

DECOMPOSITION OF CARTESIAN PRODUCT AND CORONA PRODUCT OF PATH AND SOME STANDARD GRAPHS

Vimala Roshni. S

Research Scholar

Reg. No. 22211202092004

Department of Mathematics

Sri Parasakthi College for Women, Courtallam.

Affiliated to Manonmaniam Sundaranar University.

Tirunelveli – 627 012, Tamil Nadu, India.

E-mail: roshniqueen333@gmail.com

Chithra Devi. P

Assistant Professor

Department of Mathematics

Sri Parasakthi College for Women, Courtallam.

Affiliated to Manonmaniam Sundaranar University.

Tirunelveli – 627 012, Tamil Nadu, India.

E-mail: chithradevip@sriparasakthicollege.edu.in

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Abstract

For a graph G , if $E(G)$ can be partitioned into $E_1, E_2, \dots, E_m$ such that the subgraph of G induced by E_i is H_i for all $1 \leq i \leq m$, then $H_1, H_2, \dots, H_m$ decompose G and is written as $E(G) = E(H_1) \cup E(H_2) \cup \dots \cup E(H_m)$. If for $1 \leq i \leq m$, $H_i \cong H$, we say that G has a H -decomposition. In this paper, we show the necessary and sufficient condition for the decomposition of cartesian product and corona product of path and some standard graphs into paws, stars, cycles, and paths.

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Introduction

All graphs considered here are finite and undirected unless otherwise stated. Let P_r be the path on r vertices. In [4], S. Arumugam, I. Sahul Hamid and V. M. Abraham gave the decomposition of a graph G on n vertices (not necessarily connected) into $\lfloor n/2 \rfloor$ paths and cycles. For any two graphs G and H , their cartesian product, denoted by $G \square H$ has vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = \{(g, h)(g', h') : g = g', hh' \in E(H), \text{ or } gg' \in E(G), h = h'\}$. It is well known that the cartesian product is commutative and associative. R. Frucht and F. Harary in their paper [2], introduced a new and simple operation on two graphs G_1 and G_2 called their corona product. For any two graphs G_1 and G_2 of orders m and n respectively,

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their corona product $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and m copies of G_2 such that the i^{th} vertex of G_1 is connected to every vertex in the i^{th} copy of G_2 . In [3], M. Illayaraja, K. Sowndhariya and A. Muthusamy, 2020, Decomposition of Product Graphs. Here in this paper, we focus on the decomposition of cartesian product and corona product of path and some standard graphs.

Theorem 1.1. *For any two positive integers m, n with $m, n \geq 2$, $P_m \square P_n$ is decomposed into p copies of P_{2m} and q copies of P_2 if and only if $(2n - 1)[p - (n - 1)] + q = m - 1$.*

Proof:

Let $G = P_m \square P_n$.

Let $V(G) = \{x_{i,j} / 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$E(G) = \{x_{i,j}x_{i,j+1}, x_{k,l}x_{k+1,l} / 1 \leq i \leq m, 1 \leq j \leq n - 1, 1 \leq k \leq m - 1, 1 \leq l \leq n\}$.

Suppose that G is decomposed into p copies of P_{2m} and q copies of P_2 .

Since $|E(G)| = (2n - 1)p + q$,

we have $(2n - 1)p + q = (2n - 1)(n - 1) + (m - 1)$.

Therefore, $(2n - 1)[p - (n - 1)] + q = m - 1$.

Conversely, suppose that $(2n - 1)[p - (n - 1)] + q = m - 1$.

Case (i): m is even.

When m is even, $x_{1,j}x_{1,j-1}x_{2,j-1}x_{2,j}x_{m-1,j-1}x_{m,j-1}x_{m,j}$ forms $P_{2m} \forall j = 2, 3, \dots, n$.

Also, $x_{i,1}x_{i+1,1} \cong P_2$ where $i = 2, 4, \dots, m - 2$.

Also, $x_{i,n}x_{i+1,n} \cong P_2$ where $i = 1, 3, \dots, m - 1$.

Case (ii): m is odd.

