Some Results about the Exponential Of Stochastic Ordering Models

Sukumaran D[1], Senthil Murugan C[2], Niranjan kumar K A[3], Suresh M L[4], Venkata Rami Reddy[5], Vijayabalan D[1*]

[1]Department of Mathematics, Vel Tech High Tech Dr.Rangarajan and Dr.Sakunthala Engineering College, Avadi, Chennai-620062.
[2]Department of Mathematics, Swami Dayananda College of Arts & Science college, Manjakkudi-612610.
[3]Department of Mathematics, S.A. Engineering College, Thiruverkadu, Chennai-600077.
[4]Department of Mathematics, Vel Tech Rangarajan Dr.Sagunthala R&D Institute of Science and Technology, Chennai-600062.
[5]Department of Mathematics, Malla Reddy University, Telangana-500043.
Email: vijayabalantqb@gmail.com
*Corresponding Author

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ABSTRACT
This paper explores a new approach for comparing the expectations of stochastic order in fuzzy settings and defines a new expectation ordering that allows for stochastic comparisons of fuzzy random variables. In order to achieve this goal, we first conduct stochastic comparisons of two distinct fuzzy random variables under orders with varying baseline distributions and mixing distributions for the likelihood ratio, hazard rate, mean residual life, and mean inactivity time. The primary benefit of the paper is to comprehend the novel ideas of stochastic comparison of stochastic order built on the exponential order. Proved the preservation properties and theorem by using the fuzzy stochastic orders and created a new definition and applications involving stochastic orders are presented.

Keywords: Fuzzy set, Random variables, Stochastic orders.

1.INTRODUCTION

Numerous applications in applied probability, statistics, dependability, operation research, economics, and allied domains have demonstrated the value of stochastic ordering. Over the years, a variety of stochastic ordering and related features have been quickly produced. Let X be a positive random variable that represents a system's lifespan with density function f, distribution coefficient F, and survival parameter $\bar{F} = 1 - F$. Given that the system has already survived up to t, its residual life after t is shown by the conditional random variable $X_t = (X_t, X > t), t \geq 0$. The anticipated value of X_t, is the mean leftover life (MRL) function of X and can be obtained by,

$$\mu_X(t) = \begin{cases} \int_t^{\infty} \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right) ds, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

In several disciplines, namely survival analysis, actuarial research, and reliability engineering, the MRL function is a significant feature. It has received a great deal of attention in the literature, particularly when it comes to binary systems those in which there are just two conceivable states: successful or unsuccessful. The hazard rate (HR) function of X, which is provided by, is an additional helpful reliability metric.

