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The non-archimedean pseudo-differential operator $A\varphi^{\vartheta},k^{1,\vartheta 2}$ involving coupled fractional Fourier transform connected to negative definite functions.

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Abstract

The chief interest of this article is to discuss non-archimedean pseudodifferential operator connected to coupled fractional Fourier transform. In this article, we some classes of p-adic complete inner product spaces,

$B_{\varphi,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$, $0 \leq k < \infty$, connected to negative definite, radial and continuous functions $\varphi : \mathbb{Q}_p \to \mathbb{C}$. In this article, we also define the nonarchimedean pseudo-differential operator $A_{\phi,k}^{\vartheta_1,\vartheta_2}$ involving coupled fractional Fourier transform connected to negative definite functions.

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1 Introduction

The connections of non-archimedean pseudo-differential operators with certain p-adic pseudo-differential equations that describe certain physical models [1–5]. Therefore, non-archimedean pseudo-differential operators have received a lot of attention in two decades.Non-archimedean pseudo-differential operators have gained popularity in recent years due to their utility in studying certain equations associated with new physical models/Models in physical form [6–12]. The interest in pseudo-differential operators in the p-adic context has grown significantly in recent years as a result of their utility in modelling various types of physical phenomena.

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10.48047/jocaaa.2024.33.08.420

For example, modelling geological processes (such as the formation of petroleum micro-scale reservoirs and fluid flows in porous media such as rock); the dynamics of complex systems such as macromolecules, glasses, and proteins; the study of Coulomb gases, etc. [13–17]. Nonlocal diffusion problems arise in a wide range of applications in the archimedean setting, including biology, image processing, particle systems, and coagulation models.

My research work is motivated/ inspried by the works of Ismael Gutiérrez García and Anselmo Torresblanca-Badillo [10,16,18].

The interaction of non-archimedean pseudo-differential operators and stochastic processes on p-adics has received a lot of attention in recent decades because of the connection of the p-adic pseudo-differential equations associated with certain physical models, see [19].

2 MathematicalbackgroundofcoupledfractionalFourier analysis on $\mathbb{Q}_p \times \mathbb{Q}_p$

Definition 1. *The field of p-adic numbers: Let p be a prime number. Through out this manuscript p will denote a prime number. Firstly we define p-adic norm* $|.|_p$ *on* Q *as follows*

$$|\eta|_p = \begin{cases} 0, & \text{if } \eta = 0, \\ p^{-\tau}, & \text{if } \eta = p^{\tau}\frac{\rho}{\sigma}, \end{cases}$$

where ρ and σ are integers coprime with p. The integer $\tau := ord(\eta)$, *with* $ord(0):=+\infty$, *is called the p-adic order of η.*

The unique expansion of any p-adic number $\eta \neq 0$ *is of the form*

$$\eta = p^{ord(\eta)} \sum_{i=0}^{\infty} \eta_i p^i, \tag{1}$$

where $\eta_i \in \{0,1,2,\dots\dots\dots, p-1\}$ *and* $\eta_0 \neq 0$. *Using (1), we define the fractional part of* $\eta \in \mathbb{Q}_p$, *denoted by* $\{\eta\}_p$, *as the rational number*

$$\{\eta\}_p = \begin{cases} 0, & \text{if } \eta = 0 \text{ or } ord(\eta) \geq 0, \\ p^{ord(\eta)} \sum_{i=0}^{ord_p(\eta)-1} \eta_i p^i, & \text{if } ord(\eta) < 0. \end{cases}$$

Extention of the p-adic norm on $\mathbb{Q}_p$ *is given by*

$$||\eta||_p = |\eta|, \quad \forall \eta \in \mathbb{Q}_p.$$

10.48047/jocaaa.2024.33.08.420

Let $r_0 \in Z$ and $a_0 \in Q_p$. We consider $I_{r0}(\eta_0) = \{\eta \in Q_p : ||\eta - \eta_0||_p \leq p^{r_0}\}$. The empty set and the points are the only connected subsets of Q_p. Therefore, the topological space $(Q_p, ||.||_p)$ is totally disconnected. The necesssary and sufficient condition for the compactness of a subset of Q_p is that bounded and closed subdet of Q_p.

