mathcal{H}, \mathfrak{I}, \xi)$ or simply $(\mathcal{H}, \mathfrak{I}_{\xi})$

In $(\mathcal{H}, \mathfrak{I}_{\xi})$, the elements in $\mathfrak{I}_{\xi}$ are called ξ - nano-opensets (ξ - open set) and whose complements are ξ-nano -closedset (ξ - closed).

Remark 2.2. In nano-topology $(\mathcal{H}, \mathfrak{I}_{\mathcal{N}})$, it is clear that each n-open set of $\mathfrak{I}_{\xi}$ open. Therefore, -ξ is $\mathfrak{I}_{\mathcal{N}}$ $\mathfrak{I}_{\xi} \subseteq \mathfrak{I}_{\mathcal{N}}$. That is $\mathfrak{I}_{\mathcal{N}}$ is finer than

Definition 2.4. [8] Let $(\mathcal{H}, \mathfrak{I}_{\xi})$ is ξ-nano space, H ⊆ $\mathcal{H}$ then

1. The ξ-nano-interior of a subset H ⊆ $\mathcal{H}$ is largest ξ-open set inside $\mathcal{H}$ and referred as ξ_{int} (H).
2. The ξ-nano-closure of a subset H ⊆ $\mathcal{H}$ is the smallest ξ-closed sets including and referred as ξ_{cl} (H).
3. The ξ-nano-exterior of H is indicated by ξ_E (H) = ξ_{int} ($\mathcal{H}$ − H).
4. The ξ-nano-frontier of H is represented by ξ_F (H) = ξ_{cl} (H) − ξ_{cl} ($\mathcal{H}$ − H).

Remark 2.3. The basis for $\mathfrak{I}_{\xi}$ is represented by $\mathcal{B}_{\mathfrak{I}_{\xi}}$ = { $\mathcal{H}$, $\mathcal{L}_{\mathcal{R}}$ (**X**), . $\mathcal{B}_{\mathfrak{I}_{\mathcal{N}}}$ }= $\mathcal{B}_{\mathcal{R}}$ (**X**)

3 ξ-NANO IDEAL TOPOLOGICAL SPACE

Definition 3.1. *L*et $(\mathcal{H}, \mathfrak{I}_{\xi})$ be a ξ-nano topological space having an ideal I ⊆ $\mathcal{H}$ is referred as ξ-nano Ideal topological space $(\mathcal{H}, \mathfrak{I}_{\xi}, I)$. Simply we write ξI -topological space or ξI - space .

Definition 3.2. Let $(\mathcal{H}, \mathfrak{I}_{\xi})$ -be a ξ- nano-topological space and (.)∗ known to be a kuratowski set operator from $\mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ is a power set of $\mathcal{H}$. For H ⊆ $\mathcal{H}$,

$\mathcal{H}^*_{\xi}(I, \mathfrak{I}_{\xi})$ = {x ∈ $\mathcal{H}$: V∩ $\mathcal{H}$ ∉ **I**, ∀ V ∈ ξ(x), where ξ(x) is the set of all open neighbourhood of x }. is denoted as ξ -local function of $\mathcal{H}$ modulo an ideal I and $\Im_{\xi}$. We shortly express as $\mathcal{H}_{\xi}^{*}$ for $\mathcal{H}_{\xi}^{*}(I,\Im_{\xi})$

Definition 3.3. The ξ*-closure of H ⊆ $\mathcal{H}$, define ξ − cl∗ (H) = H ∪ H_{ξ}^{*}. The ξ*-closure will produces a new topology called ξ - nano * topology given by

$\Im_{\xi}^{*}(\mathcal{I},\Im_{\xi})$ = {W ⊆ $\mathcal{H}$: ξ − cl∗ ($\mathcal{H}$ − W) = $\mathcal{H}$ − W }

and simply we write $\Im_{\xi}^{*}$ for $\Im_{\xi}^{*}(\mathcal{I},\Im_{\xi})$ and $\Im_{\xi}^{*}$ is finer than $\Im_{\xi}$.

Remark 3.1. If $(\mathcal{H},\Im_{\xi},I)$ is ξ***I*** -space, H ⊆ $\mathcal{H}$. Then,

1. The members of $\Im_{\xi}^{*}$ are termed as ξ - nano *- opensets (ξ*-openset) and the complements of ξ*-openset are called ξ - nano * - closed set (ξ *- closed).

2. The ξ-nano interior (respectively ξ-nano closure) of a subset H ⊆ $\mathcal{H}$ in $\Im_{\xi}^{*}$ are termed as $\xi_{\text{int}}^{*}(H)$ (respectively $\xi_{cl}^{*}(H)$

3. If I = ∅ then $\mathcal{H}_{\xi}^{*}$ = $\xi_{cl}(H)$ = $\xi_{cl}^{*}(H)$ and $\Im_{\xi}^{*}$ =.if I = $\mathcal{P}(\mathcal{H})$∅

Theorem 3.1. Let $(\mathcal{H},\Im_{\xi},I)$ is ξ***I***-space include an ideal I and H,S are subset of $\mathcal{H}$.
Then,

1. H ⊆ S ⇒ $H_{\xi}^{*} \subseteq \mathcal{S}_{\xi}^{*}$.
2. I ⊆ J ⇒ H_{ξ}^{*} (J) ⊆ $\mathcal{S}_{\xi}^{*}$ (I).
3. $H_{\xi}^{*} = \xi_{cl}(H_{\xi}^{*}) \subseteq \xi_{cl}(H)$.
4. $(H_{\xi}^{*})^{*}_{\xi} \subseteq H_{\xi}^{*}$.
5. $H_{\xi}^{*} \cup S_{\xi}^{*} = (H \cup S)^{*}_{\xi}$.
6. $H_{\xi}^{*} - S_{\xi}^{*} = (H - S)^{*}_{\xi} - S_{\xi}^{*} \subseteq (H - S)^{*}_{\xi}$.
7. V ∈ .$^{*}_{\xi}$ ⊆ (V ∩ H)$^{*}_{\xi}$ = V∩ (V ∩ H) H_{ξ}^{*} ⊆ $_{\xi}$ ⇒ Vב
8. $\mathcal{I}$ ∈ I ⇒ $(H \cup \mathcal{I})^{*}_{\xi} = H^{*}_{\xi} = (H - \mathcal{I})^{*}_{\xi}$.

Proof.

1. Let H ⊆ S and x ∈ H_{ξ}^{*} . Suppose that x ∉ S^{*}_{ξ} , then W ∩ S ∈ I, for some W ∈ ξ(x). Since W ∩ H ⊆ W ∩ S ∈ I implies W ∩ H ∈ I, for some W ∈ ξ(x). We have x ∉ H^{*}_{ξ} , which is a contradiction to x ∈ H^{*}_{ξ} . Therefore $H^{*}_{\xi} \subseteq S^{*}_{\xi}$, if H ⊆ S.

