A general composite iterative algorithm for monotone mappings and pseudocontractive mappings in Hilbert spaces

Jong Soo Jung

Department of Mathematics, Dong-A University, Busan 49315, Korea
E-mail: jungjs@dau.ac.kr

Abstract

In this paper, we introduce a general composite iterative algorithm for finding a common element of the set of solutions of variational inequality problem for a hemicontinuous monotone mapping and the set of fixed points of a hemicontinuous pseudocontractive mapping in a Hilbert space. Under suitable control conditions, we establish strong convergence of the sequence generated by the proposed iterative algorithm to a common element of two sets, which is the unique solution of a certain variational inequality related to a boundedly Lipschitzian and strongly monotone mapping. As a consequence, we obtain the unique minimum-norm common point of two sets.

MSC: 47H06, 47H09, 47H10, 47J20, 49J40, 47J25, 47J05.

Key words: Iterative algorithm, Hemicontinuous monotone mapping, Hemicontiunous pseudocontractive mapping, Boundedly Lipschitzian, η-Strongly monotone mapping, Variational inequality, Fixe points.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be self-mapping on C. We denote by $Fix(S)$ the set of fixed points of S.

Let A be a nonlinear mapping of C into H. The variational inequality problem (shortly, VIP) is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denote the set of solutions of the VIP (1.1) by $VI(C, A)$. The variational inequality problem has been extensively studied in the literature; see [4,14,15,24] and the references therein.

A fixed point problem (shortly, FPP) is to find a fixed point z of a nonlinear mapping $T : C \to C$ with property:

$$z \in C, \quad Tz = z. \tag{1.2}$$

Fixed point theory is one of the most powerful and important tools of modern mathematics and may be considered a core subject of nonlinear analysis.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall that a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *k-strictly pseudocontractive* ([3]) if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx - Ty\| \le \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings as a subclass, and the class of k-strictly pseudocontractive mappings falls into the one between the class of nonexpansive mappings and the class of pseudocontractive mappings. Moreover, this inclusion is strict due to Example 5.7.1 and Example 5.7.2 in [1].

Recently, in order to study the VIP (1.1) coupled with the FPP (1.2), many authors have introduced some iterative algorithms for finding a common element of the set of the solutions of the VIP (1.1) for an inverse-strongly monotone mapping A and the set of fixed points of a nonexpansive mapping T; see [6,8,9,12,19] and the references therein. Also, some iterative algorithms for finding a common element of the set of the solutions of the VIP (1.1) for a continuous monotone mapping A more general than an inverse-strongly monotone mapping and the set of fixed points of a continuous pseudocontractive mapping T more general than a nonexpansive mapping were considered by many authors: see [20,22,26] and the references therein.

In 2001, Yamada [24] introduced the hybrid steepest descent method for the nonexpansive mapping to solve a variational inequality related to a Lipschitzian and strongly monotone mapping. Since then, in 2009, He and Xu [11] invented a hybrid iterative algorithm for the nonexpansive mapping to obtain the unique solution to the VIP (1.1) related to a boundedly Lipschitzian and strongly monotone mapping. As the result, He and Xu [11] were able to relax the global Lipschitz condition on the mapping to the weaker bounded Lipschitz condition, and improved the Yamada's result [24]. In 2010, He and Liang [10] considered the hybrid steepest descent algorithm for the strict pseudocontractive mapping more general than the nonexpansive mapping to solve a variational inequality related to a boundedly Lipschitzian and strongly monotone mapping, and extended the corresponding results in He and Xu [11].

On the other hand, by using ideas of Yamada [24], Tien [21] and Ceng *et al.* [5] provided general iterative algorithms for finding a fixed point of the nonexpansive mapping, which solves a certain variational inequality related to a Lipschitzian and strongly monotone mapping. Jung [13] gave a general iterative algorithm for finding a fixed point of the k-strictly pseudocontractive mapping.