3 Few functional spaces

A function $f : Q_p \times Q_p \to C$ is called locally constant if for any $\eta, \xi \in Q_p$ there exists an integer $r(\eta,\xi) \in Z$ such that $f(\eta + \eta^0) = f(\eta)$ for all $\eta^0 \in I_{r(\eta)}$.

A function $f : Q_p \times Q_p \to C$ is called a test function (or a Bruhat-Schwartz function) if it is a compact support with locally constant. The set of all complex valued test functions on $Q_p \times Q_p$ is denoted by $D(Q_p \times Q)$ or simply D. The set of all distributions (all continous functionals) on D is denoted by $D^0(Q_p \times Q_p)$ or simply

D^0. The mapping $\langle U, \psi \rangle : \mathscr{D}'(\mathbb{Q}_p) \times \mathscr{D}(\mathbb{Q}_p) \to \mathbb{C}$ for $U \in D^0(Q_p \times Q_p)$ and $\psi \in D(Q_p \times Q_p)$ is defined as follows:

$$\langle U, \psi \rangle = \int_{Q_p}\int_{Q_p} U(\zeta,\xi)\psi(\zeta,\xi)d\zeta d\xi.$$

Definition 2. *Regular Distribution: Let M be an arbitrary compact subset of $Q_p \times Q_p$. i.e. $M \subset Q_p \times Q_p$. Then $L^1_{loc}(\mathbb{Q}_p \times \mathbb{Q}_p) = \{\phi | \phi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ such that $\varphi \in L^1(M)\}$. A distribution $\varphi \in D(Q_p \times Q_p)$ is defined by every function $\phi \in L^1_{loc}(\mathbb{Q}_p \times Q_p)$ according to the formula*

$$\langle \phi, \psi \rangle = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \varphi(\zeta,\eta)\psi(\zeta,\eta)d\zeta d\eta.$$

This type of distributions is known as regular distributions.

Let $\sigma \in [0, \infty)$. Then the set

$L^\sigma(Q_p \times Q_p, dxdy) = \{h : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ such that $\int_{Q_p}\int_{Q_p} |h(x,y)|^\sigma dxdy < \infty\}$, the set $L^\infty(Q_p \times Q_p, dxdy) = \{h : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ such that essential sup. of $|h| < \infty$, the set $C(\mathbb{Q}_p \times \mathbb{Q}_p, \mathbb{C}) = \{h : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ and h is a continous function , and the set

$$C_0(\mathbb{Q}_p \times \mathbb{Q}_p, \mathbb{C}) = \{h : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C} \text{ and } \lim_{(||\zeta||_p, ||\eta||) \to (\infty,\infty)} h(\zeta, \eta) = 0\}$$

are complex vector space under the binary operation vector addition (+) and scalar multiplication (.). It also implies that $(C_0(\mathbb{Q}_p \times \mathbb{Q}_p, \mathbb{C}), ||\cdot||_{L^\infty})$ is a Banach space.

10.48047/jocaaa.2024.33.08.420

4 Coupled fractional Fourier transform

Fourier Transform:-

The set $\chi_p(y)= e^{2\pi iy}$ for $y \in Q_p$. The map $\chi_p(.\ )$ is an additive character on Q_p, i.e. a continuous map from $(Q_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_\circ + x_1)= \chi_p(x_\circ)\chi_p(x_1)$, $x_\circ, x_1 \in Q_p$. The additive characters of Q_p from an Abelian group which is isomorphic to $(Q_p, +)$, the isomorphism is given by $\xi \to \chi_p(\xi x)$. Given $x, \xi \in Q_p$, if $f \in L_1(\mathrm{R})$ its Fourier transform is defined by

$$(F f)(\xi) = \int_{Q_p} \chi_p(\xi x) f(x) dx \quad \text{for } \xi \in Q_p.$$

Fractional Fourier Transform on Q_p:-

In this chapter, we introduce the definition of fractional Fourier transform on the field of p-adic numbers Q_p. Firstly, the map $\chi_p^{\vartheta}(\cdot,\cdot)$ is defined on Q_p as follows:

$$\forall \zeta,\ \eta \in \mathbb{Q}_p,\ \chi_p^{\vartheta}(\zeta,\eta) = \begin{cases} C^{\vartheta} e^{\frac{i(\zeta^2+\eta^2)\cot\vartheta}{2} - i\zeta\eta\csc\vartheta}, & \vartheta \neq n\pi, n \in \mathbb{Z} \\ \frac{1}{\sqrt{2\pi}} e^{-i\zeta\eta}, & \vartheta = \frac{\pi}{2}, \end{cases}$$

$C^{\vartheta} = \sqrt{\frac{1-i\cot\vartheta}{2\pi}}$.

