Commutative ideals of BCK-algebras based on makgeolli structures

Seok-Zun Song [1,*], Hee Sik Kim [2], Sun Shin Ahn [3], Young Bae Jun [4]

[1]Department of Mathematics, Jeju National University, Jeju 63243, Korea
e-mail: szsong@jejunu.ac.kr

[2]Research Institute for Natural Science, Department of Mathematics, Hanyang University, Seoul 04763, Korea
e-mail: heekim@hanyang.ac.kr

[3]Department of Mathematics Education, Dongguk University, Seoul 04620, Korea
e-mail: sunshine@dongguk.edu

[4]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea
e-mail: skywine@gmail.com
*Correspondences: S. Z. Song (szsong@jejunu.ac.kr)

April 9, 2023

Abstract The purpose of this paper is to study by applying the makgeolli structure to commutative ideal in BCK-algebras. The notion of commutative makgeolli ideal is introduced, and their properties are investigated. The relationship between makgeolli ideal and commutative makgeolli ideal is discussed. Example to show that a makgeolli ideal may not be a commutative makgeolli ideal is provided, and then the conditions under which a makgeolli ideal can be a commutative makgeolli ideal are explored. A new commutative makgeolli ideal is established using the given commutative makgeolli ideal, and characterizations of a commutative makgeolli ideal are displayed. Finally, the extension property for a commutative makgeolli ideal is established.

Keywords: BCK-soft universe, makgeolli structure, makgeolli ideal, commutative makgeolli ideal.

1

2020 Mathematics Subject Classification. 03G25, 06F35, 08A72.

1 Introduction

Many of the problems that need to be solved in the real world often include inherently inaccurate, uncertain, and ambiguous elements. The fuzzy set by Zadeh [26, 27, 28] is useful tool as a means of effectively controlling uncertainty, which is an attribute of information. Uncertainty is limited in handling using traditional mathematical tools, but can be handled using a wide range of theories such as probability theory, (intuitionistic) fuzzy set theory, theory of interval mathematics, vague set theory, rough set theory, and soft set theory etc. Molodtsov [21] introduced the concept of a soft set as a new tool for dealing with uncertainties beyond the difficulties that plagued general theoretical approaches, and he suggested several directions for the application of the soft set. Globally, interest in soft set theory and its application has been growing rapidly in recent years. Following this trend, research in the field of algebraic structure is also showing the use of soft sets. For example, groups, rings, fields and modules etc. (see [1, 3, 4, 5, 12]), and BCK/BCI-algebras etc. (see [9, 10, 11, 13, 14, 15, 16, 17, 22, 24]). In 2019, Ahn et al. [2] introduced the notion of makgeolli structures as a hybrid structure based on fuzzy set and soft set theory, and applied it to BCK/BCI-algebras. Kologani et al. [18] applied the makgeolli structure to hoops, and Song et al. [25] studied positive implicative makgeolli ideals of BCK-algebras.

In this paper, we apply the makgeolli structure to the commutative ideal of BCK-algebras. We introduce the notion of commutative makgeolli ideal, and investigate their properties. We discuss the relationship between makgeolli ideal and commutative makgeolli ideal. We provide example to show that any makgeolli ideal may not be a commutative makgeolli ideal, and then we explore the conditions under which makgeolli ideal can be commutative makgeolli ideal. We make a new commutative makgeolli ideal using the given commutative makgeolli ideal. We explore the characterization of commutative makgeolli ideal and establish the extension property for commutative makgeolli ideal.

2 Preliminaries

2.1 Preliminaries on BCK-algebras

BCI/BCK-algebra is an important type of logical algebra introduced by K. Iséki (see [7] and [8]), and it has been extensively investigated by several researchers. See the books [6, 20] for further information regarding BCI-algebras and BCK-algebras. In this section, we recall the definitions and basic results required in this paper.

Let L be a set with a special element "0" and a binary operation "$*$". If it satisfies the following conditions:

(I1) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\ (((\mathfrak{a} * \mathfrak{b}) * (\mathfrak{a} * \mathfrak{c})) * (\mathfrak{c} * \mathfrak{b}) = 0)$,

(I2) $(\forall \mathfrak{a}, \mathfrak{b} \in L)\ ((\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) * \mathfrak{b} = 0)$,

(I3) $(\forall \mathfrak{a} \in L)\ (\mathfrak{a} * \mathfrak{a} = 0)$,

(I4) $(\forall \mathfrak{a}, \mathfrak{b} \in L)\ (\mathfrak{a} * \mathfrak{b} = 0,\ \mathfrak{b} * \mathfrak{a} = 0\ \Rightarrow\ \mathfrak{a} = \mathfrak{b})$,

(K) $(\forall \mathfrak{a} \in L)\ (0 * \mathfrak{a} = 0)$,

then it is called a *BCK-algebra*, and it is denoted by $(L, *, 0)$.

The order relation " $\leq$ " in a BCK-algebra $(L, *, 0)$ is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} \leq \mathfrak{b}\ \Leftrightarrow\ \mathfrak{a} * \mathfrak{b} = 0). \tag{2.1}$$

Every BCK/BCI-algebra $(L, *, 0)$ satisfies the following conditions (see [19, 20]):

$$(\forall \mathfrak{a} \in L)\,(\mathfrak{a} * 0 = \mathfrak{a})\,, \tag{2.2}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\,(\mathfrak{a} \leq \mathfrak{b}\ \Rightarrow\ \mathfrak{a} * \mathfrak{c} \leq \mathfrak{b} * \mathfrak{c},\ \mathfrak{c} * \mathfrak{b} \leq \mathfrak{c} * \mathfrak{a})\,, \tag{2.3}$$

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = (\mathfrak{a} * \mathfrak{c}) * \mathfrak{b})\,. \tag{2.4}$$

Every BCI-algebra $(L, *, 0)$ satisfies (see [6]):

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)\,(\mathfrak{a} * (\mathfrak{a} * (\mathfrak{a} * \mathfrak{b})) = \mathfrak{a} * \mathfrak{b})\,, \tag{2.5}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)\,(0 * (\mathfrak{a} * \mathfrak{b}) = (0 * \mathfrak{a}) * (0 * \mathfrak{b}))\,. \tag{2.6}$$

A BCK-algebra $(L, *, 0)$ is said to be *commutative* (see [20]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} * (\mathfrak{a} * \mathfrak{b}) = \mathfrak{b} * (\mathfrak{b} * \mathfrak{a})). \tag{2.7}$$

A subset $\mathcal{R}$ of a BCK/BCI-algebra $(L, *, 0)$ is called

- a *subalgebra* of $(L, *, 0)$ (see [6, 20]) if it satisfies:

$$(\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{R})(\mathfrak{a} * \mathfrak{b} \in \mathcal{R}), \tag{2.8}$$