In this paper, inspired and motivated by the above mentioned results, we introduce a general composite iterative algorithm for finding a common point of the set of solutions of the VIP (1.1) for a hemicontinuous monotone mapping A and the set of fixed points of a hemicontinuous pseudocontractive mapping T. We establish strong convergence of the sequence generated by the proposed iterative algorithm to a common point of the above two sets, which solves a certain variational inequality related to a boundedly Lipschitzian and strongly monotone mapping. As a direct consequence, we find the unique solution of the minimum-norm problem: find $x^* \in Fix(T) \cap VI(C, A)$ such that

$$\|x^*\| = \min\{\|x\| : x \in Fix(T) \cap VI(C, A)\}.$$

Our results extend and unify the corresponding results of Ceng *et al.* [5], Chen *et al.* [6], Iiduka and Takahashi [8], Jung [12], Su et al. [16], Tian [21], Wangkeeree and Nammanee [22], Zegeye [25], Zegeye and Shahzad [26], and some recent results in the literature.

2. Preliminaries and Lemmas

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. We denote by $S(x : R)$ the closed ball with center $x \in H$ and radius $R > 0$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \le \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. $P_C(x)$ is characterized by the property:

$$u = P_C(x) \Longleftrightarrow \langle x - u, u - y\rangle \ge 0, \quad \forall x \in H,\ y \in C. \tag{2.1}$$

We recall that a mapping A of H into H is called

3

(i) *monotone* if $\langle x-y, Ax-Ay\rangle \geq 0, \quad \forall x,\ y \in H$;
(ii) *α-inverse-strongly monotone* ([9,14]) if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x,\ y \in H;$$

(iii) *strongly monotone* if there exists a positive real number η such that

$$\langle x-y, Ax-Ay\rangle \geq \eta\|x-y\|^2, \quad \forall x,\ y \in H;$$

(iv) *Lipschitzian continuous* if there exists $L>0$ such that

$$\|Ax-Ay\| \leq L\|x-y\|, \quad \forall x,\ y \in H;$$

(v) *hemicontinuous* ([1,17]) if, for all $x, y \in H$, the mapping $g : [0,1] \to H$ defined by $g(t) = A(tx+(1-t)y)$ is continuous, where H has a weak topology;
(vi) *boundedly Lipschitzian* on C, if for each nonempty bounded subset S on C, there exists a positive constant $k_S > 0$ depending only on the set S such that $\|Ax-Ay\| \leq k_S\|x-y\|, \quad \forall x,\ y \in S$.

We note that (i) if A is a monotone mapping, then $T = I - A$ is a pseudocontractive mapping, and (ii) the class of the Lipschitzian mappings is a proper subclass of the class of the boundedly Lipschitzian mappings. It is easy to see that if $T : C \to H$ is continuous on C, then T is hemicontinuous on C and bounded on any line segment of C, but the converse is not true (see Example 1.10.14 in [1]).

The following lemmas can be easily proven, and therefore, we omit the proofs (see [10,24]).

Lemma 2.1. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$, and let $F : H \to H$ be a boundedly Lipschitzian and η-strongly monotone mapping with constant $\eta > 0$. Take $x_0 \in H$ arbitrarily and set $\widehat{C} = S(x_0, R)$ for some $R > 0$. Denote by $\widehat{\kappa}$ the Lipschitz constant of F on $\widehat{C}$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x-y\rangle \geq (\mu\eta - \gamma l)\|x-y\|^2, \quad \forall x,\ y \in \widehat{C}.$$

That is, $\mu F - \gamma V$ is strongly monotone on $\widehat{C}$ with constant $\mu\eta - \gamma l$.