If $\psi \in L^1(Q_p)$, its fractional Fourier transform of one dimension [20,21] is defined as follows:

$$(\mathscr{F}_{\vartheta}\psi)(\eta) = \widehat{\psi}_{\vartheta}(\eta) = \int_{\mathbb{Q}_p} \chi_p^{\vartheta}(\zeta,\eta)\psi(\zeta)d\zeta, \quad \text{for } \eta \in Q_p. \tag{2}$$

The inverse fractional Fourier transform of a map $\varphi \in L^1(Q_p)$ is

$$(\mathscr{F}_{\vartheta}^{-1}\phi)(\zeta) = \int_{Q_p} \chi_p^{-\vartheta}(\zeta,\eta)\phi(\eta)d\eta, \quad \text{for } \zeta \in Q_p. \tag{3}$$

The fractional Fourier transform is an isomorphism, continuous and linear map of $D(Q_p)$ onto itself holding

$$(\mathscr{F}_{\vartheta}(\mathscr{F}_{\vartheta}^{-1}\phi))(\zeta) = (\mathscr{F}_{\vartheta}^{-1}(\mathscr{F}_{\vartheta}\phi))(\zeta) = \phi(\zeta), \tag{4}$$

for every $\varphi \in D(Q_p)$.

Exploiting the tensor product of n copies of the one-dimensional fractional Fourier transform each of order ϑ_l, l = 1, 2, 3,...,n ,the fractional Fourier transform has been extended to the higher-dimensional transform.

We assume that $\vartheta = (\vartheta_1, \vartheta_2), \mathbf{x} = (x,\eta), \mathbf{y} = (y,\zeta), \chi_p^{\vartheta}(\mathbf{x},\mathbf{y}) = \chi_p^{\vartheta_1}(x,\eta).\chi^{\vartheta_2}(y,\zeta) =$

10.48047/jocaaa.2024.33.08.420

$\chi_p^{\vartheta_1,\vartheta_2}(x,y,\eta,\zeta)$, where $\chi_p^{\vartheta_1}(x,\eta)$ and $\chi_p^{\vartheta_2}(y,\zeta)$ defined as above.

Coupled Fractional Fourier Transform on $\mathbb{Q}_p \times \mathbb{Q}_p$:-

The coupled fractional Fourier transform is defined as follows

$$[F_{\vartheta}\varphi](\eta,\zeta) = [F_{\vartheta_1,\vartheta_2}\varphi](\eta,\zeta) = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \chi_p^{\vartheta}(\mathbf{x},\mathbf{y})\phi(x,y)\,dxdy$$

$$= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \chi_p^{\vartheta_1}(x,\eta)\chi_p^{\vartheta_2}(y,\zeta)\phi(x,y)\,dxdy$$

$$= \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \chi_p^{\vartheta_1,\vartheta_2}(x,y,\eta,\zeta)\phi(x,y)\,dxdy. \quad (5)$$

The corresponding inversion formula of (5) is defined as follows

$$\varphi(x,y) = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \overline{\chi_p^{\vartheta_1,\vartheta_2}(x,y,\eta,\zeta)}[\mathscr{F}_{\vartheta_1,\vartheta_2}\phi](\eta,\zeta)\,d\eta d\zeta. \quad (6)$$

It is easy to observe that for $\vartheta_1 = \vartheta_2 = \frac{\pi}{2}$, the two-dimensional fractional Fourier transform $F_{\vartheta_1,\vartheta_2}$ becomes a classical two-dimensional Fourier transform.