$$r_X(t) = \frac{f(x)}{\bar{F}(x)}, \quad t \geq 0.$$

The HR function is very helpful in characterizing how the probability of witnessing the event varies over time and in identifying the proper failure distributions using qualitative data regarding the failure mechanism. The MRL function has been shown to be more effective in replacement and repair procedures, even though the shape of the HR function is still significant. The HR function only accounts for any potential of a sudden failure at any given time. Stochastic comparisons of residual lives and inactive periods at quantiles can be used to distinguish a family of stochastic orderings known as transform stochastic orderings in the literature, according to A.Arriaza, M.A. Sordo, and A. Surez Llorenz [1]. Following that was a brand-new stochastic order known as the star order, which falls in between the convex order and the other two transform orderings. A novel idea for the comprehensive and straightforward characterization of situations where one Beta distribution is smaller than another based on the convex transform order has been proposed by I. Arab, PE. Oliveira, and T. Wiklund [3]. They derive monotonicity properties for the probability of a random variable that is Beta distributed and exceeds its distribution's mean or mode as an application. Stochastic comparisons of vectors with a multivariate skew-normal distribution are made possible by Arevalillo and H. Navarro [2]. The novel ordering is based on the canonical transformation linked with the multivariate skew-normal distribution and the well-known convex transform order applied to the single skewed component of that canonical transformation. Three functional measures of the shape of univariate distributions are proposed by A.Arriaza, A.Crescenzo, MA. Sordo, and Suárez-Llorens [4]. These metrics are appropriate with respect to the convex transform order. To close a gap in the literature, F.Belzunce, C.Martinez-Riquelme, and M.Perera [5] concentrate on giving sufficient conditions for a few well-known stochastic orders in dependability while handling their discrete forms. In particular, based on the likelihood ratio's unimodality, they found comparison criteria for two discrete random variables in specific stochastic orders. The mean residual life, the bending property of the failure rate, the reversed hazard rate, and the mean inactive duration in mixtures have all been explored by J.H. Cha and F.G. Badia [6]. The idea of relative spacings was first developed by F. Belzunce, C. Martinez-Riquelme, and M. Perera [7]. They demonstrate the relevance of this idea in several situations, such as economy and reliability, and we offer various results for evaluating relative spacings among two populations. Numerous shifting and proportional stochastic orders have been used by Belzunce.F, Ruiz.JM, and Ruiz.M.C [8] to compare certain coherent structures that were formed from a set of components or from two sets of components. A new type of stochastic order has been proposed and explored by Izadkhah.S and Kayid.M [9]. Several fundamental and afterwards fundamental preservation properties of the new stochastic order under convolution, mixture, and shock model reliability procedures are investigated. A thorough overview of the theory and applications of aging and reliance in the application of mathematical techniques to survival and reliability studies is provided by Xie, M., and Lai, C.D [10]. The study of getting older properties of residual lifetime mixture models and stochastic comparisons has been enhanced by A. Patra and C. Kundu [11]. They performed stochastic comparisons of two distinguish mixture models under the likelihood ratio, hazard rate, mean residual life, and variance residual life orders employing two different mixing distributions and two different baseline distributions. Recent discussions regarding the stochastic comparison and aging properties of RLRT(ITRT) based on variance residual life have led to new findings regarding the stochastic ageing qualities, as noted by A. Patra and C. Kundu [12]. Sufficient standards for the residual life and inactive time's log-concavity and log-convexity have been given by Misra.N, Gupta.N, and Dhariyal.I.D [13]. In addition, we do stochastic comparisons between the inactivity time and residual life in terms of the typical stochastic order, the mean residual life order, and the failure rate order. A well-known MRL order has been introduced and examined in the literature, based on the MRL function by Nanda.K, Bhattacharjee.S and N. Balakrishnan [15]. Numerous writers have studied the MRL order's uses in survival and reliability analysis throughout the years (see Shaked and Shanthikumar [17] and Muller and Stoyan [14]). However, several existing concepts of stochastic comparisons of random variables are thought to be generalized by the proportional stochastic order, according to the literature. Proportional stochastic orders have been explored by numerous researchers as enlarged versions of the prominent stochastic orders prevalent in the literature right now, such as Ramos- Romero and Sordo-Diaz [16]. Nanda et al. examined reliability models adopting the MRL function and conducted a fresh analysis on various partial ordering effects relevant to the MRL order [15]. The probability distribution of potential outcomes is its primary concern. Examples include Shaked.M and Shanthikumar.J.G [17], Markov modals, and regression models. The modal functions as a realistic case simulation to gain a deeper understanding of the system, investigate unpredictability, and evaluate uncertain scenarios that delineate all possible outcomes and the trajectory of the system's evolution. Thus, in order to optimize profitability, experts and investors can develop their business practices and make better management decisions with the aid of this modeling technique.

An Introduction to Stochastic Orders discusses this helpful tool, which may be used to assess probabilistic models in a range of domains, including finance, economics, survival analysis, risks associated with stock trading, and reliability. For academics and students wishing to use this data as a tool for their own research, it provides a general foundation on the subject. Along with applications to probabilistic models and discussions of basic properties of many stochastic orders in the univariate and multivariate scenarios, detailed proofs of the principal results in several sectors of interest are provided. In applied probability, stochastic ordering among random variables has shown to be an effective method for comparing system reliability. Stochastic orderings are viewed as a key tool for marketing decision-making in the face of uncertainty. In order to create a mathematical or financial model that can find every possible outcome for a particular circumstance or issue, stochastic modeling uses random input variables. In the present study, we improve upon several findings on the exponential of models of stochastic ordering. We review recent studies on stochastic orderings of random variables in section 1. We give a basic overview of fuzzy random variables in section 2. We get the main idea of this article in section 3. In this case, we verify the preservation properties and theorems and analyze the stochastic ordering of fuzzy random variables. In the final section, we offer a succinct conclusion and suggestions for applying this research to other studies.

2. Preliminaries

2.1. Definition

Let χ be a set of all values. Next a fuzzy set $\widetilde{A} = \{(x, \mu_A(x))/x\epsilon\chi\}$ of χ is determined by the role it plays in membership $\mu_A: \chi \to [0,1]$.

2.2. Definition

The α-cut of the set of $\widetilde{A}$ is indicated by it's for every $(0 \le \alpha \le 1)$ $\widetilde{A}_\alpha = \{x \in \chi; \mu_{\widetilde{A}}(x) \ge \alpha\}$.