When m is odd, $x_{1,j}x_{1,j-1}x_{2,j-1}x_{2,j}x_{m-1,j-1}x_{m,j-1}x_{m,j}$ forms $P_{2m} \forall j = 2, 3, \dots, n$.

Also, $x_{i,1}x_{i+1,1} \cong P_2$ where $i = 2, 4, \dots, m - 2$.

Also, $x_{i,n}x_{i+1,n} \cong P_2$ where $i = 1, 3, \dots, m - 1$.

In both the cases, $E(G) = [E(P_{2m}) \cup E(P_{2m}) \cup \dots \cup E(P_{2m})(n - 1) - \text{times}]$

$$\cup [E(P_2) \cup E(P_2) \cup \dots \cup E(P_2)(m-1) - \text{times}]$$

Theorem 1.2. For any two positive integers m, n with $m, n \geq 2$ and n is even, $P_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ and one copy of C_3 if and only if $2p = mn - 2$.

Proof:

$$\text{Let } G = P_m \odot P_n.$$

Let $V(G) = \{v_i, v_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$, where $v_{i1}, v_{i2}, \dots, v_{in}$ are the vertices in the copy of the path corresponding to each v_i .

Suppose that G is decomposed into p copies of $K_{1,3} + e$ and one copy of C_3 .

$$\text{Since } |E(G)| = 4p + 3,$$

$$\text{we have } 4p + 3 = 4 \left(\frac{mn}{2} - 1 \right) + 3.$$

$$\Rightarrow 4p = 2(mn - 2).$$

$$\text{Therefore, } 2p = mn - 2.$$

Conversely, suppose that $2p = mn - 2$.

Now, the induced subgraph $\langle \{v_{i2}, v_{i1}, v_i, v_{i+1}\} \rangle \cong K_{1,3} + e \forall 1 \leq i \leq m-1$.

The induced subgraph $\langle \{v_{i(j-1)}, v_{ij}, v_i\} \rangle$ together with the edge $v_{i(j-2)}, v_{i(j-1)}$, where $j = 4, 6, \dots, n$ forms $K_{1,3} + e \forall 1 \leq i \leq m$.

Also, the induced subgraph $\langle \{v_{m(j-1)}, v_{mj}, v_m\} \rangle$ together with the edge $v_{mj}, v_{m(j+1)}$, where $j = 2, 4, 6, \dots, n-2$ forms $K_{1,3} + e$.

And also, the induced subgraph $\langle \{v_m, v_{m(n-1)}, v_{mn}\} \rangle \cong C_3$.

$$\text{Thus } E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \dots \cup E(K_{1,3} + e) \left(\frac{mn}{2} - 1 \right) - \text{times}]$$

$$\cup E(C_3).$$

Hence $P_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ and one copy of C_3 .

Theorem 1.3. For any two positive integers m, n with $m \geq 2, n \geq 3$ and n is odd,

$P_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,2}$ and one copy of P_2

if and only if $p + \frac{q}{2} = \frac{mn-1}{2}$.

Proof:

Let $G = P_m \odot P_n$.

Let $V(G) = \{v_i, v_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$, where $v_{i1}, v_{i2}, \dots, v_{in}$ are the vertices in the copy of the path corresponding to each v_i .

Suppose that G is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,2}$ and one copy of P_2 .

Since $|E(G)| = 4p + 2q + 1$,

we have $4p + 2q + 1 = 4\left(\frac{m(n-1)}{2}\right) + 2(m-1) + 1$.

$\Rightarrow 4p + 2q = 2(mn - 1)$.

Therefore, $p + \frac{q}{2} = \frac{mn-1}{2}$.

Conversely, suppose that $p + \frac{q}{2} = \frac{mn-1}{2}$.

Now, the induced subgraph $\langle \{v_{i(j-1)}, v_{ij}, v_i\} \rangle$ together with the edge $v_{ij} v_{i(j+1)}$, where $j = 2, 4, \dots, n-1$ forms $K_{1,3} + e \forall 1 \leq i \leq m$.

Also, the induced subgraph $\langle \{v_{in} v_i, v_i v_{i+1}\} \rangle \cong K_{1,2} \forall 1 \leq i \leq m-1$.