5 Non-archimedeanpseudo-differentialoperatorswithcoupled fractional Fourier Transform and negative definite functions on Sobolev spaces

Definition 3. *[12] A mapping* $\phi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ *is said to be positive definite, if*

$$\sum_{l,m=1}^{n} \varphi(\eta_l - \eta_m, \xi_l - \xi_m) z_l \overline{z_m} \geq 0 \quad (7)$$

for all $n \in \mathbb{N}$, $\eta_1,\ldots,\eta_n \in \mathbb{Q}_p$ *and* $\xi_1,\ldots,\xi_n \in \mathbb{Q}_p$, $z_1,\ldots,z_n \in \mathbb{C}$.

Definition 4. *[12] A mapping* $\varphi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ *is said to be negative definite, if*

$$\sum_{l,m=1}^{n} \left(\phi(\zeta_l,\xi_l) + \overline{\phi(\zeta_m,\xi_m)} - \phi(\zeta_l-\zeta_m, \xi_l-\xi_m)\right) z_l \overline{z_m} \geq 0 \quad (8)$$

for all $n \neq 0 \in \mathbb{N}$, $\zeta_1,\ldots,\zeta_n \in \mathbb{Q}_p$, *and* $\xi_1,\ldots,\xi_n \in \mathbb{Q}_p$, $z_1,\ldots,z_n \in \mathbb{C}$.

10.48047/jocaaa.2024.33.08.420

The set of all negative definite functions on $\mathbb{Q}_p \times \mathbb{Q}_p$ is denoted by $N(\mathbb{Q}_p \times \mathbb{Q}_p)$ and the set of all continuous negative definite functions on $\mathbb{Q}_p \times \mathbb{Q}_p$ is $CN(\mathbb{Q}_p \times \mathbb{Q}_p)$, throughout this manuscript.

Example 1. *We consider the set* $\mathbb{R}_+ = \{\eta \in \mathbb{R} : \eta \geq 0\}$ *and define a continuous function* $J : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $J(\eta,\xi) = J(||\eta||_p, ||\xi||_p)$ *for* $\eta, \xi \in \mathbb{Q}_p$. *Then* $J(\eta,\xi)$ *is a radial function on* $\mathbb{Q}_p \times \mathbb{Q}_p$. *In addition, we assume that* $\int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} J(\eta,\xi) d\eta d\xi = 1$. *This motivates that the copuled fractional Fourier transform of a continuous radial function is also a continuous radial function. The coupled fractional Fourier transform of* $J(||\eta||_p, ||\xi||_p)$
i.e. $[F_{\vartheta_1,\vartheta_2} J](||\eta||_p, ||\xi||_p)$ *is a positive definite function and the coupled fractional Fourier transform of* $J(0,0) - J(||\eta||_p, ||\xi||_p)$, *i.e.*
$[F_{\vartheta_1,\vartheta_2} J](0,0) - [F_{\vartheta_1,\vartheta_2} J](||\eta||_p, ||\xi||_p)$ *is a negative definite function.*

Definition 5. *Let* $\varphi : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ *be a negative definite and continuous radial function. We consider* $\psi \in D(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and a real number* $k \geq 0$. *Now, the norm is defined as follows:*

$$^{\vartheta_1,\vartheta_2}||\psi||_{\phi,\,k} = \left(\int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left[max\{1,\ |\phi(||\eta||_p, ||\xi||_p)|\} \right]^{2k} |\widehat{\psi}_\vartheta(\eta)|^2 d\eta d\xi \right)^{\frac{1}{2}}.$$

One verifies that the function space $B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$, $k \geq 0$ is the completion of $D(\mathbb{Q}_p \times \mathbb{Q}_p)$ with respect to the norm $^{\vartheta_1,\vartheta_2}||\psi||_{\varphi,\,k}$.

Remark 1. *If* $k^0 \geq k \geq 0$ *then* $^{\vartheta_1,\vartheta_2}||.||_{\varphi,\,k} \leq {}^{\vartheta_1,\vartheta_2}||.||_{\varphi,\,k^0}$ *and* $B_{\varphi,\,k^0}(\mathbb{Q}_p \times \mathbb{Q}_p) \subseteq B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$. *Therefore* $B_{\varphi,\,k^0}(\mathbb{Q}_p \times \mathbb{Q}_p) \hookrightarrow B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *if* $k^0 \geq k \geq 0$.
Moreover, we notice that $B_{\varphi,\,0}(\mathbb{Q}_p \times \mathbb{Q}_p) = L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and for* $k > 0$ *we obtain that* $B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \subset L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$.