2.3. Definition

1. For each $\alpha \epsilon (0,1]$ both $\left[\chi_\alpha^U, \chi_\beta^U\right]$ defined as $\chi_\alpha^L(\omega)(x) = \inf\{x \in \Omega;\ \chi_\alpha^L(\omega)(x) \ge \alpha\}$ and $\chi_\beta^U = \sup\{x \in \Omega;\ \chi_\alpha^U(\omega)(x) \ge \beta\}$ are finite real valued random variables defined on such(Ω, A, P) that the mathematical expectations $E(\chi_\alpha^L)$ and $E(\chi_\alpha^L)$ exist.
2. For each, $\omega\epsilon\ \Omega$ and $\alpha\epsilon(0,1]$, $\chi_\alpha^L(\omega)(x) \ge \alpha$ and $\chi_\alpha^U(\omega)(x) \ge \beta$.

2.4. Definition

If $\tilde{\chi}$ and $\widetilde{\Upsilon}$ be fuzzy random variables with fuzzy cumulative distribution function $\widetilde{F}$ and $\widetilde{G}$ respectively then

1. $\chi \le_{st} \Upsilon \Leftrightarrow \widetilde{F}(t) \ge \widetilde{G}(t)\ \forall\ t$.
2. $\chi \le_{st} \Upsilon \Leftrightarrow \{P(\chi_\alpha^L \ge t) \vee P(\chi_\alpha^U \ge t)\} \le \{P(\Upsilon_\alpha^L \ge t) \vee P(\Upsilon_\alpha^U \ge t)\}$
3. $\chi \le_{st} \Upsilon \Leftrightarrow E[f(\chi_\alpha^L)] \vee E[f(\chi_\alpha^U)] \le E[f(\Upsilon_\alpha^L)] \vee E[f(\Upsilon_\alpha^U)]$, for all increasing functions f.

3. Stochastic comparison of the exponential orders

3.1. Definition

Consider two consecutive sequence set of fuzzy random variables $\{\chi_1, \chi_2, \chi_3, \dots, \chi_n\}$ and $\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n\}$ such that, $\mathbf{e^{t\chi}} E[\varphi\{\chi_1, \chi_2, \chi_3, \dots, \chi_n\}] \le \mathbf{e^{t\Upsilon}} E[\varphi\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n\}]$, for all convex functions φ, provided expectations exists. Then the sequence $\{\chi_1, \chi_2, \chi_3, \dots, \chi_n\}$ is said to be stochastically dominant of $\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n\}$ in the convex order denoted as$\chi \le_{FCO} \Upsilon$.

Where $\chi = [\underset{\alpha\le\beta\le1}{\wedge\&\vee}\{x_\alpha^L, x_\beta^U\}]$ and $\Upsilon = [\underset{\alpha\le\beta\le1}{\wedge\&\vee}\{y_\alpha^L, y_\beta^U\}]$.

3.4. Definition

Both χ and Υ are two continuous nonzero fuzzy random factors with the following attributes: f and g are their probability density functions; F and G are their distribution functions; and $\overline{F}$ and $\overline{G}$ are their survival values. After that, the progression of the,

(i) Exponential of the likelihood ratio defined by $\chi \le_{ELRO} \Upsilon$,

$$\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{f_x}{g_x}\left(x_\alpha^l, x_\beta^u\right)\right) \le_{ELR} \underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{f_y}{g_y}\left(y_\alpha^l, y_\beta^u\right)\right)$$

(ii) Exponential order of mean inactivity as stated by $\chi \le_{EMITO} \Upsilon$,

$$\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t F_x(x_\alpha^l, x_\beta^u)dx}{F_t(x_\alpha^l, x_\beta^u)}\right) \le_{EMITO} \underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t G_y(y_\alpha^l, y_\beta^u)dx}{G_t(y_\alpha^l, y_\beta^u)}\right)$$

(iii) The reversed Hazard rate order's exponential, stated by $\chi \le_{EMITO} \Upsilon$,

$$\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\overline{F_x\left(x_\alpha^l, x_\beta^u\right)}}{f_x\left(x_\alpha^l, x_\beta^u\right)}\right) \le_{ERHRO} \underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\overline{G_x\left(y_\alpha^l, y_\beta^u\right)}}{g_y\left(y_\alpha^l, y_\beta^u\right)}\right)$$

(iv) Exponential of the Hazard rate order defined by $\chi \le_{EMITO} \Upsilon$,

$$\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{f_x(x_\alpha^l, x_\beta^u)}{\overline{F_x}(x_\alpha^l, x_\beta^u)}\right) \le_{EHRO} \underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{g_y(y_\alpha^l, y_\beta^u)}{\overline{G_y}(y_\alpha^l, y_\beta^u)}\right)$$