2. If I ⊆ J and x ∈ H^{*}_{ξ} (J), then W ∩ H ∉ J, ∀W ∈ ξ(x). Then by hypothesis W ∩ H ∉ I ⊆ J, ∀ W ∈ ξ(x) and so x ∈ H^{*}_{ξ} (I). Thus H^{*}_{ξ} (J) ⊆ S^{*}_{ξ} (I).

3. Let x ∈ H^{*}_{ξ} . Then ∀ W ∈ ξ(x), W ∩ H ∉ I. This implies that W ∩ H = ∅. Hence x ∈ ξ_{cl} (H).

4. $H^{*}_{\xi} \subseteq \xi_{cl}(H)$. Then $(H^{*}_{\xi})^{*}_{\xi} \subseteq \xi_{cl}(H^{*}_{\xi}) = H^{*}_{\xi}$ (since, H^{*}_{ξ} is a ξ-closed set of ξ_{cl} (H)).

5. Let x ∉ $H^{*}_{\xi} \cup S^{*}_{\xi}$. Then x ∉ H^{*}_{ξ} and x ∉ S^{*}_{ξ} . Then W ∩ H ∈ I, some W ∈ ξ(x) and W ∩ S ∈ I, some W ∈ ξ(x) and so (W ∩ H) ∪ (W ∩ S) ∈ I, some W ∈ ξ(x), we have (W ∩ (H ∪ S)) ∈ I, for some W ∈ ξ(x). Hence, x ∉ $(H \cup S)^{*}_{\xi}$.

On the other side, Suppose that x ∉ $(H \cup S)^{*}_{\xi}$. Then for some W ∈ ξ(x), W ∩(H ∪ S) ∈ I and so (W ∩ H) ∪

(W ∩ S) ∈ I, for some W ∈ ξ(x). This implies that W ∩ H ∈ I or W ∩ S ∈ I, for some W ∈ ξ(x). We have x ∉ H^*_ξ or x ∉S^*_ξ . Therefore x ∉ H^*_ξ ∪ S^*_ξ .
The evidence for the remaining conditions is equally clear.

Remark 3.2. The reverse implications of (1), (2), and (3) in Theorem 3.1 may not always hold true, as seen in the below Example 3.1.

Example 3.1. Let $\mathcal{H}$ = {l_α , l_β , l_γ , l_δ , l_ν } be the universe and ***X*** = { l_α , l_β , l_γ } ⊆ $\mathcal{H}$ with

$\mathcal{H}$ /$\mathcal{R}$ = {{ l_α }, {l_β }, {l_γ , l_δ , l_ν }}. Then }. β, l α}}. Let ξ = { l ν, l δ, l γ}, {l β, l α, ∅, { l$\mathcal{H}$= { $\Im_N$

Then = {∅}. $\mathcal{I}$}} with ν, l δ, l γ, l β}, {l ν, l δ, l γ, l α}, { l ν, l δ, l γ}, {l β}, {l α, { l$\mathcal{H}$= {{∅}, $\Im_\xi$

(1) For U = { l_α , l_γ } and V = { l_α , l_ν }. Then U^*_ξ = { l_α } and V^*_ξ = { l_α , lγ , l_δ , lν } and so

U^*_ξ ⊆ V^*_ξ but U ⊈V.
(2) For U = { l_α , l_ν } with an ideals $\mathcal{I}$ = {∅, {l_γ }} and $\mathcal{I}'$= {∅, {l_ν }}. It is clear that

U^*_ξ ($\mathcal{I}'$) ⊆ U^*_ξ ($\mathcal{I}$) but $\mathcal{I}$ ⊈ $\mathcal{I}'$

(3) For U = {l_α , l_γ } and I = {∅, {l_γ }}. Then ξ_{cl} (U) = {l_α , l_γ , l_δ , l_ν } and U^*_ξ = {l_α }

and so ξ_{cl} (U^*_ξ) = {l_α }. Hence, ξ_{cl} (U) ⊈ U^*_ξ ⊆ ξ_{cl} (U^*_ξ).

Theorem 3.2. Let $(\mathcal{H}, \Im_\xi, I)$ is an ξ***I***-space along an ideal $\mathcal{I}$ on $\mathcal{H}$, U ⊆ U^*_ξ . Then U^*_ξ = ξ_{cl} (U^*_ξ) = ξ_{cl}(U).

Proof. For any U ⊆ $\mathcal{H}$, we have by Theorem 3.1 (3) U ⊆ U^*_ξ implies that ξ_{cl} (U) ⊆ ξ_{cl} (U^*_ξ) and so U^*_ξ = ξ_{cl} (U^*_ξ) = ξ_{cl} (U).

Theorem 3.3. The set operator ξ − cl∗ possesses the subsequent characteristics:
1. U ⊆ ξ – cl* (U)
2. ξ – cl* (∅) = ∅ and ξ – cl* (H) = H
3. If U ⊆ V , then ξ – cl* (U) ⊆ ξ – cl* (V)
4. ξ – cl* (U) ∪ ξ – cl* (V) = ξ – cl* (U ∪ V)
5. ξ – cl* (ξ – cl* (U)) = ξ – cl* (U)

Proof. The evidence for the Theorem is apparent from Theorem 3.1 and the definition of ξ − cl∗

Remark 3.3.

1. If I = ∅, then U^*_ξ = ξ_{cl} (U) but, in this case ξ^*_{cl} (U) = ξ_{cl} (U).
2. If $(\mathcal{H}, \Im_\xi, I)$ is an ξ***I***- space including an ideal $\mathcal{I}$= ∅, then $\Im^*_\xi = \Im_\xi$

Here, we rise a question that, given a ξ-nano ideal topological space $(\mathcal{H}, \Im_\xi, I)$, we can form $[\Im^*_\xi(I)]^* = \Im^{**}_\xi$ and it is clear that $\Im^{**}_\xi$ is finer than $\Im^*_\xi$ but, it is necessary that $\Im^{**}_\xi$ is strictly finer than $\Im^*_\xi$?
The answer of the above question is the negative which is given in the following Theorem 3.4. But we notice that I ∩ $\mathcal{J}$ and I ∨ $\mathcal{J}$ = {I ∪ $\mathcal{J}$: I ∈ $\mathcal{I}$ and J ∈ $\mathcal{J}$} are becomes an ideal on $\mathcal{H}$ provided $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\mathcal{H}$.