By the Parseval-Steklov equality satisfy, we can verify $||g||_{L^2(\mathbb{Q}_p \times \mathbb{Q}_p)} \leq {}^{\vartheta_1,\vartheta_2}||g||_{\varphi,\,k}$, $\forall g \in B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and for* $k \geq 0$, *so that we obtain that*

$$B^{\phi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p),\ {}^{\vartheta_1,\vartheta_2}||\cdot||_{\phi,\,k}) \hookrightarrow \left(L^2(\mathbb{Q}_p \times \mathbb{Q}_p),\ ||\cdot||_{L^2(\mathbb{Q}_p \times \mathbb{Q}_p)}\right).$$

Using the density property of $D(\mathbb{Q}_p \times \mathbb{Q}_p)$ *in* $B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *and* $L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$, *it implies that* $B_{\varphi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ *is dense in* $L^2(\mathbb{Q}_p \times \mathbb{Q}_p)$.

10.48047/jocaaa.2024.33.08.420

Definition 6. *We consider ϕ : $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ is a negative definite and continous radial function. We define for $k \geq 0$*

$$L_2(d_k) = \left\{ g \subset L^1_{loc} : d_k g \subset L^2(\mathbb{Q}_p \times \mathbb{Q}_p) \right\},$$

where $d_k(\eta) = \left[max\{1,\ |\varphi(||\eta||_p)|\} \right]^k,\ \eta \in \mathbb{Q}_p$. One can verify that $L_2(d_k)$ is a Hilbert space with the inner product

$$\langle g, h \rangle_{L_2(d_k)} = \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} d_k(\zeta,\eta) g(\zeta,\eta) \overline{d_k(\zeta,\eta) h(\zeta,\eta)} d\zeta d\eta = \langle d_k g, d_k h \rangle_{L_2(\mathbb{Q}_p \times \mathbb{Q}_p)}. \tag{9}$$

Theorem 1. *We consider ϕ : $\mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$ is a negative definite and continous radial function. We define for $k \geq 0$*

$$L_2(d_k) = \left\{ g \in L^1_{loc} : d_k g \in L^2(\mathbb{Q}_p \times \mathbb{Q}_p) \right\},$$

where $d_k(\eta) = \left[max\{1,\ |\varphi(||\eta||_p)|\} \right]^k,\ \eta \in \mathbb{Q}_p$. Then

$$B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) = L_2(d_k).$$

Proof. Applying the Parseval-Steklov equality, we obtain that

$$\begin{aligned}
& g \in B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \\
\iff & \left(\int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left[\max\{1,\ |\varphi(||\eta||_p, ||\xi||_p)|\} \right]^{2k} |[\mathscr{F}_{\vartheta_1,\vartheta_2} g](\eta,\xi)|^2 d\eta d\xi \right)^{\frac{1}{2}} < \infty \\
\iff & \left(\int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \left| \left[\max\{1,\ |\varphi(||\eta||_p, ||\xi||_p)|\} \right]^{k} [\mathscr{F}_{\vartheta_1,\vartheta_2} g](\eta,\xi) \right|^2 d\eta d\xi \right)^{\frac{1}{2}} < \infty \\
\iff & || \left[\max\{1,\ |\varphi(||\eta||_p)|\} \right]^k g ||_{L_2(d_k)} < \infty \\
\iff & ||d_k g||_{L_2(d_k)} < \infty \\
\iff & g \in L_2(d_k).
\end{aligned}$$

It implies that $B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) = L_2(d_k)$. Hence the theorem is proved. □

Remark 2. *Since every function $g \in L_{loc}^1$ defines a regular distribution $g \in D^0(\mathbb{Q}_p \times \mathbb{Q}_p)$, we obtain that*

$$B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) = \{g \in D^0(\mathbb{Q}_p \times \mathbb{Q}_p) : ||g||_{\phi,k} < \infty\}.$$

From a Gel'fand triple the spaces $D(\mathbb{Q}_p \times \mathbb{Q}_p) \subset B_{\phi,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \subset D^0(\mathbb{Q}_p \times \mathbb{Q}_p)$.