(v) The Mean Residual Life Order exponential, as described by $\chi \le_{EMRO} \Upsilon$,

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right) \leq_{EMRLO} \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \overline{G_y(y_\alpha^l,y_\beta^u)dx}}{\overline{G_t(y_\alpha^l,y_\beta^u)}}\right)$$

(vi) Exponential serves as the decreasing order typical residual life order computed by $\chi \leq_{DEMRO} \chi_\nabla$,

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right) \leq_{EDMRLO} \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \overline{G_x(x_\alpha^l,x_\beta^u)dx}}{\overline{G_t(x_\alpha^l,x_\beta^u)}}\right)$$

Following figure describes the Graphical representation of stochastic orders of Fuzzy random variables.

3.5. Preservation properties

Dependability theory places importance on an order's preservation properties under certain dependability operations. Some features of the exponential of the inactivity order of a random variable are covered in this section.

Let χ and Υ be two continuous nonnegative fuzzy random factors that have the following traits that distribution functions F and G, survival measures $\bar{F}$ and $\bar{G}$, and probability density functions f and g, respectively.

$\psi^*_{\chi_t}(x_\alpha^l,x_\alpha^u) = \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\chi})\left(\frac{\int_0^t \overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right)$ and

$$\psi^*_{\Upsilon_t}(y_\alpha^l,y_\alpha^u) = \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\Upsilon})\left(\frac{\int_0^t \overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right)$$

This condition holds true by the previous definition, then $\chi \leq_{EMITO} \Upsilon \Leftrightarrow \psi^*_{\chi_t}(x_\alpha^l,x_\alpha^u) \leq_{EMITO} \psi^*_{\Upsilon_t}(y_\alpha^l,y_\alpha^u)$.

3.6. Proposition

Imagine two continuous nonnegative fuzzy random parameters, χ and Υ, with the following traits that probability density functions f and g, distribution functions F and G, and inheriting functions $\bar{F}$ and $\bar{G}$,respectively. Then $\chi \leq_{EMITO} \Upsilon \Leftrightarrow$

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right) \leq_{EMITO} \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \overline{G_y(y_\alpha^l,y_\beta^u)dx}}{\overline{G_t(y_\alpha^l,y_\beta^u)}}\right)$$

Is decreasing in $t \in (0,t_x)\cap(0,t_y)$, for all $t > 0$.

Proof:

Let us observed that

$$\psi^*_{\chi_t}(x_\alpha^l,x_\alpha^u) = \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\chi})\left(\frac{\int_0^t F_x(x_\alpha^l,x_\beta^u)dx}{F_t(x_\alpha^l,x_\beta^u)}\right)$$

$= \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\chi})\left(\frac{\int_0^t F_x(x_\alpha^l,x_\beta^u)dx}{\frac{\partial}{\partial x}F_t(x_\alpha^l,x_\beta^u)}\right)$; therefore given $t > 0$, by previous equations $\chi \leq_{EMITO} \Upsilon \Leftrightarrow$

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\chi})\left(\frac{\int_0^t F_x(x_\alpha^l,x_\beta^u)dx}{\frac{\partial}{\partial x}F_t(x_\alpha^l,x_\beta^u)}\right) \leq_{EMITO} \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\Upsilon})\left(\frac{\int_0^t G_y(y_\alpha^l,y_\beta^u)dy}{\frac{\partial}{\partial x}G_t(y_\alpha^l,y_\beta^u)}\right) \Leftrightarrow$$

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\chi})\left(\frac{\int_0^t \frac{\partial}{\partial x}F_t(y_\alpha^l,y_\beta^u)dy}{G_y(y_\alpha^l,y_\beta^u)}\right) \leq_{EMITO} \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(e^{-t\Upsilon})\left(\frac{\int_0^t \frac{\partial}{\partial x}F_t(x_\alpha^l,x_\beta^u)dx}{G_y(y_\alpha^l,y_\beta^u)}\right) \Leftrightarrow$$

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(\boldsymbol{e^{t\chi}})\left(\frac{\int_0^t \overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right) \leq_{EMITO} \underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(\boldsymbol{e^{t\chi}})\left(\frac{\int_0^t \overline{G_x(x_\alpha^l,x_\beta^u)dx}}{\overline{G_t(x_\alpha^l,x_\beta^u)}}\right)$$

Is decreasing in $t \in (0,t_x)\cap(0,t_y)$, for all $t > 0$.