Theorem 3.4. Let $(\mathcal{H}, \Im_\xi, I)$ is an ξ*I*-space along an ideals $\mathcal{I}$ and $\mathcal{J}$ on H and U ⊆ H.
Then,
1. U^*_ξ ($\mathcal{I}$∩$\mathcal{J}$) = U^*_ξ ($\mathcal{I}$) ∪ U^*_ξ ($\mathcal{J}$)
2. U^*_ξ ($\mathcal{I}$∨$\mathcal{J}$,)). $\mathcal{I}$($\Im^*_\xi$, $\mathcal{J}$($^*_\xi$)) ∩ U $\mathcal{J}$($\Im^*_\xi$ (I, $^*_\xi$ (I)) = U $\Im_\xi$

Proof.

1. Let x∉ U^{*}_{ξ} ($\mathcal{I}$∩$\mathcal{J}$), then W ∩ U ∈ I ∩ J , for some W ∈ ξ(x) such that W ∩ U ∈ I, for some W ∈ ξ(x) and W ∩ U ∈ J , for some W ∈ ξ(x).

This implies that x ∉ U^{*}_{ξ} ($\mathcal{I}$) and x ∉ U^{*}_{ξ} ($\mathcal{J}$) and so x ∉ U^{*}_{ξ} ($\mathcal{I}$) ∪ U^{*}_{ξ} ($\mathcal{J}$). Hence U^{*}_{ξ} ($\mathcal{I}$) ∪ U^{*}_{ξ} ($\mathcal{J}$) ⊆ U^{*}_{ξ} ($\mathcal{I}$∩$\mathcal{J}$).

On the contrary, we have U^{*}_{ξ} ($\mathcal{I}$∩$\mathcal{J}$) ⊆ U^{*}_{ξ} ($\mathcal{I}$) ∪ U^{*}_{ξ} ($\mathcal{J}$) and (1) follows.

2. Let x ∉ U^{*}_{ξ} ($\mathcal{I}$∨$\mathcal{J}$, such that W $\mathcal{J}$ ∈ Jand $\mathcal{I}$ ∈ ξ(x). Let I ∈ , for some W $\mathcal{J}$∨$\mathcal{I}$ ∈ (I))Then W ∩ U $\mathfrak{I}_{\xi}$. Then we have (W ∩U)−I = J and (W ϕ = U = I ∪ J. Because of Heredity of $\mathcal{I}$, it is possible to infer that I ∩$\mathcal{J}$∩ . Similarly, (W −J)∩U = (W ∩U)−(J ∩U) $\mathcal{I}$ ∈ U)−J = I and so (W −I)∩U =(W ∩U)−(I ∩U) = (W ∩U)−I = J∩ .$\mathcal{I}$ ∈ (W ∩U)−J = I=

This implies that x ∉ U^{*}_{ξ} (I, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{J}$))or x ∉ U^{*}_{ξ} ($\mathcal{J}$, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{I}$)) (ie., x ∈ I or x ∈ J but not both). Hence, x ∉ U^{*}_{ξ} (I, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{J}$)) ∩ U^{*}_{ξ} ($\mathcal{J}$, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{I}$)). We have shown U^{*}_{ξ} (I, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{J}$)) ∩ U^{*}_{ξ} ($\mathcal{J}$, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{I}$)).⊆ U^{*}_{ξ} ($\mathcal{I}$∨$\mathcal{J}$,). $\mathfrak{I}_{\xi}$

On the contrary, let x ∉ U^{*}_{ξ} (I, $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{J}$)).Then there is some W ∈ ξ(x) and J ∈ $\mathcal{J}$ such that (W − J) ∩ U ∈ $\mathcal{I}$. it is possible to that, due to heredity of $\mathcal{J}$ that, J ⊂ U . Now, define I = (W − J) ∩ U and we have W ∩ U = I ∪ J ∈ $\mathcal{I}$∨$\mathcal{J}$. This implies that x ∉ U^{*}_{ξ} ($\mathcal{I}$∨$\mathcal{J}$, imilarly, we)). S $\mathcal{J}$($\mathfrak{I}^{*}_{\xi}$ (I, $^{*}_{\xi}$)⊆ U $\mathfrak{I}_{\xi}$, $\mathcal{J}$∨$\mathcal{I}$($^{*}_{\xi}$) and so U $\mathfrak{I}_{\xi}$ have U^{*}_{ξ} ($\mathcal{I}$∨$\mathcal{J}$, $\mathfrak{I}_{\xi}$)) $\mathcal{I}$($\mathfrak{I}^{*}_{\xi}$, $\mathcal{J}$($^{*}_{\xi}$ U ⊆)

Hence we establish the result.

Corollary 3.1. Let $(\mathcal{H}, \mathfrak{I}_{\xi}, I)$ is an ξI - space modulo an ideal $\mathcal{I}$ on $\mathcal{H}$. Then U^{*}_{ξ} ($\mathcal{I}$, .) = $\mathfrak{I}_{\xi}$ U^{*}_{ξ} ($\mathcal{I}$, . $\mathfrak{I}^{**}_{\xi}$ = $\mathfrak{I}^{*}_{\xi}$.) and hence $\mathfrak{I}^{*}_{\xi}$

Proof. By taking $\mathcal{I}$=$\mathcal{J}$ in Theorem.3.4 (2), we have

U^{*}_{ξ} ($\mathcal{I}$∨ I ,)). $\mathcal{I}$($\mathfrak{I}^{*}_{\xi}$, $\mathcal{I}$($^{*}_{\xi}$)) ∩ U$\mathcal{I}$($\mathfrak{I}^{*}_{\xi}$ (I, $^{*}_{\xi}$)= U $\mathfrak{I}_{\xi}$

U^{*}_{ξ} ($\mathcal{I}$,)). $\mathcal{I}$($\mathfrak{I}^{*}_{\xi}$, $\mathcal{I}$($^{*}_{\xi}$)= U $\mathfrak{I}_{\xi}$

This implies $\mathfrak{I}_{\xi}$ = $\mathfrak{I}^{*}_{\xi}$ and hence $\mathfrak{I}^{*}_{\xi}$ = $\mathfrak{I}^{**}_{\xi}$.

The Theorem 3.3 above provides a straightforward explanation of the following outcome.

Theorem 3.5. Let $(\mathcal{H}, \mathfrak{I}_{\xi})$ is an ξ***I***-space modulo an ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathcal{H}$,

1. $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{I}$∨$\mathcal{J}$) = [$\mathfrak{I}^{*}_{\xi}$ ($\mathcal{J}$)]* ($\mathcal{I}$) = [) $\mathcal{J}$(*)]$\mathcal{I}$($\mathfrak{I}^{*}_{\xi}$
2. $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{I}$∩$\mathcal{J}$) = $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{I}$) ∩ $\mathfrak{I}^{*}_{\xi}$ ($\mathcal{J}$).

Proof. (1) Theorem 3.3 (2) and (2) leads to (1).