Lemma 2.2. *Let H be a real Hilbert space H. Let $F : H \to H$ be a boundedly Lipschitzian and η-strongly monotone mapping with constant $\eta > 0$. Take $x_0 \in H$ arbitrarily and set $\widehat{C} = S(x_0, R)$ for some $R > 0$. Denote by $\widehat{\kappa}$ the Lipschitz constant of F on $\widehat{C}$ Let $0 < \mu < \frac{2\eta}{\widehat{\kappa}^2}$ and $0 < t < \rho \leq 1$. Then $G := \rho I - t\mu F$ restricted to $\widehat{C}$ is a contractive mapping with constant $\rho - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\widehat{\kappa}^2)}$.*

By a similar arguments in [2], we obtain the following lemma for the hemicontinuous monotone mapping, which extends Lemma 2.3 of Zegeye [25].

Lemma 2.3. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a hemicontinuous monotone mapping. Suppose that for each $x, y \in C$, there exists $\tau_{xy} > 0$ such that $A(tx + (1-t)y) < \tau_{xy}$ for all $t \in [0, 1]$; that is, A is bounded on any line segment on C. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Proof. Since $A : C \to H$ is a hemicontinuous mapping, for $x, y \in C$, the mapping $g : [0, 1] \to H$ defined by $g(t) = A(tx + (1-t)y)$ is continuous, where H has a weak topology, and so A is bounded on any line segment on C. Thus, by taking $f(z, y) = \langle y - z, A(z)\rangle$ as a bifunction $f : C \times C \to \mathbb{R}$ in [2], the result follows from a similar argument in [2].

Moreover, by a similar argument in [7,18] together with Lemma 2.3, we have the following lemma, which improves Lemma 2.4 of Zegeye [25].

Lemma 2.4. *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a hemicontinuous monotone mapping. Suppose that for each $x, y \in C$, there exists $\tau_{xy} > 0$ such that $A(tx + (1-t)y) < \tau_{xy}$ for all $t \in [0, 1]$; that is, A is bounded on any line segment on C. For $\lambda > 0$ and $x \in H$, define $A_\lambda : H \to C$ by*

$$A_\lambda x = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\}.$$

Then the following hold:

(i) *A_λ is single-valued;*
(ii) *A_λ is firmly nonexpansive, that is,*

$$\|A_\lambda x - A_\lambda y\|^2 \leq \langle x - y, A_\lambda x - A_\lambda y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(A_\lambda) = VI(C, A)$;*
(iv) *$VI(C, A)$ is a closed convex subset of C*

Proof. Let $f(z, y) = \langle y - z, Az\rangle$ as a bifunction $f : C \times C \to \mathbb{R}$ in [7]. Then the result follows from similar arguments in [2] and [7].

Applying Lemma 2.3 and lemma 2.4, we get the following lemmas for the hemicontinuous pseudocontractive mapping, which generalize Lemma 3.1 and Lemma 3.2 of Zegeye [25], respectively.

Lemma 2.5. Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a hemicontinuous pseudocontractive mapping. Suppose that T is bounded on any line segment on C. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

Proof. Let $A := I - T$, where I is the identity mapping on C. Then, T is a hemicontinuous pseudocontractive mapping and T is bounded on any line segment of C, A is clearly hemicontinuous monotone mapping and bounded on any line segment of C. Thus, by Lemma 2.3, there exists $z \in C$ such that $\langle y - z, Az\rangle + (1/r)\langle y - z, z - x\rangle \geq 0$ for all $y \in C$. But this is equivalent to $\langle y - z, Tz\rangle - (1/r)\langle y - z, (1 + r)z - x\rangle \leq 0$ for all $y \in C$. Hence the result holds.

Lemma 2.6. *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a hemicontinuous pseudocontractive mapping. Suppose that T is bounded on any line segment on C. For $r > 0$ and $x \in H$, define $T_r : H \to C$ by*

$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*
(iv) *$Fix(T)$ is a closed convex subset of C*

Proof. We note that $\langle y - z, Tz\rangle - (1/r)\langle y - z, (1 + r)z - x\rangle \leq 0$, for all $y \in C$, is equivalent to $\langle y - z, Az\rangle + (1/r)\langle y - z, z - x\rangle \geq 0$, for all $y \in C$, where $A := I - T$ is a hemicontinuous monotone mapping and I is the identity mapping on C. Moreover, as T is a self-mapping, we get that $VI(C, A) = Fix(T)$. Thus, by Lemma 2.4, the conclusions of (i)–(iv) hold.