3.7.Theorem

Let us take χ_1, χ_2 and $\mathcal{Z}$ be three continuous non negative fuzzy random variable with probability density function f and g and ? distribution function F and G and H survival functions $\bar{F}$ and $\bar{G}$ and $\bar{H}$ respectively. Then $\chi_1 \leq_{EMITO} \chi_2$ and Z is log-concave then $\chi_1 + \mathcal{Z} \leq_{EMITO} \chi_2 + \mathcal{Z}$.

Proof: By the previous preposition, it is enough to show that for all $0 \leq t_1 \leq t_2$ and $x > 0$.

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(\boldsymbol{e^{t\chi}})\left(\int_0^\infty\int_{-\infty}^{t_1}\frac{P[\chi_1 \leq u-(x_\alpha^l,x_\alpha^u)]\{f(t_1-u)dudx}{P[\chi_2 \leq u-(x_\alpha^l,x_\alpha^u)]\{g(t_2-u)dudx}\right) \geq_{EMITO}$$

$$\underset{\alpha\leq\beta\leq 1}{\wedge\&\vee} E(\boldsymbol{e^{t\chi}})\left(\int_0^\infty\int_{-\infty}^{t_1}\frac{P[\chi_1 \leq u-(x_\alpha^l,x_\alpha^u)]\{f(t_2-u)dudx}{P[\chi_2 \leq u-(x_\alpha^l,x_\alpha^u)]\{g(t_1-u)dudx}\right)$$

Since Z is non negative then $g(t-u)=0$ when $t<u$, hence the above inequality is equivalent to

$$\underset{\alpha\le\beta\le 1}{\wedge\&\vee} E(e^{tX})\left(\int_0^{\infty}\int_{-\infty}^{t_1}\frac{[F_{\chi_1}\le u-(x_\alpha^l,x_\alpha^u)]\{f(t_1-u)dudx}{[F_{\chi_2}\le u-(x_\alpha^l,x_\alpha^u)]\{g(t_2-u)dudx}\right)\ge_{EMITO}$$

$$\underset{\alpha\le\beta\le 1}{\wedge\&\vee} E(e^{tX})\left(\int_0^{\infty}\int_{-\infty}^{t_1}\frac{[G_{\chi_1}\le u-(x_\alpha^l,x_\alpha^u)]\{f(t_2-u)dudx}{[G_{F_{\chi_2}}\le u-(x_\alpha^l,x_\alpha^u)]\{g(t_2-u)dudx}\right)$$

$0\le t_1\le t_2$ or equivalently,

$$\underset{\alpha\le\beta\le 1}{\wedge\&\vee}\begin{vmatrix}\int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_2-u)dudx\}] & \int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_2-u)dudx\}]\\ \int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_1-u)dudx\}] & \int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_1-u)dudx\}]\end{vmatrix}$$

$$\ge 0 \quad \text{and}$$

$$\underset{\alpha\le\beta\le 1}{\wedge\&\vee}\begin{vmatrix}\int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_2-u)dudx\}] & \int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_2-u)dudx\}]\\ \int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_1-u)dudx\}] & \int_0^{\infty}\int_{-\infty}^{t_1}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u-(x_\alpha^l,x_\alpha^u))]\{g(t_1-u)dudx\}]\end{vmatrix}$$

$$\ge 0$$

By the well known basic composition formula

$$\underset{\alpha\le\beta\le 1}{\wedge\&\vee}\int_{u_1<u_2}^{\infty}\int_{u_1<u_2}^{\infty}\begin{vmatrix}g(t_2-u_1) & g(t_2-u_2)\\ g(t_1-u_1) & g(t_1-u_2)\end{vmatrix}$$

$$\times\begin{vmatrix}\int_0^{\infty}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u_1-(x_\alpha^l,x_\alpha^u))] & \int_0^{\infty}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_1}(u_1-(x_\alpha^l,x_\alpha^u))]\\ \int_0^{\infty}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_2}(u_2-(x_\alpha^l,x_\alpha^u))] & \int_0^{\infty}\boldsymbol{E}(\boldsymbol{e}^{tX})[F_{\chi_1}(u_2-(x_\alpha^l,x_\alpha^u))]\end{vmatrix}du_1\,du_2$$

Seeing that the first determinate is non-positive because of g's log-concavity and the second determinant being non-positive due to $X_1\le_{EMITO}\ X_2$ leads us to the conclusion.
Where $\chi_1=\left\{\underset{\alpha\le\beta\le 1}{\wedge\&\vee}F_{\chi_1}(x_\alpha^l,x_\beta^u)\right\}$ *and* $\chi_2=\ \left\{\underset{\alpha\le\beta\le 1}{\wedge\&\vee}\ F_{\chi_2}(x_\alpha^l,x_\beta^u)\right\}$. Which is complete the proof.