10.48047/jocaaa.2024.33.08.420

Theorem 2. *If* $\frac{1}{[max\{1,|\varphi(||\zeta||_p,||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p)$ *for some* $k \in \mathbb{N}$ *then* $B_{\phi,k}(\mathbb{C}\times\mathbb{C}) \hookrightarrow C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$.

Proof. We assume that the fixed $k \in \mathbb{N}$ such that $\frac{1}{[\max\{1,|\varphi(||\zeta||_p,||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p)$. Using the Hölder inequality, for $g \in B_{\phi,k}(\mathbb{C}\times\mathbb{C})$ we obtain that

$$
\begin{aligned}
&\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} |[\mathscr{F}_{\vartheta_1,\vartheta_2} g](\eta,\xi)| d\zeta d\xi \\
= \ & \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{1}{[\max\{1,|\varphi(||\zeta||_p,||\xi||_p)|\}]^{\frac{k}{2}}} \\
&\times [\max\{1,|\varphi(||\zeta||_p,||\xi||_p)|\}]^{\frac{k}{2}} |\mathscr{F}_{\vartheta_1,\vartheta_2} g](\eta,\xi)| d\zeta d\xi \\
\leq \ & \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \frac{d\zeta d\xi}{[\max\{1,|\varphi(||\zeta||_p,||\xi||_p)|\}]^{k}}\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} [\max\{1,|\varphi(||\zeta||_p,||\xi||_p)|\}]^{k} |[\mathscr{F}_{\vartheta_1,\vartheta_2} g](\eta,\xi)|^2 d\zeta d\xi\right)^{\frac{1}{2}} \\
\leq \ & K||g||_{\varphi,k} < \infty,
\end{aligned}
$$

Where K is a constant. It implies that $\mathcal{F}_{\vartheta_1,\vartheta_2} g \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p)$. Applying RiemannLebesgue theorem, we obtain that $g \in C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$. We also obtain that

$$||g||_{L^\infty} \leq ||\widehat{g}_\vartheta||_{L^1} \leq ||g||_{\phi,k}.$$

It implies that $B_{\phi,k^0}(\mathbb{C}) \hookrightarrow C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$. Hence the theorem is proved. □

Remark 3. *From* $k \in \mathbb{N}$ *in the Theorem2 we get that* $B_{\phi,k^0}(\mathbb{C}\times\mathbb{C}) \hookrightarrow C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$, $\forall\, k^0 > k$. *We recall that* $D^0(\mathbb{Q}_p \times \mathbb{Q}_p)$ *is dense in* $C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$, *we get that* $B_{\phi,k^0}(\mathbb{C}\times\mathbb{C})$ *is dense in* $C_0(\mathbb{Q}_p \times \mathbb{Q}_p)$, $\forall\, k^0 \geq k$.

Example 2. *(a) The mapping* $\varphi(||\zeta||_p,||\xi||_p) = \delta(||\zeta||_p^\gamma + ||\zeta||_p^\gamma),\ \delta,\ \gamma > 0$ *holds*

$$\frac{1}{[max\{1,|\phi(||\zeta||_p+||\xi||_p)|\}]^k} \in L^1(\mathbb{Q}_p \times \mathbb{Q}_p),\ \ \forall\, k > \frac{1}{\gamma}.$$

(b) Let $g(\zeta,\xi)$ *be an elliptic polynomial of degree* n_1. *Then, we consider* $\phi(\zeta,\xi) = |g(\zeta,\xi)|^\delta$, $\zeta, \xi \in \mathbb{Q}_p$ *and for any fixed* $\delta > 0$. *We obtain that*

10.48047/jocaaa.2024.33.08.420

$\{1,|\phi(||^{1}\zeta||_{p,||\xi||_p})|\}]^{k} \in L^{1}(Q^{p} \times Q_{p}),$

$\forall\ k > n\_\_\ ^{1}{}_{1}\gamma.$

$[max$

Examples 1 and 2, in this manuscript we assume that subclassses of nonconstant negative definite functions, motivated by .

There are three types of negative definite function.