We also need the following lemmas for the proof of our main results.

Lemma 2.7. *In a real Hilbert space H, there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Lemma 2.8. ([23]) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \beta_n + \gamma_n, \quad \forall n \ge 1,$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=1}^{\infty}(1-\lambda_n) = 0$*;*
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\lambda_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\beta_n| < \infty$;
(iii) $\gamma_n \ge 0$ $(n \ge 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$;
- C is a nonempty closed convex subset of H;
- $A : C \to H$ is a hemicontinuous monotone mapping with $VI(C,A) \neq \emptyset$ and is bounded on any line segment of C;
- $T : C \to C$ is a hemicontinuous pseudocontractive mapping with $Fix(T) \neq \emptyset$ and is bounded on any line segment of C;
- $A_{\lambda_n} : H \to C$ is a mapping defined by

  $$A_{\lambda_n}x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{\lambda_n}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C \right\},$$

  where $\{\lambda_n\} \subset (0,\infty)$;
- $T_{r_n} : H \to C$ is a mapping defined by

  $$T_{r_n}x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n}\langle y - z, (1+r)z - x \rangle \le 0, \quad \forall y \in C \right\},$$

  where $\{r_n\} \subset (0,\infty)$;
- $F : H \to H$ is a boundedly Lipschitzian and η-strongly monotone mapping with constant $\eta > 0$;
- $V : H \to H$ is an l-Lipschitzian mapping with constant $l > 0$;
- $\Omega := VI(C,A) \cap Fix(T) \neq \emptyset$

By Lemma 2.4 and Lemma 2.6, we note that A_{λ_n} and T_{r_n} are firmly nonexpansive and so nonexpansive, and $VI(C,A) = Fix(A_{\lambda_n})$ and $Fix(T_{r_n}) = Fix(T)$.

Now, we present a new composite iterative algorithm for hemicontinuous monotone mappings and hemicontinuous pseudocontractive mappings and establish strong convergence of this algorithm.

Theorem 3.1. *Let $x_0 \in \Omega$ be chosen arbitrarily. Set $\widehat{C} = S(x_0, \frac{\gamma\|Vx_0\|+\mu\|Fx_0\|}{\tau-\gamma l}) \cap C$ and denote by $\widehat{\kappa}$ the Lipschitz constant of F on $\widehat{C}$, where the constants μ, γ and τ are such that $0 < \mu < \frac{2\eta}{\widehat{\kappa}^2}$, $0 \le \gamma l < \tau$ and, $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\widehat{\kappa}^2)}$, respectively. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n, \\ x_{n+1} = (1-\beta_n) y_n + \beta_n T_{r_n} A_{\lambda_n} y_n, \quad \forall n \ge 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1)$ and $\{\lambda_n\}$, $\{r_n\} \subset (0,\infty)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the conditions:

(C1) $\alpha_n \to 0$ $(n \to \infty)$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(C4) $\beta_n \subset [0,a)$ *for all* $n \ge 0$ *and for some* $a \in (0,1)$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(C5) $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(C6) $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to $q \in \Omega$, which is a solution of the following variational inequality

$$\langle (\gamma V - \mu F) q, q - p \rangle \ge 0, \quad \forall p \in \Omega. \tag{3.2}$$

Proof. Note that from the condition (C1), without loss of generality, we assume that $2\alpha_n(\tau - \gamma l) < 1$ and $\alpha_n < 1 - \beta_n - \alpha_n$ for $n \ge 1$. For $K = P_\Omega$, it follows that $K(I + \gamma V - \mu F)$ is a contractive mapping of $\widehat{C}$ into Ω. In fact, from Lemma 2.2, we have, for any $x, y \in \widehat{C}$,

$$\begin{aligned} \|K(I + \gamma V - \mu F)x - &(I + \gamma V - \mu F)y\| \\ &\le \|(I + \gamma V - \mu F)x - (I + \gamma V - \mu F)y\| \\ &\le \gamma\|Vx - Vy\| + \|(I - \mu F)x - (I - \mu F)y\| \\ &\le \gamma l\|x - y\| + (1 - \tau)\|x - y\| \\ &= (1 - (\tau - \gamma l))\|x - y\|. \end{aligned}$$