And also, the induced subgraph $\langle \{v_m v_{mn}\} \rangle \cong P_2$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \dots \cup E(K_{1,3} + e) \left(\frac{m(n-1)}{2}\right) - \text{times}]$

$\cup [E(K_{1,2}) \cup E(K_{1,2}) \dots \cup E(K_{1,2})(m-1) - \text{times}]$

$\cup E(P_2)$.

Hence $P_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,2}$ and one copy of P_2 .

Theorem 1.4. For any two positive integers m, n with $m \geq 3$, $n \geq 2$ and n is even,

$C_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ if and only if $2p = 2mn$.

Proof:

Let $G = C_m \odot P_n$.

Let $V(G) = \{v_i, v_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$, where $v_{i1}, v_{i2}, \dots, v_{in}$ are the vertices in the copy of the path corresponding to each v_i .

Suppose that G is decomposed into p copies of $K_{1,3} + e$.

Since $|E(G)| = 4p$,

we have $4p = 4 \left(\frac{mn}{2}\right)$.

Therefore, $2p = mn$.

Conversely, suppose that $2p = mn$.

Now, the induced subgraph $\langle \{v_{i(n-1)}, v_{in}, v_i, v_{i+1}\} \rangle \cong K_{1,3} + e \forall 1 \leq i \leq m-1$.

Also, the induced subgraph $\langle \{v_{m(n-1)}, v_{mn}, v_m, v_1\} \rangle \cong K_{1,3} + e$.

And also, the induced subgraph $\langle \{v_{i(j-1)}, v_{ij}, v_i\} \rangle$ together with the edge $v_{ij} v_{i(j+1)}$, where $j = 2, 4, \dots, n-2$ forms $K_{1,3} + e \forall 1 \leq i \leq m$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \dots \cup E(K_{1,3} + e) \left(\frac{mn}{2}\right) - \text{times}]$

Hence $C_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$.

Theorem 1.5. For any two positive integers m, n with $m, n \geq 3$ and n is odd, $C_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$ if and only if $p + \frac{q}{2} = \frac{mn}{2}$.

Proof:

Let $G = C_m \odot P_n$.

Let $V(G) = \{v_i, v_{ij} / 1 \leq i \leq m, 1 \leq j \leq n\}$, where $v_{i1}, v_{i2}, \dots, v_{in}$ are the vertices in the copy of the path corresponding to each v_i .

Suppose that G is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$.

Since $|E(G)| = 4p + 2q$,

we have $4p + 2q = 4\left(\frac{m(n-1)}{2}\right) + 2m$.

Therefore, $p + \frac{q}{2} = \frac{mn}{2}$.

Conversely, suppose that $p + \frac{q}{2} = \frac{mn}{2}$.

Now, the induced subgraph $\langle \{v_{i1'}, v_{i2'}, v_i, v_{i+1}\} \rangle \cong K_{1,3} + e \forall 1 \leq i \leq m-1$.

The induced subgraph $\langle \{v_{m1'}, v_{m2'}, v_m, v_1\} \rangle \cong K_{1,3} + e$.

Also, the induced subgraph $\langle \{v_{ij'}, v_{i(j+1)'}, v_i\} \rangle$ together with the edge $v_{i(j-1)'} v_{ij'}$, where $j = 3, 5, \dots, n-2$ forms $K_{1,3} + e \forall 1 \leq i \leq m$.

And also, the induced subgraph $\langle \{v_{i(n-2)'}, v_n, v_n v_i\} \rangle \cong K_{1,2} \forall 1 \leq i \leq m$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \dots \cup E(K_{1,3} + e) \left(\frac{m(n-1)}{2}\right) - \text{times}]$

$\cup [E(K_{1,2}) \cup E(K_{1,2}) \dots \cup E(K_{1,2}) m - \text{times}]$

Hence $C_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ and q copies of $K_{1,2}$.

Theorem 1.6. For any two positive integers m, n with $m, n \geq 2$ and n is even, $S_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ and one copy of C_3 if and only if $2p = mn - 2$.

Proof:

Let $G = S_m \odot P_n$.