Theorem 3.6. Let $(\mathcal{H}, \mathfrak{I}_{\xi}, I)$ is a ξI - space modulo an ideal $\mathcal{I}$ on $\mathcal{H}$, U ⊆ $\mathcal{H}$. If U ⊆ U^{*}_{ξ} ,

1. ξ_{cl} (U) = ξ^{*}_{cl} (U)
2. ξ_{int} ($\mathcal{H}$− U) = ξ^{*}_{int} ($\mathcal{H}$ − U).

Proof. 1. From theorem 3.2, we conclude that

$U^{*}_{cl} = \xi_{cl}(U^{*}_{cl}) = \xi_{cl}(U)$

$U \cup U^{*}_{cl} = U \cup \xi_{cl}(U)$ and so $\xi^{*}_{cl}(U) = \xi_{cl}(U)$.

2. If $U \subseteq U^{*}_{\xi}$, then by (1) $\xi^{*}_{cl}(U) = \xi_{cl}(U)$

$\mathcal{H} - \xi^{*}_{cl}(U) = \mathcal{H} - \xi_{cl}(U)$ and so $\xi^{*}_{int}(\mathcal{H} - U) = \xi_{int}(\mathcal{H} - U)$.

Definition 3.4. In a ξI- space $(\mathcal{H}, \Im_{\xi}, I)$, a subset U is ξ* - dense-in-itself (or ξ* - perfect, ξ *-closed) if U satisfies $U \subseteq U^{*}_{\xi}$ (or $U = U^{*}_{\xi}$, $U^{*}_{\xi} \subseteq U$).
We have an implication arrow-diagram shows the interlink between the sets are defined above.
ξ* – dense in itself ⇐ ξ* – perfect ⇒ ξ* – closed.
The converse implications of Theorem 3.1, specifically (1), (2), and (3), may not always hold true, which is demonstrated in the given Example 3.2.

Example 3.2. If $\mathcal{H} = \{m_\alpha, m_\beta, m_\gamma, m_\delta\}$ be the universe and $\mathcal{X} = \{m_\alpha, m_\beta\}$ with $\mathcal{H}/\mathcal{R} = \{\{m_\alpha\}, \{m_\gamma\}, \{m_\beta, m_\delta\}\}$. Then $\Im_N = \{\mathcal{H}, \phi, \{m_\alpha\}, \{m_\beta, m_\delta\}, \{m_\alpha, m_\beta, m_\delta\}\}$.

Let $\xi = \{m_\beta, m_\delta\}$. Then $\Im_\xi = \{\mathcal{H}, \phi, \{m_\alpha\}, \{m_\beta\}, \{m_\delta\}, \{m_\alpha, m_\beta\}, \{m_\alpha, m_\delta\}, \{m_\beta, m_\delta\}, \{m_\alpha, m_\beta, m_\delta\}\}$ and $\mathcal{I} = \{\phi, \{m_\alpha\}\}$

1. For $U = \{m_\beta, m_\delta\}$, $U^{*}_{\xi} = \{m_\beta, m_\gamma, m_\delta\}$, then $U \subseteq U^{*}_{\xi}$ and so U is ξ* -dense in itself but not ξ* -perfect.

2. For $U = \{m_\alpha, m_\gamma\}$, then $U^{*}_{\xi} = \{m_\gamma\}$ and so $U^{*}_{\xi} \subseteq U$. Therefore, U is closed under ξ* but not perfect under ξ*.

Lemma 3.1. Let $(\mathcal{H}, \Im_{\xi}, I)$ is an ξ***I***- space, $U \subseteq \mathcal{H}$. Then $U^{*}_{\xi} = \xi_{cl}(U^{*}_{\xi}) = \xi_{cl}(U) = \xi^{*}_{cl}(U)$ provided U is ξ* - dense-in-itself set.

Proof. Let U is ξ* - dense-in-itself and $U \subseteq U^{*}_{\xi}$. Then by Theorem 3.6, we have $U^{*}_{\xi} = \xi_{cl}(U^{*}_{\xi}) = \xi_{cl}(U) = \xi^{*}_{cl}(U)$

Definition 3.5. [7] An ideal I in an ξI - space $(\mathcal{H}, \Im_{\xi}, I)$ is referred as }.ϕ∩ I = { $\Im_\xi$ boundary if - $\Im_\xi$

Theorem 3.7. If $(\mathcal{H}, \Im_{\xi}, I)$ is an ξI - space and I is . $^{*}_{\xi}\ \mathcal{H} = \mathcal{H}$boundary ideal. Then - $\Im_\xi$

Proof. It is true that, $\mathcal{H}^{*}_{\xi} \subseteq \mathcal{H}$. On the other way, consider if $x \in$ H but $x \notin \mathcal{H}^{*}_{\xi}$, then $W \cap \mathcal{H} \in \mathcal{I}$ and so $W \in \mathcal{I}$ and contradicts to the that $\Im_\xi \cap \mathcal{I} = \{\phi\}$. Therefore, $\mathcal{H} = \mathcal{H}^{*}_{\xi}$.

Theorem 3.8. Let $(\mathcal{H}, \Im_{\xi}, I)$ is an ξI -space. The followings are equivalent.

1. $\mathcal{H} = \mathcal{H}^{*}_{\xi}$
2. $\mathcal{I}$ is ξ-boundary ideal
3. If $\boldsymbol{I} \in \mathcal{I}$, then $\xi_{int}(I) = \phi$
4. $\forall$ U $\in$. $^{*}_{\xi}$ U $\subseteq$, U $\Im_\xi$

4 ξI -OPEN AND ξI -CLOSED SETS

Definition 4.1. A subset U of a ξI -space $(\mathcal{H}, \Im_{\xi}, I)$ including an ideal $\mathcal{I}$ on $\boldsymbol{\mathcal{H}}$ is referred to be ξ-nano $\mathcal{I}$-openset (in-short ξI-open set),if $U \subseteq \xi_{int}(U^{*}_{\xi})$. We denote $\xi IO(\mathcal{H}, \Im_\xi) = \{U \subseteq \mathcal{H} : U \subseteq \xi_{int}(U^{*}_{\xi})\}$ or simply we write ξI O($\mathcal{H}$) for ξ I O($\mathcal{H}$, $\Im_\xi$) to avoid confusion.

ξ-opensets and ξI-opensets with respect to an ideal in ξI-space are different from one another and is demonstrated from the following Example.