- an *ideal* of $(L, *, 0)$ (see [6, 20]) if it satisfies:

$$0 \in \mathcal{R}, \tag{2.9}$$

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} * \mathfrak{b} \in \mathcal{R},\ \mathfrak{b} \in \mathcal{R}\ \Rightarrow\ \mathfrak{a} \in \mathcal{R}). \tag{2.10}$$

A subset $\mathcal{R}$ of a BCK-algebra $(L, *, 0)$ is called a *commutative ideal* of $(L, *, 0)$ (see [20]) if it satisfies (2.9) and

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L)((\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} \in \mathcal{R},\ \mathfrak{c} \in \mathcal{R}\ \Rightarrow\ \mathfrak{a} * (\mathfrak{b} * (\mathfrak{b} * \mathfrak{a})) \in \mathcal{R}). \tag{2.11}$$

Lemma 2.1 ([20])**.** *A nonempty subset $\mathcal{R}$ of a BCK-algebra $(L, *, 0)$ is a commutative ideal of $(L, *, 0)$ if and only if $\mathcal{R}$ is an ideal of $(L, *, 0)$ that satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b} \in L)(\mathfrak{a} * \mathfrak{b} \in \mathcal{R}\ \Rightarrow\ \mathfrak{a} * (\mathfrak{b} * (\mathfrak{b} * \mathfrak{a})) \in \mathcal{R}). \tag{2.12}$$

2.2 Preliminaries on makgeolli structures

Let L be a universal set and $\mathbb{E}$ a set of parameters. We say that the pair $(L, \mathbb{E})$ is a *soft universe*.

Definition 2.2 ([2]). Let $(L, \mathbb{E})$ be a soft universe and let $\mathcal{R}$ and $\mathcal{S}$ be subsets of $\mathbb{E}$. A *makgeolli structure* over $(L, \mathbb{E})$ (related to $\mathcal{R}$ and $\mathcal{S}$) is a structure of the form:

$$\mathbb{M}_{(\mathcal{R},\mathcal{S},L)} := \{\langle(\mathfrak{a},\mathfrak{b},z); f_{\mathcal{R}}(\mathfrak{a}), g_{\mathcal{S}}(\mathfrak{b}), \xi(z)\rangle \mid (\mathfrak{a},\mathfrak{b},z) \in \mathcal{R}\times\mathcal{S}\times L\} \tag{2.13}$$

where $f_{\mathcal{R}} := (f, \mathcal{R})$ and $g_{\mathcal{S}} := (g, \mathcal{S})$ are soft sets over L and ξ is a fuzzy set in L.

A fuzzy set ξ in a set L of the form

$$\xi(\mathfrak{b}) := \begin{cases} t \in (0,1] & \text{if } \mathfrak{b} = \mathfrak{a}, \\ 0 & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be a *fuzzy point* with support $\mathfrak{a}$ and value t and is denoted by $\langle \mathfrak{a}_t \rangle$.

For a fuzzy set ξ in a set L, we say that a fuzzy point $\langle \mathfrak{a}_t \rangle$ is

(i) *contained* in ξ, denoted by $\langle \mathfrak{a}_t \rangle \in \xi$, (see [23]) if $\xi(\mathfrak{a}) \geq t$.

(ii) *quasi-coincident* with ξ, denoted by $\langle \mathfrak{a}_t \rangle \, q \, \xi$, (see [23]) if $\xi(\mathfrak{a}) + t > 1$.

For the sake of simplicity, the makgeolli structure in (2.13) will be denoted by $\mathbb{M}_{(\mathcal{R},\mathcal{S},L)} := (f_{\mathcal{R}}, g_{\mathcal{S}}, \xi)$. The makgeolli structure $\mathbb{M}_{(\mathcal{R},\mathcal{R},L)} := (f_{\mathcal{R}}, g_{\mathcal{R}}, \xi)$ over $(L, \mathbb{E})$ related to a subset $\mathcal{R}$ of $\mathbb{E}$ is simply denoted by $\mathbb{M}_{(\mathcal{R},L)} := (f_{\mathcal{R}}, g_{\mathcal{R}}, \xi)$. If $\mathcal{R} = \mathcal{S} = \mathbb{E}$, we use the notation $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ as the makgeolli structure over $(L, \mathbb{E})$.

We say that a soft universe $(L, \mathbb{E})$ is a *BCK/BCI-soft universe* if L and $\mathbb{E}$ are BCK/BCI-algebras with binary operations "$*$" and "$\oslash$", respectively.

Definition 2.3 ([2]). Let $(L, \mathbb{E})$ be a BCK/BCI-soft universe. A makgeolli structure $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is called a *makgeolli ideal* of $(L, \mathbb{E})$ if it satisfies:

$$\begin{cases} (\forall \mathfrak{a} \in \mathbb{E}) \left(f_{\mathbb{E}}(0) \supseteq f_{\mathbb{E}}(\mathfrak{a}),\ g_{\mathbb{E}}(0) \subseteq g_{\mathbb{E}}(\mathfrak{a}) \right). \\ (\forall z \in L) \left(\langle 0/\xi(z) \rangle \in \xi \right). \end{cases} \tag{2.14}$$

$$\begin{cases} (\forall \mathfrak{a},\mathfrak{b} \in \mathbb{E}) \begin{pmatrix} f_{\mathbb{E}}(\mathfrak{a}) \supseteq f_{\mathbb{E}}(\mathfrak{a} \oslash \mathfrak{b}) \cap f_{\mathbb{E}}(\mathfrak{b}) \\ g_{\mathbb{E}}(\mathfrak{a}) \subseteq g_{\mathbb{E}}(\mathfrak{a} \oslash \mathfrak{b}) \cup g_{\mathbb{E}}(\mathfrak{b}) \end{pmatrix}. \\ (\forall x, y \in L)(\forall t, r \in (0,1]) \begin{pmatrix} \langle (x*y)/t \rangle \in \xi,\ \langle y/r \rangle \in \xi \\ \Rightarrow\ \langle x/\min\{t,r\} \rangle \in \xi \end{pmatrix}. \end{cases} \tag{2.15}$$

Lemma 2.4 ([2]). *Let $(L, \mathbb{E})$ be a BCK/BCI-soft universe. Every makgeolli ideal $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ of $(L, \mathbb{E})$ satisfies the following assertions.*