3.8. Lemma

If $\chi_1\le_{EMITO}\ \Upsilon_1$ and $\chi_2\le_{EMITO}\ \Upsilon_2$ where X_1 is independent fuzzy random variable of χ_2 and Υ_1 is independent fuzzy random variable of Υ_2 with probability density function f and g distribution function F and G and survival functions $\bar{F}$ and $\bar{G}$ and respectively. Then the following statements hold:

(i) If χ_1 and Υ_2 have log-concave densities, then $\chi_1+\chi_2\le_{EMITO}\ \Upsilon_1+\Upsilon_2$
(ii) If χ_2 and Υ_1 have log-concave densities, then $\chi_1+\chi_2\le_{EMITO}\ \Upsilon_1+\Upsilon_2$

Proof(i)

$$\underset{\alpha\le\beta\le 1}{\wedge\&\vee}E(e^{tX})\left(\frac{\int_0^t\overline{F_x(x_\alpha^l,x_\beta^u)dx}}{\overline{F_t(x_\alpha^l,x_\beta^u)}}\right)\le_{EMITO}\ \underset{\alpha\le\beta\le 1}{min}E(e^{tX})\left(\frac{\int_0^t\overline{G_x(x_\alpha^l,x_\beta^u)dx}}{\overline{G_t(x_\alpha^l,x_\beta^u)}}\right)$$

The following chain inequality, which is establishing (1), follows by theorem 2.1:

$$\chi_1+\chi_2\le_{EMITO}\ \chi_1+\Upsilon_2\le_{EMITO}\ \Upsilon_1+\Upsilon_2$$

Where, $\chi=\underset{\alpha\le\beta\le 1}{\wedge\&\vee}E(e^{tX})\left(\frac{\int_0^t F_x(x_\alpha^l,x_\beta^u)dx}{F_t(x_\alpha^l,x_\beta^u)}\right)$ and $\Upsilon=\underset{\alpha\le\beta\le 1}{\wedge\&\vee}E(e^{t\Upsilon})\left(\frac{\int_0^t G_y(y_\alpha^l,y_\beta^u)dy}{G_t(y_\alpha^l,y_\beta^u)}\right)$

Proof (ii)

The evidence for (ii) is analogous. The following outcome can be obtained by repeatedly employing lemma 3.8 and the closure property of log-concaves under convolution.

3.9. Theorem

Let us consider $\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}$and $\{\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots,\Upsilon_n\}$be two sequence sets of fuzzy random variables, X_i is said to be smaller than Y_i in the exponential order denoted as, $\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}\le\{\Upsilon_1,\Upsilon_2,\Upsilon_3,\dots,\Upsilon_n\}$and have log-concave densities for all i, then $\sum_{i=1}^{n}\chi_i\ \le_{EMITO}\ \sum_{i=1}^{n}\Upsilon_i$, $i=1,2,3,\dots,n$.

Proof

We shall employ induction to demonstrate the theorem. Certainly, the result stays true for $n=1$.
Assume that the result is true for $q=n-1$, that is

$$\sum_{i=1}^{n-1}\chi_i\ \le_{EMITO}\ \sum_{i=1}^{n-1}\Upsilon_i$$

Where, $\chi=\underset{\alpha\le\beta\le 1}{\wedge\&\vee}E(e^{tX})\left(\frac{\int_0^t F_x(x_\alpha^l,x_\beta^u)dx}{F_t(x_\alpha^l,x_\beta^u)}\right)$ and $\Upsilon=\underset{\alpha\le\beta\le 1}{\wedge\&\vee}E(e^{t\Upsilon})\left(\frac{\int_0^t G_y(y_\alpha^l,y_\beta^u)dy}{G_t(y_\alpha^l,y_\beta^u)}\right)$

Note that each of the two sides of above equation has a log-concave density. Applying previous lemma the results follows. The following concepts will be used in the sequel.