Example 4.1. Let $\mathcal{H}$ = {m_α , m_β , m_γ , m_δ , m_ν } be the universe and $\mathcal{X}$ = {m_α , m_β , m_γ }

with $\mathcal{H}/\mathcal{R}$ = {{m_α }, {m_β }, {m_γ , m_δ , m_ν }}. Then }. β, m α}}. Let ξ = {m ν, m δ, m γ}, {m β, m α, {mϕ, $\mathcal{H}$= { $\Im_N$

$\mathcal{I}$}} and take ν, m δ, m γ, m β}, {m ν, m δ, m γ, m α}, {m ν, m δ, m γ}, {m β, m α{m}, β}, {m α, {mϕ, $\mathcal{H}$= { $\Im_\xi$ Then }.ϕ, {m_α }= {

1. For U = {m_β , m_ν } ∈ ξ ***I*** O($\mathcal{H}$) but U∉ ξO(H), because U^{*}_{ξ} = {m_β , m_γ , m_δ , m_ν } and ξ_{int} (U^{*}_{ξ}) = {m_β , m_γ , m_δ , m_ν } and so U ⊆ ξ_{int} (U^{*}_{ξ}).

2. For U = {m_α , m_β }. Then U ∈ ξO($\mathcal{H}$) but U∉ ξIO($\mathcal{H}$) because U^{*}_{ξ} = {m_β } and ξ_{int} (U^{*}_{ξ})= {m_β } and so U⊈ ξ_{int} (U^{*}_{ξ}).

Theorem 4.1. Any arbitrary union of ξ I -opensets is also ξ I -open.

Proof. If $(\mathcal{H},\Im_\xi,I)$ is an ξ I -space and W_α ∈ ξ I O($\mathcal{H}$,), for α ∈ Δ, where Δ is an indexed set. That is, $\Im_\xi$

) and so ξ) α((W intξ ⊆ αΔ, W ∈ for each α

$$\bigcup_{\alpha\in\Delta} W_\alpha \subseteq \bigcup_{\alpha\in\Delta} (\xi_{int}((W\alpha)^{*}_{\xi}) \subseteq \xi_{int}\left(\bigcup_{\alpha\in\Delta}(W_\alpha)^{*}_{\xi}\right) = \xi_{int}\left(\left(\bigcup_{\alpha\in\Delta} w_\alpha\right)^{*}_{\xi}\right)$$

Hence, $\bigcup_{\alpha\in\Delta} W_\alpha$ ∈ ξ I O($\mathcal{H}$).

Finite Intersection of ξ I -opensets not necessarily ξ I -open, as illustrated by the Example 4.2.

Example 4.2. From example 4.1,

$\Im_\xi$ = {$\mathcal{H}$, ϕ, {m_α }, {m_β }, {m_α , m_β }, {m_γ , m_δ , m_ν }, {m_α , m_γ , m_δ , m_ν }, {m_β , m_γ , m_δ , m_ν }} and an ideal I = {{ ϕ }, {m_γ }}.

For U = {m_β , m_γ , m_δ }, U^{*}_{ξ} = {m_β , m_γ , m_δ , m_ν } andso U ∈ ξIO($\mathcal{H}$).

For V = {m_β, m_γ, m_ν }, V^{*}_{ξ} = {m_β , m_γ , m_δ , m_ν } and so V ∈ ξIO($\mathcal{H}$).

But, U ∩ V = {m_β , m_γ }, $(U \cap V)^{*}_{\xi}$ = {m_β }, ξ_{int} ($(U \cap V)^{*}_{\xi}$) = {m_β } and so U ∩ V ⊈ξ_{int} ($(U \cap V)^{*}_{\xi}$). Hence, U ∩ V ∉ ξIO($\mathcal{H}$).

Theorem 4.2. Let $(\mathcal{H},\Im_\xi,I)$ be a ξI-space, U ⊆ $\mathcal{H}$.

1. If $\mathcal{I}$ = {ϕ}, then U^{*}_{ξ} = ξ_{cl}(U)

2. If $\mathcal{I}$ = $\mathcal{P}(\mathcal{H})$, then U^{*}_{ξ} = ϕ and hence U is ξ I -openset if and only if U = ϕ.

Theorem 4.3. For an ξ I -openset U of a ξ I -space $(\mathcal{H},\Im_\xi,I)$, we have U^{*}_{ξ} = $(\xi_{int}(U^{*}_{\xi}))^{*}_{\xi}$.

Definition 4.2. Any subset F of a ξ I -space $(\mathcal{H},\Im_\xi,I)$ is referred as ξ-nano ideal - closed (in short ξ ***I*** - closedset) if its complement is ξ I -openset.

Remark 4.1. If U ⊆ $(\mathcal{H},\Im_\xi,I)$,we have $\{(\xi_{int}(U^{*}_{\xi}))\}^c \neq \xi_{int}((U^c)^{*}_{\xi})$ in general (seen from the following Example 4.3), where U^c is a complement of U .

Example 4.3. From example 4.1, U = {m_γ, m_δ , m_ν }, $(\xi_{int}(U))^{*\xi}$ = {m_γ , m_δ , m_ν } and so $\{(\xi_{int}(U))^{*\xi}\}^c$ = {m_α , m_β }. Also $(U^c)^{*\xi}$ = {m_β } and so $\xi_{int}((U^c)^{*\xi})$ = {m_β }.

Theorem 4.4. If U ⊆ $(\mathcal{H}, \Im_\xi, I)$ is ξ I -closed, then U ⊃ $(\xi int(U^c))^{*\xi}$.

Proof. It is evident from the definition of ξI-closed and theorem 3.4.

Theorem 4.5. Let U ⊆ $(\mathcal{H}, \Im_\xi, I)$ is ξ I -closed and $\mathcal{H}$ − $(\xi_{int}(U))^{*\xi}$ = $(\xi int(\mathcal{H} - U))^{*\xi}$ .Then U is ξ ***I*** -closed iff U ⊃ $(\xi_{int}(U))^{*\xi}$.

Theorem 4.6. $(\mathcal{H}, \Im_\xi, I)$ is an ξI-space and U, V ⊆ $\mathcal{H}$. Then

1. If U ∈ ξ ***I*** O($\mathcal{H}$),V ∈ ξ O($\mathcal{H}$).then U ∩ V ∈ ξ ***I*** O($\mathcal{H}$).

2. If U ∈ ξ ***I*** O($\mathcal{H}$).and V ∈ ξ O($\mathcal{H}$), then U ∩ V ⊂ $\xi_{int}(V \cap (V \cap U)^{*\xi})$.

Proof. 1. U ⊆ $\xi_{int}(U^{*\xi})$ and so U ∩ V ⊂ $\xi_{int}(U^{*\xi}) \cap V = \xi_{int}(U^{*\xi} \cap V)$, from Theorem 3.4 (7), we have U ∩ V ⊂ $\xi_{int}((U \cap V)^{*\xi})$.

2. It derives straightforwardly from Theorem 3.4 (7).

Corollary 4.1. Any ξ I -closedset union with ξ-closedset is also ξ I -closedsets.