(i) $$\begin{cases} (\forall \mathfrak{a},\mathfrak{b} \in \mathbb{E}) \left(\mathfrak{a} \leq \mathfrak{b} \ \Rightarrow \begin{cases} f_{\mathbb{E}}(\mathfrak{a}) \supseteq f_{\mathbb{E}}(\mathfrak{b}) \\ g_{\mathbb{E}}(\mathfrak{a}) \subseteq g_{\mathbb{E}}(\mathfrak{b}) \end{cases} \right). \\ (\forall x, y \in L) \left(x \leq y \ \Rightarrow\ \xi(x) \geq \xi(y) \right). \end{cases}$$

4

$$\text{(ii)} \begin{cases} (\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathbb{E}) \left(\mathfrak{a} \oslash \mathfrak{b} \leq \mathfrak{c} \;\Rightarrow \begin{cases} f_{\mathbb{E}}(\mathfrak{a}) \supseteq f_{\mathbb{E}}(\mathfrak{b}) \cap f_{\mathbb{E}}(\mathfrak{c}) \\ g_{\mathbb{E}}(\mathfrak{a}) \subseteq g_{\mathbb{E}}(\mathfrak{b}) \cup g_{\mathbb{E}}(\mathfrak{c}) \end{cases} \right). \\ (\forall x, y, z \in L)\,(x * y \leq z \;\Rightarrow\; \xi(x) \geq \min\{\xi(y), \xi(z)\}). \end{cases}$$

Let $(L, \mathbb{E})$ be a BCK/BCI-soft universe. Given a makgeolli structure $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ over $(L, \mathbb{E})$, consider the following sets:

$$\begin{aligned} f_{\mathbb{E}}(\mathbb{E}; \alpha) &:= \{\mathfrak{a} \in \mathbb{E} \mid f_{\mathbb{E}}(\mathfrak{a}) \supseteq \alpha\}, \\ g_{\mathbb{E}}(\mathbb{E}; \delta) &:= \{\mathfrak{b} \in \mathbb{E} \mid g_{\mathbb{E}}(\mathfrak{b}) \subseteq \delta\}, \\ \xi(L; t) &:= \{z \in L \mid \xi(z) \geq t\} \end{aligned}$$

where α and δ are subsets of L and $t \in [0, 1]$.

Lemma 2.5 ([2]). *A makgeolli structure $\mathbb{M}_{(L,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ over a BCK/BCI-soft universe $(L, \mathbb{E})$ is a makgeolli ideal of $(L, \mathbb{E})$ if and only if the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, and the nonempty set $\xi(L; t)$ is an ideal of $(L, *, 0)$ for all subsets α and δ of L and $t \in [0, 1]$.*

3 Commutative makgeolli ideals

In what follows, let $(Y, \mathbb{E})$ be a BCK-soft universe unless otherwise specified.

Definition 3.1. A makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is called a *commutative makgeolli ideal* of $(Y, \mathbb{E})$ if it satisfies (2.14) and

$$(\forall \check{x}, \check{y}, \check{z} \in \mathbb{E}) \begin{pmatrix} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \\ g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \end{pmatrix}, \tag{3.1}$$

$$(\forall x, y, z \in Y)(\forall t, r \in (0, 1]) \begin{pmatrix} \langle ((x * y) * z)/t \rangle \in \xi, \; \langle z/r \rangle \in \xi \\ \Rightarrow \; \langle (x * (y * (y * x)))/\min\{t, r\} \rangle \in \xi \end{pmatrix}. \tag{3.2}$$

Example 3.2. Consider a BCK-soft universe $(Y, \mathbb{E})$ where $Y := \{0, 1, 2, 3, 4\}$ and $\mathbb{E} := \{0, 1, 2, 3\}$ have binary operations "$*$" and "$\oslash$", respectively, given by Table 1.

Table 1: Cayley tables for the binary operations "$*$" and "$\oslash$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

$\oslash$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ be a makgeolli structure over $(Y, \mathbb{E})$ defined as follows:

$$f_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} Y & \text{if } x = 0, \\ \{3,4\} & \text{if } x = 1, \\ \{1,3,4\} & \text{if } x = 2, \\ \{1,2,3,4\} & \text{if } x = 3, \end{cases}$$

$$g_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} \{4\} & \text{if } x = 0, \\ \{0,1,4\} & \text{if } x = 1, \\ \{1,4\} & \text{if } x = 2, \\ \{0,1,3,4\} & \text{if } x = 3, \end{cases}$$

and

$$\xi : Y \to [0,1],\ y \mapsto \begin{cases} 0.79 & \text{if } y = 0, \\ 0.62 & \text{if } y = 1, \\ 0.62 & \text{if } y = 2, \\ 0.45 & \text{if } y = 3, \\ 0.67 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$.

We discuss the relationship between the commutative makgeolli ideal and the makgeolli ideal.

Theorem 3.3. *Every commutative makgeolli ideal is a makgeolli ideal.*

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ be a commutative makgeolli ideal of $(Y, \mathbb{E})$. If we put $\breve{y} = 0 = y$ in (3.1) and (3.2) and use (K) and (2.2), then we get (2.15). Hence $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$. □

The following example informs the existence of the makgeolli ideal, not the commutative makgeolli ideal.

Example 3.4. Consider a BCK-soft universe $(Y, \mathbb{E})$ in which $Y = \{0, 1, 2, 3, 4\} = \mathbb{E}$ with binary operations "$*$" and "$\oslash$", respectively, given by Table 2.

Table 2: Cayley tables for the binary operations "$*$" and "$\oslash$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

$\oslash$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

6

Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli structure on $(Y, \mathbb{E})$ defined as follows:

$$f_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} Y & \text{if } x = 0, \\ \{1,2,4\} & \text{if } x = 1, \\ \{0,1,3,4\} & \text{if } x = 2, \\ \{1,4\} & \text{if } x = 3, \\ \{0,2\} & \text{if } x = 4, \end{cases}$$

$$g_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} \{4\} & \text{if } x = 0, \\ \{0,2,4\} & \text{if } x = 1, \\ \{1,4\} & \text{if } x = 2, \\ \{0,2,4\} & \text{if } x = 3, \\ \{0,1,2,4\} & \text{if } x = 4, \end{cases}$$

and

$$\xi : Y \to [0,1],\ y \mapsto \begin{cases} 0.73 & \text{if } y = 0, \\ 0.63 & \text{if } y = 1, \\ 0.54 & \text{if } y = 2, \\ 0.42 & \text{if } y = 3, \\ 0.42 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$. But it is not a commutative makgeolli ideal of $(Y, \mathbb{E})$ since

$$f_{\mathbb{E}}(2 \oslash (4 \oslash (4 \oslash 2))) = f_{\mathbb{E}}(2) = \{0,1,3,4\} \not\supseteq \{1,2,4\} = f_{\mathbb{E}}((2 \oslash 4) \oslash 1) \cap f_{\mathbb{E}}(1)$$

and/or $\langle ((2 * 3) * 0)/0.71 \rangle \in \xi$ and $\langle 0/0.65 \rangle \in \xi$, but

$$\langle (2 * (3 * (3 * 2)))/\min\{0.71, 0.65\} = \langle 2/0.65 \rangle \,\overline{\in}\, \xi.$$

We explore the conditions for the makgeolli ideal to be the commutative makgeolli ideal.