- A negative definite function $\phi : Q_p \times Q_p \to C$ is said to be type I, if ∃ a positive constant K=K(ϕ) such that

$$|\phi(||\zeta||_p,||\xi||_p)| \le K, \qquad \forall\ \zeta, \xi \in Q_p.$$

- A negative definite function $\phi : Q_p \times Q_p \to C$ is said to be type II, if ∃ a positive constant $K_1 = K_1(\phi)$, $K_2 = K_2(\phi)$, $\delta_1 = \delta_1(\phi)$ and $\delta_2 = \delta_2(\phi)$, $0 < \delta_1 \le \delta_2$, $K_1 \le K_2$, such that

$$K_1[\max\{1, ||\zeta||_p + ||\xi||_p\}]^{\delta_1} \le \max\{1, |\phi(||\zeta||_p,||\xi||_p)|\} \le$$
$$K_2[\max\{1, ||\zeta||_p + ||\xi||_p\}]^{\delta_2}, \forall\ \zeta, \xi \in Q_p.$$

- A negative definite function $\phi : Q_p \times Q_p \to C$ is said to be type III, if ∀ constant $\delta_3 > 0$, ∃ a positive constant $K_3 = K_3(\phi,\delta_3)$ such that

$$K_3[\max\{1, ||\zeta||_p + ||\xi||_p\}]^{\delta_3} < \max\{1, |\phi(||\zeta||_p,||\xi||_p)|\}, \forall\ \zeta, \xi \in Q_p.$$

Now we introduce for $g \in B_{\varphi,k}(Q_p \times Q_p)$

$$A^{\vartheta_1,\vartheta_2}_{\phi,k}(g)(\eta,\mu) = -\mathscr{F}^{-1}_{\vartheta_1,\vartheta_2}\left(\left[\max\{1,\ |\phi(||\zeta||_p,||\xi||_p)|\}\right]^{-\frac{k}{2}}[\mathscr{F}_{\vartheta_1,\vartheta_2}g](\zeta,\xi)\right)$$
$$= -\int_{Q_p}\int_{Q_p} \chi_p^{-\vartheta_1,-\vartheta_2}(\zeta,\xi,\eta,\mu)\left[\max\{1,\ |\phi(||\zeta||_p,||\xi||_p)|\}\right]^{-\frac{k}{2}}$$
$$\times[F_{\vartheta_1,\vartheta_2}g](\zeta,\xi)d\zeta d\xi.$$

We find the norm of $A^{\vartheta_1,\vartheta_2}_{\phi,k}(g)(\eta,\mu)$ as follows:

$$||\mathscr{A}^{\vartheta_1,\vartheta_2}_{\phi,k}(g)(\eta,\mu)||_{\phi,k}$$

10.48047/jocaaa.2024.33.08.420

$$\begin{aligned}
&= \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\left[\max\{1,\ |\phi(||\zeta||_p,||\xi||_p)|\}\right]^{k}|[\mathscr{F}_{\vartheta_1,\vartheta_2}g](\zeta,\xi)|^2 d\zeta d\xi\right)^{\frac{1}{2}}\\
&\leq \left(\int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p}\left[\max\{1,\ |\phi(||\zeta||_p,||\xi||_p)|\}\right]^{2k}|[\mathscr{F}_{\vartheta_1,\vartheta_2}g](\zeta,\xi)|^2 d\zeta d\xi\right)^{\frac{1}{2}}\\
&= ||g||_{\phi,k} < \infty.
\end{aligned}$$

The mapping $\left[\max\{1,\ |\phi(||\zeta||_p,||\xi||_p)|\}\right]^{-\frac{k}{2}},\ \zeta,\ \xi \in \mathbb{Q}_p$ is known as the symbol of $A_{\phi,k}^{\vartheta_1,\vartheta_2}$. Hence, $A_{\phi,k}^{\vartheta_1,\vartheta_2} : \mathscr{B}_{\phi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p) \to \mathscr{B}_{\phi,\,k}(\mathbb{Q}_p \times \mathbb{Q}_p)$ is a well-defined non-archimedean pseudo-differential operator.

6 Conclusion and Discussions for Further Research

In this article, we discuss following chapters: mathematical background of coupled fractional Fourier analysis on $\mathbb{Q}_p \times \mathbb{Q}_p$, few functional spaces, coupled fractional Fourier Transform on $\mathbb{Q}_p \times \mathbb{Q}_p$.

There are potential directions for future research of my manuscript by using many types of integral transformations.

7 Declarations

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