Let $V(G) = \{v_1; v_i, v_{ij'} / 1 \leq i \leq m, 1 \leq j \leq n\}$, where $v_{i1'}, v_{i2'}, \dots, v_{in'}$ are the vertices in the copy of the path corresponding to each $v_1; v_i$.

Suppose that G is decomposed into p copies of $K_{1,3} + e$ and one copy of C_3 .

Since $|E(G)| = 4p + 3$,

we have $4p + 3 = 4\left(\frac{mn}{2} - 1\right) + 3$.

Therefore, $2p = mn - 2$.

Conversely, suppose that $2p = mn - 2$.

Now, the induced subgraph $\langle \{v_{i1'}, v_{i2'}, v_i, v_1\} \rangle \cong K_{1,3} + e \forall 2 \leq i \leq m-1$.

The induced subgraph $\langle \{v_{ij'}, v_{i(j+1)'}, v_i\} \rangle$ together with the edge $v_{i(j-1)'} v_{ij'}$, where

$j = 3, 5, \dots, n - 1$ forms $K_{1,3} + e \forall 2 \leq i \leq m$.

Also, the induced subgraph $\langle \{v_{1(j-1)'}, v_{1j'}, v_1\} \rangle$ together with the edge $v_{1j'} v_{1(j+1)'}$, where $j = 2, 4, \dots, n - 2$ forms $K_{1,3} + e$.

And also, the induced subgraph $\langle \{v_1, v_{1(n-1)'}, v_{1n'}\} \rangle \cong C_3$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \dots \cup E(K_{1,3} + e) \left(\frac{mn}{2} - 1\right) - \text{times}] \cup E(C_3)$.

Hence $S_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$ and one copy of C_3 .

Theorem 1.7. For any two positive integers m, n with $m \geq 2, n \geq 3$ and n is odd,

$S_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,2}$ and one copy of P_2

if and only if $p + \frac{q}{2} = \frac{mn-1}{2}$.

Proof:

Let $G = S_m \odot P_n$.

Let $V(G) = \{v_1; v_i, v_{ij'} / 1 \leq i \leq m, 1 \leq j \leq n\}$, where $v_{i1'}, v_{i2'}, \dots, v_{in'}$ are the vertices in the copy of the path corresponding to each $v_1; v_i$.

Suppose that G is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,2}$ and one copy of P_2 .

Since $|E(G)| = 4p + 2q + 1$,

we have $4p + 2q + 1 = 4\left(\frac{m(n-1)}{2}\right) + 2(m-1) + 1$.

Therefore, $p + \frac{q}{2} = \frac{mn-1}{2}$.

Conversely, suppose that $p + \frac{q}{2} = \frac{mn-1}{2}$.

Now, the induced subgraph $\langle \{v_{i1'}, v_{i2'}, v_i, v_1\} \rangle \cong K_{1,3} + e \forall 2 \leq i \leq m$.

The induced subgraph $\langle \{v_{ij'}, v_{i(j+1)'}, v_i\} \rangle$ together with the edge $v_{i(j-1)'} v_{ij'}$, where

$j = 3, 5, \dots, n - 2$ forms $K_{1,3} + e \forall 2 \leq i \leq m$.

The induced subgraph $\langle \{v_{1j}, v_{1(j+1)}, v_1\} \rangle$ together with the edge $v_{1(j-1)} v_{1j}$, where

$j = 2, 4, \dots, n-1$ forms $K_{1,3} + e$.

Also, the induced subgraph $\langle \{v_{i(n-1)} v_{in}, v_{in} v_i\} \rangle \cong K_{1,2} \forall 2 \leq i \leq m-1$.

And also, the induced subgraph $\langle \{v_1 v_{1n}\} \rangle \cong P_2$.

Thus $E(G) = [E(K_{1,3} + e) \cup E(K_{1,3} + e) \dots \cup E(K_{1,3} + e) \left(\frac{m(n-1)}{2}\right) - \text{times}]$

$\cup [E(K_{1,2}) \cup E(K_{1,2}) \dots \cup E(K_{1,2})(m-1) - \text{times}]$

$\cup E(P_2)$.

Hence $S_m \odot P_n$ is decomposed into p copies of $K_{1,3} + e$, q copies of $K_{1,2}$ and one copy

of P_2 .

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