Theorem 3.5. *In a commutative BCK-algebra, every makgeolli ideal is a commutative makgeolli ideal.*

Proof. Let $(Y, \mathbb{E})$ be a BCK-soft universe in which $(Y, *, 0)$ and $(\mathbb{E}, \oslash, 0)$ are commutative BCK-algebras, and let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli ideal of $(Y, \mathbb{E})$. Using (I1), (I3), (2.1), (2.4) and the commutativity of Y and $\mathbb{E}$, we have

$$(\forall \check{x}, \check{y}, \check{z} \in \mathbb{E})((\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash ((\check{x} \oslash \check{y}) \oslash \check{z}) \leq \check{z}),$$
$$(\forall x, y, z \in \mathbb{E})((x * (y * (y * x))) * ((x * y) * z) \leq z).$$

It follows from Lemma 2.4(ii) that

$$f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}),$$
$$g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}),$$

and

$$\xi(x * (y * (y * x))) \geq \min\{\xi((x * y) * z), \xi(z)\}. \tag{3.3}$$

Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle((x * y) * z)/t\rangle \in \xi$ and $\langle z/r\rangle \in \xi$. Then $\xi((x * y) * z) \geq t$ and $\xi(z) \geq r$, and so

$$\xi(x * (y * (y * x))) \geq \min\{\xi((x * y) * z), \xi(z)\} \geq \min\{t, r\}$$

by (3.3). Hence $\langle(x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi$. Therefore $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

Corollary 3.6. *If a BCK-soft universe $(Y, \mathbb{E})$ satisfies any one of the following conditions:*

$$\begin{cases} (\forall \breve{x}, \breve{y} \in \mathbb{E})\, (\breve{x} \oslash (\breve{x} \oslash \breve{y}) \leq \breve{y} \oslash (\breve{y} \oslash \breve{x}))\,, \\ (\forall x, y \in Y)\, (x * (x * y) \leq y * (y * x))\,, \end{cases} \tag{3.4}$$

$$\begin{cases} (\forall \breve{x}, \breve{y} \in \mathbb{E})\, (\breve{x} \leq \breve{y} \;\Rightarrow\; \breve{x} = \breve{y} \oslash (\breve{y} \oslash \breve{x}))\,, \\ (\forall x, y \in Y)\, (x \leq y \;\Rightarrow\; x = y * (y * x))\,, \end{cases} \tag{3.5}$$

$$\begin{cases} (\forall \breve{x}, \breve{y}, \breve{z} \in \mathbb{E})\, (\breve{x} \leq \breve{z},\, \breve{z} \oslash \breve{y} \leq \breve{z} \oslash \breve{x} \;\Rightarrow\; \breve{x} \leq \breve{y})\,, \\ (\forall x, y, z \in Y)\, (x \leq z,\, z * y \leq z * x \;\Rightarrow\; x \leq y)\,, \end{cases} \tag{3.6}$$

then every makgeolli ideal is a commutative makgeolli ideal.

Proof. Straightforward. □

Theorem 3.7. *Let $(Y, \mathbb{E})$ be a BCK-soft universe in which $(Y, *, 0)$ and $(\mathbb{E}, \oslash, 0)$ are lower semilattices with respect to the order relation "$\leq$". Then every makgeolli ideal is a commutative makgeolli ideal.*

Proof. Assume that $(Y, *, 0)$ and $(\mathbb{E}, \oslash, 0)$ are lower semilattices with respect to the order relation "$\leq$" in the BCK-soft universe $(Y, \mathbb{E})$. Let $\breve{x}, \breve{y} \in \mathbb{E}$ and $x, y \in Y$. Then $\breve{x} \oslash (\breve{x} \oslash \breve{y})$ is a common lower bound of $\breve{x}$ and $\breve{y}$; and $x * (x * y)$ is a common lower bound of x and y. Also, $\breve{y} \oslash (\breve{y} \oslash \breve{x})$ is the greatest lower bound of $\breve{x}$ and $\breve{y}$; and $y * (y * x)$ is the greatest lower bound of x and y. Hence $\breve{x} \oslash (\breve{x} \oslash \breve{y}) \leq \breve{y} \oslash (\breve{y} \oslash \breve{x})$ and $x * (x * y) \leq y * (y * x)$. Therefore every makgeolli ideal is a commutative makgeolli ideal by Corollary 3.6. □

Theorem 3.8. *If a makgeolli ideal $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ of $(Y, \mathbb{E})$ satisfies:*

$$(\forall \breve{x}, \breve{y}, \breve{z} \in \mathbb{E}) \begin{pmatrix} f_{\mathbb{E}}((\breve{x} \oslash \breve{z}) \oslash (\breve{y} \oslash (\breve{y} \oslash \breve{x}))) \supseteq f_{\mathbb{E}}(((\breve{x} \oslash \breve{y}) \oslash \breve{z}) \\ g_{\mathbb{E}}((\breve{x} \oslash \breve{z}) \oslash (\breve{y} \oslash (\breve{y} \oslash \breve{x}))) \subseteq g_{\mathbb{E}}(((\breve{x} \oslash \breve{y}) \oslash \breve{z}) \end{pmatrix}, \tag{3.7}$$

$$(\forall x, y, z \in Y)\, (\xi((x * z) * (y * (y * x))) \geq \xi(((x * y) * z))\,, \tag{3.8}$$

then it is a commutative makgeolli ideal of $(Y, \mathbb{E})$.