This is, $K(I + \gamma V - \mu F)$ is a contractive mapping with constant $(1 - (\tau - \gamma l))$. Since $\widehat{C}$ is complete, there exists a unique element $q \in \widehat{C}$ such that $q = P_\Omega(I + \gamma V - \mu F)q$. Equivalently, by (2.1), q is the unique solution of the variational inequality:

$$\langle (\gamma V - \mu F) q, q - p \rangle \ge 0, \quad \forall p \in \Omega.$$

In fact, noting that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \iff \widehat{\kappa} \ge \eta$, it follows from Lemma 2.1 that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y \rangle \ge (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu F - \gamma V$ is strongly monotone on $\widehat{C}$ for $0 \le \gamma l < \tau \le \mu\eta$. Hence the variational inequality (3.2) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the variational inequality (3.2):

From now, put $z_n = A_{\lambda_n} x_n$, $u_n = T_{r_n} z_n$, $w_n = A_{\lambda_n} y_n$, and $v_n = T_{r_n} w_n$ for every $n \ge 0$.

Now, we divide the proof into several steps.

Step 1. We show that $x_n \in \widehat{C}$ for all $n \ge 0$ by induction, and hence $\{x_n\}$ is bounded. It is obvious that $x_0 \in \widehat{C}$. First of all, from Lemma 2.4 (iii) and Lemma 2.6 (iii), we observe that $VI(C, A) = Fix(A_{\lambda_n})$ and $Fix(T) = Fix(T_{r_n})$. Then, it follows that

$$\|z_n - x_0\| = \|A_{\lambda_n} x_n - x_0\| \le \|x_n - x_0\|,$$

and

$$\|w_n - x_0\| = \|A_{\lambda_n} y_n - x_0\| \le \|y_n - x_0\|.$$

Now, suppose that we have proved $x_n \in \widehat{C}$, that is,

$$\|x_n - x_0\| \le \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}.$$

Using lemma 2.2, Lemma 2.4 (ii), and Lemma 2.6 (ii), we derive that

$$\begin{aligned}
\|y_n - x_0\| &= \|\alpha_n(\gamma V x_n - \mu F x_0) + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n - (I - \alpha_n \mu F) x_0\| \\
&\le \|(I - \alpha_n \mu F) T_{r_n} z_n - (I - \alpha_n \mu F) x_0\| + \|\alpha_n(\gamma V x_n - \mu F x_0)\| \\
&\le (1 - \tau\alpha_n)\|z_n - x_0\| + \alpha_n \gamma \|V x_n - V x_0\| + \alpha_n \|\gamma V x_0 - \mu F x_0\| \\
&\le (1 - \tau\alpha_n)\|x_n - x_0\| + \alpha_n \gamma l \|x_n - x_0\| + \alpha_n \|\gamma V x_0 - \mu F x_0\| \\
&\le (1 - (\tau - \gamma l)\alpha_n)\|x_n - x_0\| + (\tau - \gamma l)\alpha_n \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l} \\
&\le \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}.
\end{aligned}$$

This implies $y_n \in \widehat{C}$ and

$$\begin{aligned}
\|x_{n+1} - x_0\| &= \|(1 - \beta_n)(y_n - x_0) + \beta_n(T_{r_n} A_{\lambda_n} y_n - x_0)\| \\
&\le \|(1 - \beta_n)\|y_n - x_0\| + \beta_n \|T_{r_n} w_n - x_0\| \\
&\le (1 - \beta_n)\|y_n - x_0\| + \beta_n \|w_n - x_0\| \\
&\le (1 - \beta_n)\|y_n - x_0\| + \beta_n \|y_n - x_0\| \\
&= \|y_n - p\| \\
&\le \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}.
\end{aligned}$$

It prove that $x_{n+1} \in \widehat{C}$. Therefore, $x_n \in \widehat{C}$ for all $n \ge 0$. Thus, $\{x_n\}$ is bounded.