3.10. Definition

A function $F_{\chi\Upsilon}: \chi \times \Upsilon \to [0,1]$ is said to be totally positive fuzzy set of order 2. If for all $\alpha_1 < \alpha_2$ *and* $\beta_1 < \beta_2$ $\{(\alpha_1, \alpha_2)\epsilon\chi, (\beta_1, \beta_2)\epsilon\Upsilon\}$

$$\begin{vmatrix} F_{\chi\Upsilon}(\alpha_1, \beta_1) & F_{\chi\Upsilon}(\alpha_1, \beta_2) \\ F_{\chi\Upsilon}(\alpha_2, \beta_1) & F_{\chi\Upsilon}(\alpha_2, \beta_2) \end{vmatrix} \geq 0$$

Let us take $\chi(\delta)$ be a distribution function-containing fuzzy random variable $F_{\chi_{(\delta)}}$ and let $\Upsilon(\delta)$ another fuzzily distributed random variable with a distribution function. $F_{\Upsilon_{(\delta)}}$, for $i = 1,2$, and support $\mathcal{R}^+$.The following is a closure of exponential of inactivity time order under mixture.

3.11. Theorem

Let us take $X(\delta)$ is set of fuzzy random variable $\delta \,\epsilon\, \mathcal{R}^+$ and independent of $\emptyset_1 and\ \emptyset_2$. If $\emptyset_1 \leq_{FLR} \emptyset_2$ and if $\chi(\delta_1) \leq_{EMITO} X(\delta_2)$ whenever $\delta_1 \leq \delta_2$, then $\Upsilon(\emptyset_1) \leq_{EMITO} \Upsilon(\emptyset_2)$.

Proof

Let F_χ be the distribution function of $\chi(\delta_i)$ with $i = 1,2$. We know that,

$$F_{\chi_i} = \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t F_{\chi(\delta_i)}(x_\alpha^l, x_\beta^u)dx}{F_t(x_\alpha^l, x_\beta^u)}\right)$$

Again because of previous preposition, we should prove that,

$$\psi_{\chi_i}^*(x_\alpha^l, x_\alpha^u) = \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t F_{\chi(\delta_i)}\left(t-(x_\alpha^l, x_\beta^u)\right)dx}{F_t\left(t-(x_\alpha^l, x_\beta^u)\right)}\right)$$ is totally positive order 2 in (i, t). But actually

$$= \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t\int_0^\infty F_{\chi(\delta_i)}\left(t-(x_\alpha^l, x_\beta^u)\right)dxdt}{F_t\left(t-(x_\alpha^l, x_\beta^u)\right)}\right)$$

$$= \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t\int_0^\infty F_{\chi(\delta_i)}\left(t-(x_\alpha^l, x_\beta^u)\right)dG_{\Upsilon(\emptyset_i)}dx}{F_t\left(t-(x_\alpha^l, x_\beta^u)\right)}\right)$$

$$= \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t\int_0^\infty \psi_{\chi_i}^*(x_\alpha^l, x_\beta^u)dG_{\Upsilon(\emptyset_i)}dx}{\psi_{X_i}^*(x_\alpha^l, x_\beta^u)}\right)$$

$$= \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t\int_0^\infty \psi_{\chi_i}^*(x_\alpha^l, x_\beta^u)\psi_{\Upsilon_i}^*(y_\alpha^l, y_\alpha^u)dxdy}{\psi_{\chi_i}^*(x_\alpha^l, x_\beta^u)\psi_{\Upsilon_i}^*(y_\alpha^l, y_\alpha^u)}\right)$$

By assumption $\chi(\delta_1) \leq_{EMITO} \chi(\delta_2)$ whenever$\delta_1 \leq \delta_2$, we have that $\psi_{\chi_i}^*(x_\alpha^l, x_\alpha^u)$ is totally positive order2 in (δ, t), while form assumption $(\emptyset_1) \leq_{FLR} (\emptyset_2)$ follows that $\psi_{\Upsilon_i}^*(y_\alpha^l, y_\alpha^u)$ is totally positive order2 in (δ, i). Thus again assertion follow from the basic composition formula.

Let $\{\chi_1, \chi_2, \chi_3, \dots, \chi_n\}$be sets of fuzzy random variables with distributions $f_{\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}}$ and denote their survival functions by, $\bar{f}_{\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}} = \left(1 - f_{\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}}\right)$ respectively.Let $\underline{\alpha}$= $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ and $\underline{\beta}$= $\{\beta_1, \beta_2, \beta_3, \dots, \beta_n\}$ be two sets of probability vectors. A probability vector $\underline{\alpha}$= $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ with $a_i > 0$ for $i = 1,2,3, \dots, n$ is said to be smaller than probability vector $\underline{\beta}$= $\{\beta_1, \beta_2, \beta_3, \dots, \beta_n\}$ in the sense of discrete likelihood ratio order, denoted as $\alpha_i \leq_{DFLR} \beta_i$ if

$$\frac{\beta_i}{\alpha_i} \leq_{DFLR} \frac{\beta_j}{\alpha_j} \text{ for all } 1 \leq i \leq j \leq n$$