8

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli ideal of $(Y,\mathbb{E})$ that satisfies the conditions (3.7) and (3.8). Using (2.4), (2.15) and (3.7), we have

$$\begin{aligned} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &\supseteq f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \\ &= f_{\mathbb{E}}((\check{x} \oslash \check{z}) \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \cap f_{\mathbb{E}}(\check{z}) \\ &\supseteq f_{\mathbb{E}}(((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \end{aligned}$$

and

$$\begin{aligned} g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &\subseteq g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \\ &= g_{\mathbb{E}}((\check{x} \oslash \check{z}) \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \cup f_{\mathbb{E}}(\check{z}) \\ &\subseteq g_{\mathbb{E}}(((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}). \end{aligned}$$

Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle((x * y) * z)/t\rangle \in \xi$ and $\langle z/r\rangle \in \xi$. Then

$$\xi((x * (y * (y * x))) * z) = \xi((x * z) * (y * (y * x))) \geq \xi(((x * y) * z) \geq t$$

by (2.4) and (3.8), that is, $\langle((x * (y * (y * x))) * z)/t\rangle \in \xi$. It follows from (2.15) that $\langle(x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi$. Therefore $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$. □

Theorem 3.9. *A makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ over $(Y,\mathbb{E})$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$ if and only if it is a makgeolli ideal of $(Y,\mathbb{E})$ that satisfies:*

$$(\forall \check{x}, \check{y} \in \mathbb{E}) \left(\begin{array}{c} f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}(\check{x} \oslash \check{y}) \\ g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}(\check{x} \oslash \check{y}) \end{array} \right), \tag{3.9}$$

$$(\forall x, y \in Y)\,(\xi(x * (y * (y * x))) \geq \xi(x * y))\,. \tag{3.10}$$

Proof. Assume that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$. Then it is a makgeolli ideal of $(Y,\mathbb{E})$ (see Theorem 3.3). If we put $\check{z} = 0$ in (3.1) and use (2.2) and (2.14), then

$$\begin{aligned} &f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash 0) \cap f_{\mathbb{E}}(0) = f_{\mathbb{E}}(\check{x} \oslash \check{y}), \\ &g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash 0) \cup g_{\mathbb{E}}(0) = g_{\mathbb{E}}(\check{x} \oslash \check{y}). \end{aligned}$$

Let $t := \xi(x * y)$ for all $x, y \in Y$. Then $t := \xi((x * y) * 0)$, i.e., $\langle((x * y) * 0)/t\rangle \in \xi$. Since $\langle 0/t\rangle \in \xi$ by (2.14), it follows from (3.2) that $\langle(x * (y * (y * x)))/t\rangle \in \xi$. Hence $\xi(x * (y * (y * x))) \geq t = \xi(x * y)$. Therefore (3.9) and (3.10) are valid.

Conversely, let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli ideal of $(Y,\mathbb{E})$ that satisfies (3.9) and (3.10). For every $\check{x}, \check{y}, \check{x} \in \mathbb{E}$, we have

$$\begin{aligned} &f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}), \\ &g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \end{aligned}$$

9

by (3.9) and (2.15). Let $x, y, z \in Y$ and $t, r \in (0, 1]$ be such that $\langle z/r\rangle \in \xi$ and $\langle((x * y) * z)/t\rangle \in \xi$. Then $\langle(x * y)/\min\{t, r\}\rangle \in \xi$ by (2.15). It follows from (3.10) that

$$\xi((x * (y * (y * x))) \geq \xi(x * y) \geq \min\{t, r\},$$

i.e., $\langle((x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi$. Consequently, $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

Theorem 3.10. *A makgeolli structure $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ over $(Y, \mathbb{E})$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$ if and only if the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all subsets α and δ of Y, and the nonempty set $\xi(Y; t)$ is a commutative ideal of $(Y, *, 0)$ for all $t \in [0, 1]$.*

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ be a commutative makgeolli ideal of $(Y, \mathbb{E})$. Then it is a makgeolli ideal of $(Y, \mathbb{E})$ (see Theorem 3.3). Hence the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, and the nonempty set $\xi(Y; t)$ is an ideal of $(Y, *, 0)$ for all subsets α and δ of Y and $t \in [0, 1]$ by Lemma 2.5. Let $\check{x} \oslash \check{y} \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ for all $\check{x}, \check{y} \in \mathbb{E}$ and subsets α and δ of Y. Then $f_{\mathbb{E}}(\check{x} \oslash \check{y}) \supseteq \alpha$ and $g_{\mathbb{E}}(\check{x} \oslash \check{y}) \subseteq \delta$. It follows from (3.9) that

$$f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq f_{\mathbb{E}}(\check{x} \oslash \check{y}) \supseteq \alpha$$

and $g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq g_{\mathbb{E}}(\check{x} \oslash \check{y}) \subseteq \delta$. Hence $\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$, and therefore $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ by Lemma 2.1. Let $x, y \in Y$ and $t \in [0, 1]$ be such that $x * y \in \xi(Y; t)$. Then $\xi(x * y) \geq t$, and so $\xi(x * (y * (y * x))) \geq \xi(x * y) \geq t$ by (3.10), that is, $x * (y * (y * x)) \in \xi(Y; t)$. Thus $\xi(Y; t)$ is a commutative iddeal of $(Y, *, 0)$ by Lemma 2.1.

Conversely, suppose that the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all subsets α and δ of Y, and the nonempty set $\xi(Y; t)$ is a commutative ideal of $(Y, *, 0)$ for all $t \in [0, 1]$. Then $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$, and $\xi(Y; t)$ is an ideal of $(Y, *, 0)$. Thus $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$ by Lemma 2.5. Let $\check{x}, \check{y} \in \mathbb{E}$ be such that $f_{\mathbb{E}}(\check{x} \oslash \check{y}) = \alpha$ and $g_{\mathbb{E}}(\check{x} \oslash \check{y}) = \delta$. Then $\check{x} \oslash \check{y} \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$, and so $\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ by Lemma 2.1. Hence $f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq \alpha = f_{\mathbb{E}}(x \oslash y)$ and $g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq \delta = f_{\mathbb{E}}(x \oslash y)$. Let $x, y \in Y$ be such that $\xi(x * y) = t$. Then $x * y) \in \xi(Y; t)$, which implies from Lemma 2.1 that $x * (y * (y * x)) \in \xi(Y; t)$. Thus $\xi(x * (y * (y * x))) \geq t = \xi(x * y)$. Therefore $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$ by Theorem 3.9. □

Corollary 3.11. *If $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}}, g_{\mathbb{E}}, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$, then $f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ and $\xi(Y; t)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ and $(Y, *, 0)$, respectively, for all subsets α and δ of Y and $t \in [0, 1]$.*

Proof. Straightforward. □

The converse of Corollary 3.11 is not true in general as seen in the following example.

Example 3.12. Consider a BCK-soft universe $(Y, \mathbb{E})$ where $Y = \mathbb{E} := \{0, 1, 2, 3, 4\}$ has binary operation "$*(= \oslash)$" given by Table 3.