It is not difficult to verify that that the sequences $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{Vx_n\}$, $\{Fx_n\}$, $\{Fy_n\}$, $\{Fu_n\}$, are bounded. Moreover, since $\|u_n - x_0\| = \|T_{r_n}z_n - x_0\| \le \|x_n - x_0\|$ and $\|v_n - x_0\| = \|T_{r_n}w_n - x_0\| \le \|y_n - x_0\|$, $\{u_n\}$ and $\{v_n\}$ are also bounded. And, by the condition (C1), we have

$$\begin{aligned} \|y_n - u_n\| &= \|y_n - T_{r_n}z_n\| \\ &= \alpha_n\|\gamma Vx_n - \mu FT_{r_n}z_n\| \\ &\le \alpha_n(\gamma\|Vx_n\| + \mu\|Fu_n\|) \to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.3}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$. Indeed, since $z_n = A_{\lambda_n}x_n$ and $z_{n-1} = A_{\lambda_{n-1}}x_{n-1}$, we have

$$\langle y - z_n, Az_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.4}$$

and

$$\langle y - z_{n-1}, Az_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle y - z_{n-1}, z_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C, \tag{3.5}$$

Putting $y := z_{n-1}$ in (3.4) and $y := z_n$ in (3.5), we get

$$\langle z_{n-1} - z_n, Az_n\rangle + \frac{1}{\lambda_n}\langle z_{n-1} - z_n, z_n - x_n\rangle \ge 0, \tag{3.6}$$

and

$$\langle z_n - z_{n-1}, Az_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle z_n - z_{n-1}, z_{n-1} - x_{n-1}\rangle \ge 0. \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$\langle z_n - z_{n-1}, Az_{n-1} - Az_n\rangle + \left\langle z_n - z_{n-1}, \frac{z_{n-1} - x_{n-1}}{\lambda_{n-1}} - \frac{z_n - x_n}{\lambda_n}\right\rangle \ge 0,$$

which implies

$$-\langle z_n - z_{n-1}, Az_n - Az_{n-1}\rangle + \left\langle z_n - z_{n-1}, \frac{z_{n-1} - x_{n-1}}{\lambda_{n-1}} - \frac{z_n - x_n}{\lambda_n}\right\rangle \ge 0. \tag{3.8}$$

Since A is monotone, from (3.8) we get

$$\left\langle z_n - z_{n-1}, \frac{z_{n-1} - x_{n-1}}{\lambda_{n-1}} - \frac{z_n - x_n}{\lambda_n}\right\rangle \ge 0,$$

and hence

$$\left\langle z_n - z_{n-1}, z_{n-1} - z_n + z_n - x_{n-1} - \frac{\lambda_{n-1}}{\lambda_n}(z_n - x_n)\right\rangle \ge 0.$$

Without loss of generality, let us assume that there exists a real number λ such that $\lambda_n > \lambda > 0$ for all $n \geq 0$. Then we have

$$\begin{aligned} \|z_n - z_{n-1}\|^2 &\leq \left\langle z_n - z_{n-1}, x_n - x_{n-1} + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)(z_n - x_n)\right\rangle \\ &\leq \|z_n - z_{n-1}\|\left\{\|x_n - x_{n-1}\| + \left|1 - \frac{\lambda_{n-1}}{\lambda_n}\right|\|z_n - x_n\|\right\}, \end{aligned} \tag{3.9}$$

and hence from (3.9) we obtain

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{\lambda_n}|\lambda_n - \lambda_{n-1}|\|z_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_1, \end{aligned} \tag{3.10}$$

where $L_1 = \sup\{\|z_n - x_n\| : n \geq 0\} < \infty$. Using the same method, we also get

$$\|w_n - w_{n-1}\| \leq \|y_n - y_{n-1}\| + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2, \tag{3.11}$$

where $L_2 = \sup\{\|w_n - y_n\| : n \geq 0\} < \infty$.