Theorem 4.7. If $(\mathcal{H}, \Im_\xi, I)$ is an ξ***I*** -space and U ∈ ξO($\mathcal{H}$) and V ∈ ξ ***I*** O($\mathcal{H}$), there is an ξ-openset W of $\mathcal{H}$ suchthat U ∩ W = ϕ implies U ∩ V = ϕ.

Proof. Let V ∈ ξ I O($\mathcal{H}$), then V ⊆ $\xi_{int}(V^{*\xi})$, by taking W = $\xi_{int}(V^{*\xi})$ as an ξ-open set such that V ⊂ W but, U ∩ W = ϕ, then W ⊆ $\mathcal{H}$ − U implies $\xi_{cl}(W)$ ⊆ ($\mathcal{H}$ − U). Hence V ⊆ $\mathcal{H}$− U and this completes the proof.

5 ξ-NANO IDEAL CONTINUOUS FUNCTION

Definition 5.1. A function $\varphi : (\mathcal{H}, \Im_\xi, I) \rightarrow (S, \Im^l_\xi)$ is known as ξ-nano-ideal continuous (in-short ξ***I*** continuous) if for every W ∈ ℶ′ξ , φ−1 (W) ∈ ξ ***I*** O($\mathcal{H}$).

The Examples 5.1 and 5.2 below demonstrate that the notions of ξ continuity and ξ I continuity are not related.

Example 5.1. Let $\mathcal{H}$ = {l_α , l_β , l_γ , l_δ , l_ν } be the universe and $\mathcal{X}$= {l_α , l_β , l_γ } ⊆ $\mathcal{H}$ with $\mathcal{H}/\mathcal{R}$ = {{l_α}, {l_β }, {l_γ , l_δ , l_ν }}. Then $\Im_N$ = {$\mathcal{H}$, ϕ {l_α , l_β }, {l_γ, l_δ , l_ν }}. Let ξ ={l_α , l_β}. Then $\Im_\xi$ = {$\mathcal{H}$, {ϕ} {l_α }, {l_β }, {l_γ , l_δ, l_ν }, {l_α , l_γ , l_δ, l_ν }, {l_β , l_γ , l_δ, l_ν }} with $\mathcal{I}$={ ϕ, {m_α }}. Let S = {m_α , m_β , m_γ , m_δ } be the universe and Y = {m_α , m_β } ⊆ **S** with **S**/$\mathcal{R}$ = {{m_α}, {m_γ}, {m_β , m_δ }}. Then $\Im_N$ = {S, ϕ, {m_α }, {m_β , m_δ }, {m_α, m_β, m_δ}}. Let ξ = {m_α , m_β, m_δ }. Then $\Im^l_\xi$ = { ϕ, S, {m_α}, {m_β }, {m_δ }, {m_α , m_β}, {m_α , m_δ }, {m_β , m_δ}, {m_α , m_β , m_δ }}.

Define $\varphi : (\mathcal{H}, \Im_\xi) \rightarrow (S, \Im^l_\xi)$ as $\varphi(l_\alpha) = m_\gamma$, $\varphi(l_\beta) = m_\alpha$, $\varphi(l_\gamma) = m_\beta$ and $\varphi(l_\delta) = \varphi(l_\nu) = m_\delta$. Then φ is ξI -continuous but not ξ- continuous because {m_α, m_β } ∈ ξO($\mathcal{H}$) but φ^{-1} (m_α , m_β) = {l_β , l_γ} ∉ ξO($\mathcal{H}$).

Example 5.2. By Example 4.1, $\mathcal{H}$ = {m_α , m_β , m_γ , m_δ , m_ν} be the universe and

$\Im_\xi$ = {$\mathcal{H}$, ϕ, {m_α }, {m_β }, {m_α , m_β }, {m_γ , m_δ , m_ν }, {m_α , m_γ , m_δ , m_ν }, {m_β , m_γ , m_δ , m_ν}}

and an ideal $\mathcal{I}$ = {{ ϕ }, {m_α }}.

Define φ :$\mathcal{H}$→ $\mathcal{H}$, an identity function. Clearly, φ is ξ -continuity whereas ξI- continuity as {m_α , m_β } ∈ ξO($\mathcal{H}$) an φ^{-1} {m_α, m_β } ∉ ξIO($\mathcal{H}$) because $\{m_\alpha, m_\beta\}^{*\xi}$ = {m_β } and so {m_α, m_β }⊈ $\xi_{int}\{m_\alpha, m_\beta\}^{*\xi}$ = {m_β }.

Theorem 5.1. For a function $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi)$, the statements are equivalent.
1. φ is ξI-continuity.
2. Let x $\in \mathcal{H}$ and every V ∈ $\Im^l_\xi$ having φ (x), there is an U $\in$ ξ I O($\mathcal{H}$),containing x, suchthat φ (U) ⊂ V
3. Let x ∈ $\mathcal{H}$, V ∈ $\Im^l_\xi$ having φ (x) implies $(\varphi^{-1}(V))^{*_\xi}$ is an neighborhood for x.

Proof.

(1) ⇒ (2) Since, V ∈ $\Im^l_\xi$ containing φ (x), then by (1) $\varphi^{-1}(V) \in$ ξ ***I*** O($\mathcal{H}$). By taking U = $\varphi^{-1}(V)$ which containing x. Therefore φ (U) ⊂ V .

(2) ⇒ (3) Since, V ∈ $\Im^l_\xi$ containing φ (x), then by (2), there exists
U $\in$ ξ ***I*** O($\mathcal{H}$),containing x suchthat φ (U) ⊂ V . So, x $\in$ U ⊂ $\xi_{int}(U^{*_\xi}) \subset$
$\xi_{int}((\varphi^{-1}(V))^{*_\xi}) \subseteq (\varphi^{-1}(V))^{*_\xi}$ Therefore, $(\varphi^{-1}(V))^{*_\xi}$ is an neighborhood for x.

(3) ⇒ (1) If V ∈ $\Im^l_\xi$ each y $\in$ V , φ^{-1} (y) ∈ φ^{-1}(V) ⊂ H. Then by (3), $(\varphi^{-1}(V))^{*_\xi}$ is a neighborhood of φ^{-1}(y) and so $\varphi^{-1}(V) \subseteq \xi_{int}((\varphi^{-1}(V))^{*_\xi})$. Hence, ϕ is ξI- continuous.

Theorem 5.2. For $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi)$,the followings are equivalent.
1. φ is ξ I -continuity.
2. The pre-image of each ξ-closedset in S is ξ I -closedset in $\mathcal{H}$
3. $(\xi_{int}(\varphi^{-1}(V)))^{*_\xi} \subset \varphi^{-1}(V^{*_\xi})$, for each ξ * - dense-in-itself subset V ⊂ $\mathcal{H}$.
4. $\varphi((\xi_{int}(U))^{*_\xi}) \subset (\varphi(U))^{*_\xi}$, for any subset U ⊂ $\mathcal{H}$,for each ξ * - perfect subset φ (U) in **S**.