Table 3: Cayley tables for the binary operations "$*(= \oslash)$"

$*(= \oslash)$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ be a makgeolli structure over $(Y, \mathbb{E})$ defined as follows:

$$f_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} Y & \text{if } x = 0, \\ \{3, 4\} & \text{if } x = 1, \\ \{1, 3, 4\} & \text{if } x = 2, \\ \{1, 2, 3, 4\} & \text{if } x = 3, \\ \{4\} & \text{if } x = 4, \end{cases}$$

$$g_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ x \mapsto \begin{cases} \{3\} & \text{if } x = 0, \\ \{0, 3\} & \text{if } x = 1, \\ \{0, 2, 3\} & \text{if } x = 2, \\ \{0, 2, 3, 4\} & \text{if } x = 3, \\ Y & \text{if } x = 4, \end{cases}$$

and

$$\xi : Y \to [0, 1],\ y \mapsto \begin{cases} 0.82 & \text{if } y = 0, \\ 0.54 & \text{if } y = 1, \\ 0.75 & \text{if } y = 2, \\ 0.65 & \text{if } y = 3, \\ 0.42 & \text{if } y = 4. \end{cases}$$

It is routine to verify that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a makgeolli ideal of $(Y, \mathbb{E})$ and the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha) \cap g_{\mathbb{E}}(\mathbb{E}; \delta)$ and $\xi(Y; t)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ and $(Y, *, 0)$, respectively, for all subsets α and δ of Y and $t \in [0, 1]$. We have $f_{\mathbb{E}}(2 \oslash (4 \oslash (4 \oslash 2))) = f_{\mathbb{E}}(2) = \{1, 3, 4\} \not\supseteq Y = f_{\mathbb{E}}(0) = f_{\mathbb{E}}(2 \oslash 4)$ and/or $\xi(1 * (4 * (4 * 1))) = \xi(1) = 0.54 \not\geq 0.82 = \xi(0) = \xi(1 * 4)$. Hence $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is not a commutative makgeolli ideal of $(Y, \mathbb{E})$ by Theorem 3.9.

We make a new commutative makgeolli ideal using the given commutative makgeolli ideal.

Theorem 3.13. *Given a makgeolli structure* $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ *over* $(Y, \mathbb{E})$, *let* $\mathbb{M}^*_{(Y,\mathbb{E})} := (f^*_{\mathbb{E}},\, g^*_{\mathbb{E}},\, \xi^*)$ *be a new makgeolli structure over* $(Y, \mathbb{E})$ *which is*

11

defined by

$$f^*_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ \check{x} \mapsto \begin{cases} f_{\mathbb{E}}(\check{x}) & \text{if } \check{x} \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)), \\ \beta & \text{otherwise,} \end{cases}$$

$$g^*_{\mathbb{E}} : \mathbb{E} \to \mathcal{P}(Y),\ \check{x} \mapsto \begin{cases} g_{\mathbb{E}}(\check{x}) & \text{if } \check{x} \in g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w)), \\ \gamma & \text{otherwise,} \end{cases}$$

$$\xi^* : Y \to [0,1],\ x \mapsto \begin{cases} \xi(x) & \text{if } x \in \xi(Y; \xi(u)), \\ k & \text{otherwise,} \end{cases}$$

where $w \in \mathbb{E}$*,* $u \in Y$*,* $k \in [0,1]$ *and* $\beta, \gamma \in \mathcal{P}(Y)$ *with* $\beta \subsetneq f_{\mathbb{E}}(\check{x})$*,* $\gamma \supsetneq g_{\mathbb{E}}(\check{x})$ *and* $\xi(x) > k$*. If* $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\ g_{\mathbb{E}},\ \xi)$ *is a commutative makgeolli ideal of* $(Y, \mathbb{E})$*, then* $\mathbb{M}^*_{(Y,\mathbb{E})} := (f^*_{\mathbb{E}},\ g^*_{\mathbb{E}},\ \xi^*)$ *is a commutative makgeolli ideal of* $(Y, \mathbb{E})$*.*

Proof. Assume that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\ g_{\mathbb{E}},\ \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. Then the sets $f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w))$ and $g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all $w \in \mathbb{E}$, and $\xi(Y; \xi(u))$ is a commutative ideal of $(Y, *, 0)$ for all $u \in Y$. Hence $0 \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w)) \cap \xi(Y; \xi(u))$, and so $f^*_{\mathbb{E}}(0) = f_{\mathbb{E}}(0) \supseteq f_{\mathbb{E}}(\check{x}) \supset f^*_{\mathbb{E}}(\check{x})$ and $g^*_{\mathbb{E}}(0) = g_{\mathbb{E}}(0) \subseteq g_{\mathbb{E}}(\check{x}) \subseteq g^*_{\mathbb{E}}(\check{x})$ for all $\check{x} \in \mathbb{E}$. Also, we get $\xi^*(0) = \xi(0) \geq \xi(x) \geq \xi^*(x)$, i.e., $\langle 0/\xi^*(\check{x}) \rangle \in \xi^*$ for all $x \in Y$. Let $\check{x}, \check{y}, \check{z} \in \mathbb{E}$. If $(\check{x} \oslash \check{y}) \oslash \check{z} \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$ and $z \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$, then $\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w)) \cap g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$. Thus

$$\begin{aligned} f^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &= f_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \\ &\supseteq f_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f_{\mathbb{E}}(\check{z}) \\ &= f^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f^*_{\mathbb{E}}(\check{z}) \end{aligned}$$

and

$$\begin{aligned} g^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) &= g_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \\ &\subseteq g_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g_{\mathbb{E}}(\check{z}) \\ &= g^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g^*_{\mathbb{E}}(\check{z}). \end{aligned}$$

If $(\check{x} \oslash \check{y}) \oslash \check{z} \notin f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w))$ or $z \notin f_{\mathbb{E}}(\mathbb{E}; f_{\mathbb{E}}(w))$, then $f^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) = \beta$ or $f^*_{\mathbb{E}}(\check{z}) = \beta$. Hence $f^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \supseteq \beta = f^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cap f^*_{\mathbb{E}}(z)$. If $(\check{x} \oslash \check{y}) \oslash \check{z} \notin g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$ or $z \notin g_{\mathbb{E}}(\mathbb{E}; g_{\mathbb{E}}(w))$, then $g^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) = \gamma$ or $g^*_{\mathbb{E}}(\check{z}) = \gamma$. Hence $g^*_{\mathbb{E}}(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \subseteq \gamma = g^*_{\mathbb{E}}((\check{x} \oslash \check{y}) \oslash \check{z}) \cup g^*_{\mathbb{E}}(z)$. Let $x, y, z \in Y$ and $t, r \in (0,1]$ be such that $\langle ((x * y) * z)/t \rangle \in \xi^*$ and $\langle z/r \rangle \in \xi^*$. If $(x * y) * z \in \xi(Y; \xi(u))$ and $z \in \xi(Y; \xi(u))$, then $x * (y * (y * x)) \in \xi(Y; \xi(u))$ and thus