Moreover, since $u_{n-1} = T_{r_{n-1}}z_{n-1}$ and $u_n = T_{r_n}z_n$, we have

$$\langle y - u_{n-1}, Tu_{n-1}\rangle - \frac{1}{r_{n-1}}\langle y - u_{n-1}, (1 + r_{n-1})u_{n-1} - z_{n-1}\rangle \leq 0, \quad \forall y \in C, \tag{3.12}$$

and

$$\langle y - u_n, Tu_n\rangle - \frac{1}{r_n}\langle y - u_n, (1 + r_n)u_n - z_n\rangle \leq 0, \quad \forall y \in C, \tag{3.13}$$

Putting $y := u_n$ in (3.12) and $y := u_{n-1}$ in (3.13), we get

$$\langle u_n - u_{n-1}, Tu_{n-1}\rangle - \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, (1 + r_{n-1})u_{n-1} - z_{n-1}\rangle \leq 0, \tag{3.14}$$

and

$$\langle u_{n-1} - u_n, Tu_n\rangle - \frac{1}{r_n}\langle u_{n-1} - u_n, (1 + r_n)u_n - z_n\rangle \leq 0. \tag{3.15}$$

Adding (3.14) and (3.15), we obtain

$$\begin{aligned} &\langle u_n - u_{n-1}, Tu_{n-1} - T_{u_n}\rangle \\ &\quad - \left\langle u_n - u_{n-1}, \frac{(1 + r_{n-1})u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{(1 + r_n)u_n - z_n}{r_n}\right\rangle \leq 0, \end{aligned}$$

which implies that

$$\begin{aligned} &\langle u_n - u_{n-1}, (u_n - Tu_n) - (u_{n-1} - Tu_{n-1})\rangle \\ &\qquad\qquad - \left\langle u_n - u_{n-1}, \frac{u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{u_n - z_n}{r_n}\right\rangle \leq 0. \end{aligned}$$

Now, since T is pseudocontractive, we obtain

$$\left\langle u_n - u_{n-1}, \frac{u_{n-1} - z_{n-1}}{r_{n-1}} - \frac{u_n - z_n}{r_n} \right\rangle \geq 0,$$

and hence

$$\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - z_{n-1} - \frac{r_{n-1}}{r_n}(u_n - z_n)\rangle \geq 0.$$

Also, we can assume that $r_n > r > 0$ for all n and for some $r > 0$. Thus, using the method in (3.9) and (3.10), we deduce

$$\|u_n - u_{n-1}\| \leq \|z_n - z_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_3, \tag{3.17}$$

where $L_3 = \sup\{\|u_n - z_n\| : n \geq 0\}$. Also, using the same method, we have

$$\|v_n - v_{n-1}\| \leq \|w_n - w_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_4, \tag{3.18}$$

where $L_4 = \sup\{\|v_n - w_n\| : n \geq 0\}$.

Now, simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= \alpha_n\gamma V x_n + (I - \alpha_n\mu F)T_{r_n}A_{\lambda_n}x_n - \alpha_{n-1}\gamma V x_{n-1} \\ &\quad - (I - \alpha_{n-1}\mu F)T_{r_{n-1}}A_{\lambda_{n-1}}x_{n-1} \\ &= \alpha_n\gamma V x_n + (I - \alpha_n\mu F)T_{r_n}z_n - \alpha_{n-1}\gamma V x_{n-1} \\ &\quad - (I - \alpha_{n-1}\mu F)T_{r_{n-1}}z_{n-1} \\ &= (\alpha_n - \alpha_{n-1})(\gamma V x_{n-1} - \mu F u_{n-1}) + \alpha_n\gamma(V x_n - V x_{n-1}) \\ &\quad + (I - \alpha_n\mu F)u_n - (I - \alpha_n\mu F)u_{n-1}. \end{aligned}$$