Proof. (1) ⇒ (2) Let V ⊂ **S** be a ξ-closedset. Then **S** − V is ξ-openset ie., S − V ∈ $\Im^l_\xi$. By (1),
φ^{-1} (**S** − V) = φ^{-1} (**S**) − φ^{-1} (V) = $\mathcal{H}$− φ^{-1} (V)
is ξ I -open. Thus φ^{-1} (V) is ξ I -closed.

(2) ⇒ (3) Let V ⊂ S. Since, V^{*_ξ} is ξ-closed, then by (2) $\varphi^{-1}(V^{*_\xi})$ is ξI -closed. Then by **Theorem 4.4** $\varphi^{-1}(V^{*_\xi}) \supset (\xi_{int}(\varphi^{-1}(V^{*_\xi})))^{*_\xi}$. Since, V is ξ* - dense-in-itself, then
$\varphi^{-1}(V^{*_\xi}) \supset (\xi_{int}(\varphi^{-1}(V^{*_\xi})))^{*_\xi} \supset (\xi_{int}(\varphi^{-1}(V)))^{*_\xi}$.
(3) ⇒ (4) Let U ⊂ $\mathcal{H}$ and V = φ (U). Then by (3)
$\varphi^{-1}(V^{*_\xi}) \supset (\xi_{int}(\varphi^{-1}(V)))^{*_\xi} \supset (\xi_{int}(U))^{*_\xi}$.
Hence, $\varphi[(\xi_{int}(U))^{*_\xi}] \subset V^{*_\xi} = [\varphi(U)]^{*_\xi}$.

(4) ⇒ (1) Let V ∈(U) ⊂ W then by (4), φ (W), implies 1- φ closed and U = -– V is ξ **S**, W = $\Im^l_\xi$
we have $\varphi[(\xi_{int}(U))^{*_\xi}] \subset (\varphi(U))^{*_\xi} \subset W^{*_\xi} = W$. Thus $\varphi^{-1}(W) \supset (\xi_{int}(U))^{*_\xi} = (\xi_{int}(\varphi^{-1}(W)))^{*_\xi}$. Hence φ^{-1} (W) = φ^{-1} (**S**− V) is ξI-closedset and so φ^{-1} (V) is ξ I -openset in $\mathcal{H}$.Hence φ is ξ I -continuity.

Theorem 5.3. Let $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi)$, be a ξI-continuous function, H ∈ -|H is ξIϕThen $\Im_\xi$ .continuity where ϕ|H is a restriction on H

Proof. Let Z ∈ $\Im^l_\xi$. Then φ^{-1} (Z) ∈ ξ I O($\mathcal{H}$), $\varphi^{-1}(Z) \subseteq \xi_{int}(\varphi^{-1}(Z)^{*_\xi})$ and so H ∩ φ^{-1} (Z) ⊂ H ∩ $\xi_{int}(\varphi^{-1}(Z)^{*_\xi})$. Hence $(\phi|H)^{-1}(Z) \subset H \cap \xi_{int}(\varphi^{-1}(Z)^{*_\xi})$. As H ∈ $\Im_\xi$, then $(\phi|H)^{-1}(Z) = \xi_{int}(H \cap (\varphi^{-1}(Z))^{*_\xi}) \subset \xi_{int}$

$(H \cap \varphi^{-1}(Z))^{*\xi}$ = ξ_{int} $((\phi|H)^{-1}(Z))^{*\xi}$. We have, $\phi|H$ is ξ I -continuity.

Theorem 5.4. If $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi, \mathcal{J})$ is an function and $\{H_\alpha : \alpha \in \Delta\}$ be an open covering for $\mathcal{H}$. If $\varphi|H_\alpha$ is ξ I -continuity, ∀ α ∈ Δ, then φ is ξ I -continuity.

Theorem 5.5. Let $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi)$ be a ξI -continuous function and $\varphi^{-1}(V^{*\xi}) \subset (\varphi^{-1}(V))^{*\xi}$, ∀ V ⊂ **S**. Then pre-image of each ξ I -openset is ξ I -open.

The following Example 5.3 demonstrated that the composition of ξI continuity not necessarily ξ I -continuity.

Example 5.3. Let $\mathcal{H}$ = **S** = $\{m_\alpha, m_\beta, m_\gamma, m_\delta, m_\nu\}$ be the universe, $\mathcal{H}/\mathcal{R}$ = $\{\{m_\alpha\}, \{m_\beta\}, \{m_\gamma, m_\delta, m_\nu\}\}$ and $\mathcal{X}$ = $\{m_\alpha, m_\beta, m_\gamma\} \subseteq \mathcal{H}$ Then $\Im_N$ = $\{\phi, \mathcal{H}, \{m_\alpha, m_\beta\}, \{m_\gamma, m_\delta, m_\nu\}\}$ and let $\xi = \{m_\alpha, m_\beta\}$, $\Im_\xi$ = $\{\phi, \mathcal{H}, \{m_\alpha\}, \{m_\beta\}, \{m_\alpha, m_\beta\}, \{m_\gamma, m_\delta, m_\nu\}, \{m_\alpha, m_\gamma, m_\delta, m_\nu\}, \{m_\beta, m_\gamma, m_\delta, m_\nu\}\}$ and an ideal $\mathcal{I} = \{\{\phi\}, \{m_\gamma\}\}$ on $\mathcal{H}$.

Also, .**S**}} on $_\alpha$, {mϕ= {′$\mathcal{I}$and $\mathcal{H}$}} on $_\gamma$, {mϕ= {$\mathcal{I}$. Take an ideals $\Im^l_\xi = \Im_\xi$ and $\Im^l_N = \Im_N$

Let, W = $\{w_\alpha, w_\beta, w_\gamma, w_\delta\}$ be the universe and W/$\mathcal{R}''$ = $\{\{w_\alpha\}, \{w_\gamma\}, \{w_\beta, w_\delta\}\}$ with Y = $\{w_\alpha, w_\beta\} \in$ W , then }, $_\delta$w}, {$_\beta$}, {w $_\alpha$, W, {wϕ= { $\Im^{ll}_\xi$ }. Then $_\delta$, w $_\beta$= {w ′′}}. Let ξ $_\delta$, w $_\beta$, w $_\alpha$}, {w $_\delta$, w $_\beta$}, {w$_\alpha$, {wϕ= {W, $\Im^{ll}_N$ }.$\{w_\alpha, w_\beta\}, \{w_\alpha, w_\delta\}, \{w_\beta, w_\delta\}, \{w_\alpha, w_\beta, w_\delta\}$

Define $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi, \mathcal{I}^l)$ be the identity function and $\varrho : (S, \Im^l_\xi, \mathcal{I}^l) \to (W, \Im^{ll}_\xi)$ defined as $\varrho(m_\alpha) = w_\gamma$, $\varrho(m_\beta) = w_\alpha$, $\varrho(m_\gamma) = w_\beta$, $\varrho(m_\delta) = \varrho(m_\nu) = w_\delta$.