$$\begin{aligned} \xi^*(x * (y * (y * x))) &= \xi(x * (y * (y * x))) \\ &\geq \min\{\xi((x * y) * z), \xi(z)\} \\ &= \min\{\xi^*((x * y) * z), \xi^*(z)\} \\ &\geq \min\{t, r\}, \end{aligned}$$

that is, $\langle (x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi^*$. If $(x * y) * z \notin \xi(Y; \xi(u))$ or $z \notin \xi(Y; \xi(u))$, then $\xi^*((x * y) * z) = k$ or $\xi^*(z) = k$. Thus

$$\xi^*(x * (y * (y * x))) \geq k = \min\{\xi^*((x * y) * z), \xi^*(z)\} \geq \min\{t, r\},$$

and so $\langle (x * (y * (y * x)))/\min\{t, r\}\rangle \in \xi^*$. Therefore $\mathbb{M}^*_{(Y,\mathbb{E})} := (f^*_{\mathbb{E}},\, g^*_{\mathbb{E}},\, \xi^*)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. □

Note that a makgeolli ideal might not be a commutative makgeolli ideal (see Example 3.4). But we can consider the extension property for a commutative makgeolli ideal.

Theorem 3.14. *Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ and $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$ be makgeolli ideals of $(Y, \mathbb{E})$ such that $\mathbb{M}_{(Y,\mathbb{E})} \Subset \tilde{\mathbb{M}}_{(Y,\mathbb{E})}$, that is,*

(i) $f_{\mathbb{E}}(0) = \tilde{f}_{\mathbb{E}}(0),\, g_{\mathbb{E}}(0) = \tilde{g}_{\mathbb{E}}(0),\, \xi(0) = \tilde{\xi}(0)$,

(ii) $(\forall \check{x} \in \mathbb{E}, \forall x \in Y)\ (\tilde{f}_{\mathbb{E}}(\check{x}) \supseteq f_{\mathbb{E}}(\check{x}),\, \tilde{g}_{\mathbb{E}}(\check{x}) \subseteq g_{\mathbb{E}}(\check{x}),\, \tilde{\xi}(x) \geq \xi(x))$.

If $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$, then so is $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$.

Proof. Let $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ and $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$ be makgeolli ideals of $(Y, \mathbb{E})$ such that $\mathbb{M}_{(Y,\mathbb{E})} \Subset \tilde{\mathbb{M}}_{(Y,\mathbb{E})}$. Then $f_{\mathbb{E}}(\mathbb{E}; \alpha) \subseteq \tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha)$, $g_{\mathbb{E}}(\mathbb{E}; \delta) \supseteq \tilde{g}_{\mathbb{E}}(\mathbb{E}; \delta)$ and $\xi(Y; t) \subseteq \tilde{\xi}(Y; t)$ for all subsets α and δ of Y and $t \in (0, 1]$. Assume that $\mathbb{M}_{(Y,\mathbb{E})} := (f_{\mathbb{E}},\, g_{\mathbb{E}},\, \xi)$ is a commutative makgeolli ideal of $(Y, \mathbb{E})$. Then the nonempty sets $f_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $g_{\mathbb{E}}(\mathbb{E}; \delta)$ are commutative ideals of $(\mathbb{E}, \oslash, 0)$ for all subsets α and δ of Y, and the nonempty set $\xi(Y; t)$ is a commutative ideal of $(Y, *, 0)$ for all $t \in (0, 1]$ by Theorem 3.10. Since $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f}_{\mathbb{E}},\, \tilde{g}_{\mathbb{E}},\, \tilde{\xi})$ is a makgeolli ideal of $(Y, \mathbb{E})$, we know from Lemma 2.5 that the nonempty sets $\tilde{f}_{\mathbb{E}}(\mathbb{E}; \alpha)$ and $\tilde{g}_{\mathbb{E}}(\mathbb{E}; \delta)$ are ideals of $(\mathbb{E}, \oslash, 0)$ for all subsets α and δ of Y, and the nonempty set $\tilde{\xi}(Y; t)$ is an ideal of $(Y, *, 0)$ for all $t \in (0, 1]$. Let $x, y \in Y$ and $t \in (0, 1]$ be such that $x * y \in \tilde{\xi}(Y; t)$. Using (I3) and (2.4), we have $(x * (x * y)) * y = (x * y) * (x * y) = 0 \in \xi(Y; t)$. Since $\xi(Y; t)$ is a commutative ideal of $(Y, *, 0)$, using (2.4) and Lemma 2.1 leads to

$$\begin{aligned}
&(x * (y * (y * (x * (x * y))))) * (x * y) \\
&= (x * (x * y)) * (y * (y * (x * (x * y)))) \\
&\in \xi(Y; t) \subseteq \tilde{\xi}(Y; t),
\end{aligned}$$

and so $x * (y * (y * (x * (x * y)))) \in \tilde{\xi}(Y; t)$ bacause $\tilde{\xi}(Y; t)$ is an ideal of $(Y, *, 0)$.

Note that

$$\begin{aligned}
&(x*(y*(y*x)))*(x*(y*(y*(x*(x*y)))))\\
&\overset{(I1)}{\leq} (y*(y*(x*(x*y))))*(y*(y*x))\\
&\overset{(I1)}{\leq} (y*x)*(y*(x*(x*y)))\\
&\overset{(I1)}{\leq} (x*(x*y))*x\\
&\overset{(2.4)}{=} (x*x)*(x*y) \overset{(I3)\&(K)}{=} 0 \in \tilde{\xi}(Y;t).
\end{aligned}$$