Let us take χ and Υ be two continuous non negative fuzzy random variable with probability density function f and g, distribution function F and G, and survival functions $\bar{F}$ and $\bar{G}$ respectively.

$$F(\chi) = \sum_{i=1}^n \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\chi}) \left(\frac{\int_0^t \alpha_i F_{\chi_i}(x_\alpha^l, x_\beta^u)dx}{\alpha_i F_t(x_\alpha^l, x_\beta^u)}\right) \text{ and } G(\Upsilon) = \sum_{i=1}^n \underset{\alpha\leq\beta\leq1}{\wedge\&\vee} E(e^{t\Upsilon}) \left(\frac{\int_0^t \beta_i G_{Y_i}(y_\alpha^l, y_\beta^u)dy}{\beta_i G_t(y_\alpha^l, y_\beta^u)}\right)$$

Conditions under which χ and Υ are analogous with regard to the exponential inactivity time order of fuzzy random variables are established by the following discovery.

3.12. Theorem

Let $\{\chi_1, \chi_2, \chi_3, \dots, \chi_n\}$ be sets of fuzzy random variables with distributions $f_{\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}}$ and denote their survival functions by , $\bar{F}_{\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}} = \left(1 - F_{\{\chi_1,\chi_2,\chi_3,\dots,\chi_n\}}\right)$, such that $\chi_1 \leq_{EMITO} \chi_2 \leq_{EMITO} \dots \leq_{EMITO} X_n$ and let $\underline{\alpha}$= $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ and $\underline{\beta}$= $\{\beta_1, \beta_2, \beta_3, \dots, \beta_n\}$ such that $\alpha_i \leq_{DFLR} \beta_i$. Let fuzzy random variable χ and Υ have distribution function f_x *and* g_y defined by the previous equation. Then $\chi \leq_{EMITO} \Upsilon$.

Proof: Because of previous preposition, we need to establish that

$$\sum_{i=1}^{n}\left(\frac{\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{\beta_j F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)}{\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i G_{Y_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{\alpha_j G_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)}\right) \le \sum_{i=1}^{n}\left(\frac{\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{\beta_j F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)}{\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i G_{Y_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{\alpha_j G_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)}\right)$$

The aforementioned equation can be demonstrated to be equivalent to by multiplying by the denominators and eliminating equal terms.

$$\sum_{i=1}^{n}\sum_{i=1}^{n}\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{\alpha_i\beta_j F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\times\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{\alpha_i\beta_j G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)$$
$$\le$$
$$\sum_{i=1}^{n}\sum_{i=1}^{n}\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{\alpha_j\beta_i F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\times\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{\alpha_j\beta_i G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)$$

Where $i \ne j$. Now for each fixed pair (i,j) with $i<j$ we have

$$\left[\beta_i\alpha_j\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\right]$$
$$+$$
$$\left[\beta_j\alpha_i\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\right]$$
$$-$$
$$\left[\beta_i\alpha_j\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\right]$$
$$+$$
$$\left[\beta_j\alpha_i\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\right]$$

$$=\left(\beta_i\alpha_j\beta_j\alpha_i\right)\left[\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)-\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\chi})\left(\frac{\int_0^t \alpha_i\beta_j F_{X_i}\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)dx}{F_t\left(\left(x_\alpha^l,x_\beta^u\right)-t\right)}\right)\right)\left(\underset{\alpha\le\beta\le1}{\wedge\&\vee} E(e^{t\Upsilon})\left(\frac{\int_0^t \alpha_i\beta_j G_{Y_i}\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)dx}{G_t\left(\left(y_\alpha^l,y_\beta^u\right)-t\right)}\right)\right)$$

This is non-negative because both terms are non negative by assumption. This is complete the proof. The above holds true maximum value of fuzzy random variable. The same preservation properties and theorems are holds true for another models like Likelihood ratio order, Hazard rate order, Mean residual life orders.

4. CONCLUSION

In actuarial science, one of the most important roles is the exponential order of a stochastic model. We propose different preservation features under mixture and convolution reliability processes of the fuzzy random variable with exponential stochastic order in the current study. Applications such as Hazard Rate Order, Mean Residual Life Order, and Reverse Hazard Rate Order all using stochastic models are outlined. Examples are given to show how the results may be exploited to find the exponential order of mean inactivity time ordered fuzzy random variables. Our findings also have implications for dependability, risk theory, and statistics. Future studies can take into account the extra features and uses of this novel ordering.

CONFLICT OF INTEREST

There is no conflict of interest, the author argues.

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