It is clear that, φ and ϱ are ξI-continuous functions but $\varrho \circ \varphi$ is not ξ***I***-continuous because $\{w_\beta\} \in$ ξO(W) but $(\varrho \circ \varphi)^{-1}(\{w_\beta\}) = \{m\gamma\} \notin$ ξIO($\mathcal{H}$) as {mγ }⊈ξint $(\{m\gamma\}^{*}{}_\xi)$ = ϕ.

Theorem 5.6. Let $\varphi : (\mathcal{H}, \Im_\xi, I) \to (S, \Im^l_\xi)$ and $\varrho : (S, \Im^l_\xi, \mathcal{J}) \to (W, \Im^{ll}_\xi)$ Then

1. $\varrho \circ \varphi$ is ξI-continuity if φ is ξI-continuity and ϱ is ξ-continuity.

2. If φ is surjective, $\varphi^{-1}(V^{*\xi}) \subset (\varphi^{-1}(V))^{*\xi}$, for every V ⊂ **S** and both φ and ϱ are ξI-continuous, then $\varrho \circ \varphi$ is ξI-continuous.

Proof. 1. Obvious.
2. It is evident from the application of Theorem 5.5.

6 ξI -OPEN AND ξI -CLOSED MAPPING

Definition 6.1. Let $\varphi : (\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi)$ is said to be ξ-open map (resp. ξ-closed map) if image of every ξ- openset (resp. ξ-closedset)in $\mathcal{H}$ is ξ-openset(resp. ξ-closedset) in **S**.

Definition 6.2. A function $\varphi : (\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi, \mathcal{J})$ is said to be ξI-open map (ξI- closed map) if image of every ξ-opensets (resp. ξ-closedsets) in $\mathcal{H}$ is ξI-open (ξI- closedset) in $\mathcal{H}$.

Remark 6.1. The concept of ξI-open and ξ-openess of a function are distinct, as demonstrated in the following example.

Example 6.1. Let $\mathcal{H}$ = $\{l_\alpha, l_\beta, l_\gamma, l_\delta\}$ be the universe and $\mathcal{X}$ = $\{l_\alpha, l_\beta\} \subseteq \mathcal{H}$ with $\mathcal{H}/\mathcal{R}$= $\{\{l_\alpha\}, \{l_\gamma\}, \{l_\beta, l_\delta\}\}$. Then $\Im_N$ = $\{\mathcal{H}, \phi, \{l_\alpha\}, \{l_\alpha, l_\beta, l_\delta\}, \{l_\beta, l_\delta\}\}$.Let ξ = $\{l_\alpha, l_\beta, l_\delta\}$. Then $\Im_\xi$ = $\{\mathcal{H}, \phi, \{l_\alpha\}, \{l_\beta\}, \{l_\delta\}, \{l_\alpha, l_\beta\}, \{l_\alpha, l_\delta\}, \{l_\beta, l_\delta\}, \{l_\alpha, l_\beta, l_\delta\}\}$.

Let **S** = $\{m_\alpha, m_\beta, m_\gamma, m_\delta, m_\nu\}$ be the universe and Y = $\{m_\alpha, m_\beta, m_\gamma\} \subseteq$ **S** with **S**/$\mathcal{R}'$= $\{\{m_\alpha\}, \{m_\beta\}, \{m_\gamma, m_\delta, m_\nu\}\}$. Then $\Im^l_N$ = $\{\mathcal{H}, \phi, \{m_\alpha\}, \{m_\beta\}, \{m_\gamma, m_\delta, m_\nu\}\}$. Let ξ′ = $\{m_\alpha, m_\beta\}$. Then, $\Im^l_\xi$ = $\{\phi, S, \{m_\alpha\}, \{m_\beta\}, \{m_\gamma, m_\delta, m_\nu\}, \{m_\alpha, m_\gamma, m_\delta, m_\nu\}, \{m_\beta, m_\gamma, m_\delta, m_\nu\}\}$ and I =$\{\phi, \{m_\alpha\}\}$.

Define $\varphi : (\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi)$ by $\varphi(l_\alpha) = m_\beta$, $\varphi(l_\beta) = m_\gamma$, $\varphi(l_\gamma) = m_\alpha$, $\varphi(l_\delta) = m_\delta$.

Then φ is ξI-open but not ξ-open because {m_β} ∈ ξO($\mathcal{H}$) but φ (l_β) = m_γ ∉ ξO(**S**).

Theorem 6.1. Let φ : $(\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi, \mathcal{J})$ is a map. The followings are equivalent.

1. φ -is ξI-open map.
2. ∀ x ∈ $\mathcal{H}$, each neighbourhood U for x, there is an ξI-openset V ⊂ **S** having φ(x) suchthat V ⊂ φ(U).

Proof. Immediate.

Theorem 6.2. Let φ : $(\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi, \mathcal{J})$ is a ξ-open map (ξ-closed map) if V ⊂ **S** and U ⊂ $\mathcal{H}$ is ξ-closedset (ξ-openset) containing φ^{-1} (V). Then,exist an ξ***I***-closedset (ξI-openset) W ⊂ **S** having V suchthat φ^{-1} (W) ⊂ U .

Proof. It is obvious.

Theorem 6.3. Let φ : $(\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi, \mathcal{J})$ be a ξI-open. Then φ^{-1} (ξint (V))$^{*\xi}$ ⊂ (φ^{-1} (V))$^{*\xi}$ such that φ^{-1} (V) is a ξ^* -dense-in-itself, for every V ⊂ S.

Proof. It is trivial by using Theorem 6.2.

Theorem 6.4. For any bijective function φ : $(\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi, \mathcal{J})$ the followings are equivalent.

1. φ^{-1}: $(S, \Im^l_\xi, \mathcal{J}) \to (\mathcal{H}, \Im_\xi)$ is ξI-continuous
2. φ is ξI-open
3. φ is ξI-closed.

Theorem 6.5. If φ : $(\mathcal{H}, \Im_\xi) \to (S, \Im^l_\xi, \mathcal{J})$ is ξ***I***-open and for each U ⊂ $\mathcal{H}$, ϕ(U$^{*\xi}$) ⊂ (ϕ(U))$^{*\xi}$, then image of every ξI-openset is ξI-openset.

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