Hence $x*(y*(y*x)) \in \tilde{\xi}(Y;t)$, and therefore $\tilde{\xi}(Y;t)$ is a commutative ideal of $(Y,*,0)$. Let $\check{x}, \check{y} \in \mathbb{E}$ be such that $\check{x} \oslash \check{y} \in \tilde{f_{\mathbb{E}}}(\mathbb{E};\alpha) \cap \tilde{g_{\mathbb{E}}}(\mathbb{E};\delta)$. Then

$$(\check{x} \oslash (\check{x} \oslash \check{y})) \oslash \check{y} = (\check{x} \oslash \check{y}) \oslash (\check{x} \oslash \check{y}) = 0 \in f_{\mathbb{E}}(\mathbb{E};\alpha) \cap g_{\mathbb{E}}(\mathbb{E};\delta)$$

by (I3) and (2.4), and so

$$\begin{aligned}
&(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash (\check{x} \oslash (\check{x} \oslash \check{y}))))) \oslash (\check{x} \oslash \check{y})\\
&= (\check{x} \oslash (\check{x} \oslash \check{y})) \oslash (\check{y} \oslash (\check{y} \oslash (\check{x} \oslash (\check{x} \oslash \check{y}))))\\
&\in f_{\mathbb{E}}(\mathbb{E};\alpha) \cap g_{\mathbb{E}}(\mathbb{E};\delta) \subseteq \tilde{f_{\mathbb{E}}}(\mathbb{E};\alpha) \cap \tilde{g_{\mathbb{E}}}(\mathbb{E};\delta)
\end{aligned}$$

since $f_{\mathbb{E}}(\mathbb{E};\alpha)$ and $g_{\mathbb{E}}(\mathbb{E};\delta)$ are commutative ideals of $(\mathbb{E},\oslash,0)$. Using (I1), (I3), (K) and (2.4), we have

$$(\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x}))) \oslash (\check{x} \oslash (\check{y} \oslash (\check{y} \oslash (\check{x} \oslash (\check{x} \oslash \check{y}))))) \leq 0.$$

Since $\tilde{f_{\mathbb{E}}}(\mathbb{E};\alpha)$ and $\tilde{g_{\mathbb{E}}}(\mathbb{E};\delta)$ are ideals of $(\mathbb{E},\oslash,0)$, it follows that

$$\check{x} \oslash (\check{y} \oslash (\check{y} \oslash \check{x})) \in \tilde{f_{\mathbb{E}}}(\mathbb{E};\alpha) \cap \tilde{g_{\mathbb{E}}}(\mathbb{E};\delta).$$

Hence $\tilde{f_{\mathbb{E}}}(\mathbb{E};\alpha)$ and $\tilde{g_{\mathbb{E}}}(\mathbb{E};\delta)$ are commutative ideals of $(\mathbb{E},\oslash,0)$ by Lemma 2.1. Consequently, $\tilde{\mathbb{M}}_{(Y,\mathbb{E})} := (\tilde{f_{\mathbb{E}}}, \tilde{g_{\mathbb{E}}}, \tilde{\xi})$ is a commutative makgeolli ideal of $(Y,\mathbb{E})$. □

Acknowledgment

The author, Seok-Zun Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

References

[1] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, *Computers & Mathematics with Applications*, **59** (2010), 3458–3463.

[2] S. S. Ahn, S. Z. Song, Y. B. Jun and H. S. Kim, Makgeolli Structures and Its Application in BCK/BCI-Algebras, *Mathematics* 2019, **7**, 784; doi: 10.3390/math7090784

[3] H. Aktaş and N. Çağman, Soft sets and soft groups, *Information Sciences*, **177** (2007), 2726–2735.

[4] A. O. Atagün and A. Sezgin, Soft substructures of rings, fields and modules, *Computers & Mathematics with Applications*, **61** (2011), 592–601.

[5] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, *Computers & Mathematics with Applications*, **56** (2008), 2621–2628.

[6] Y. S. Huang, BCI-algebra, Science Press: Beijing, China, 2006.

[7] K. Iséki, On BCI-algebras, *Mathematics seminar notes*, **8** (1980), 125–130.

[8] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, *Mathematica Japonica*, **23** (1978), 1–26.

[9] Y. B. Jun, Soft BCK/BCI-algebras, *Computers & Mathematics with Applications*, **56** (2008), 1408–1413.

[10] Y. B. Jun, Union soft sets with applications in BCK/BCI-algebras, *Bulletin of the Korean Mathematical Society*, **50**(6) (2013), 1937-–1956. http://dx.doi.org/10.4134/BKMS.2013.50.6.1937

[11] Y. B. Jun, H. S. Kim and J. Neggers, Pseudo d-algebras, *Information Sciences*, **179** (2009), 1751–1759.

[12] Y. B. Jun, K. J. Lee and A. Khan, Soft ordered semigroups, *Mathematical Logic Quarterly*, **56** (2010), 42–50.

[13] Y. B. Jun, K.J. Lee and C. H. Park, Soft set theory applied to ideals in d-algebras, Comput. Math. Appl. **57** (2009), 367–378.

[14] Y. B. Jun, K. J. Lee and M. S. Kang, Intersectional soft sets and applications to BCK/BCI-algebras, *Computers & Mathematics with Applications*, **28** (2013), no. 1, 11-–24 http://dx.doi.org/10.4134/CKMS.2013.28.1.011

[15] Y. B. Jun, K. J. Lee and E. H. Roh, Intersectional soft BCK/BCI-ideals, *Annals of Fuzzy Mathematics and Informatics*, **4**(1) (2012), 1–7.

[16] Y. B. Jun, K. J. Lee and J. Zhan, Soft p-ideals of soft BCI-algebras, *Computers & Mathematics with Applications*, **58** (2009), 2060–2068.

[17] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, *Information Sciences*, **178** (2008), 2466–2475.

[18] M. A. Kologani, M. M. Takallo, Y. B. Jun and R. A. Borzooei, Makgeolli structure on hoops, *Applied and Computational Mathematics*, **21**(2) (2022), 178–192. DOI:10.30546/1683-6154.21.2.2022.178

[19] J. Meng, On ideals in BCK-algebras, *Mathematica Japonica*, 40(1) (1994), 143–154.

[20] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co.: Seoul, Korea, 1994.

[21] D. Molodtsov, Soft set theory - First results, *Computers & Mathematics with Applications*, **37** (1999), 19–31.

[22] C. H. Park, Y. B. Jun and M. A. Öztürk, Soft WS-algebras, *Communications of the Korean Mathematical Society*, **23** (2008), 313–324.

[23] P. M. Pu and Y. M. Liu, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, *Journal of Mathematical Analysis and Applications*, **76** (1980), 571–599.

[24] S. Z. Song, K. J. Lee and Y. B. Jun, Intersectional soft sets applied to subalgebras/ideals in BCK/BCI-algebras, *Advanced Studies in Contempory Mathematics*, **23**(3) (2013), 509–524.

[25] S. Z. Song, M. A. Özt urk and Y. B. Jun, Positive implicative makgeolli ideals of BCK-algebras, European Journal of Pure and Applied Mathematics, **15** (2022), no. 4, 1498–1511. DOI:https://doi.org/10.29020/nybg.ejpam.v15i4.4522

[26] L. A. Zadeh, "From Circuit Theory to System Theory," in Proceedings of the IRE, vol. 50, no. 5, pp. 856-865, May 1962, doi:10.1109/JRPROC.1962.288302

[27] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8**(3) (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

[28] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU) - an outline, *Information Sciences*, **172** (2005), 1–40.