By (3.17) and Lemma 2.2, we obtain

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq |\alpha_n - \alpha_{n-1}|(\gamma\|V x_{n-1}\| + \mu\|F u_{n-1}\|) \\ &\quad + \alpha_n\gamma l\|x_n - x_{n-1}\| + (1 - \tau\alpha_n)\|u_n - u_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|(\gamma\|V x_{n-1}\| + \mu\|F u_{n-1}\|) + \alpha_n\gamma l\|x_n - x_{n-1}\| \\ &\quad + (1 - \tau\alpha_n)\|z_n - z_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_3. \end{aligned} \tag{3.19}$$

Also, observe that

$$\begin{aligned} x_{n+1} - x_n &= (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(T_{r_{n-1}}w_{n-1} - y_{n-1}) \\ &\quad + \beta_n(T_{r_n}w_n - T_{r_{n-1}}w_{n-1}) \\ &= (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(v_{r_{n-1}} - y_{n-1}) \\ &\quad + \beta_n(v_n - v_{n-1}). \end{aligned} \tag{3.20}$$

By (3.10), (3.11), (3.18), (3.19), and (3.20), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \beta_n\|v_n - v_{n-1}\| \\
&\le (1-\beta_n)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \beta_n\|w_n - w_{n-1}\| + \frac{1}{r}|r_n - r_{n-1}|L_4 \\
&\le (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2 + \frac{1}{r}|r_n - r_{n-1}|L_4 \\
&= \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2 + \frac{1}{r}|r_n - r_{n-1}|L_4 \qquad (3.21) \\
&\le \gamma l\alpha_n\|x_n - x_{n-1}| + (1-\tau\alpha_n)\|z_n - z_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\gamma\|x_{n-1}\| + \mu\|Fu_{n-1}\|) + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|L_2 + \frac{1}{r}|r_n - r_{n-1}|(L_3 + L_4) \\
&\le (1-(\tau-\gamma l)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\gamma\|Vx_{n-1}\| + \mu\|Fu_{n-1}\|) \\
&\quad + |\beta_n - \beta_{n-1}|(\|v_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \frac{1}{\lambda}|\lambda_n - \lambda_{n-1}|(L_1 + L_2) + \frac{1}{r}|r_n - r_{n-1}|(L_3 + L_4) \\
&\le (1-(\tau-\gamma l)\alpha_n)\|x_n - x_{n-1}\| + M_1|\alpha_n - \alpha_{n-1}| + M_2|\beta_n - \beta_{n-1}| \\
&\quad + M_3|\lambda_n - \lambda_{n-1}| + M_4|r_n - r_{n-1}|,
\end{aligned}$$

where $M_1 = \sup\{\gamma\|Vx_n\| + \mu\|Fu_n\| : n \ge 0\}$, $M_2 = \sup\{\|v_n\| + \|y_n\| : n \ge 0\}$, $M_3 = \frac{1}{\lambda}(L_1 + L_2)$ and $M_4 = \frac{1}{r}(L_3 + L_4)$. From the conditions (C1) – (C6), it is easy to see that

$$\lim_{n\to\infty}(\tau-\gamma l)\alpha_n = 0, \quad \sum_{n=1}^{\infty}(\tau-\gamma l)\alpha_n = \infty,$$

and

$$\sum_{n=2}^{\infty}(M_1|\alpha_n - \alpha_{n-1}| + M_2|\beta_n - \beta_{n-1}| + M_3|\lambda_n - \lambda_{n-1}| + M_4|r_n - r_{n-1}|) < \infty.$$

Applying Lemma 2.8 to (3.21), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Moreover, by (3.10) and (3.19), we also have

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Indeed,

$$\begin{aligned}\|x_{n+1} - y_n\| &= \beta_n\|v_n - y_n\| \\ &\le \beta_n(\|v_n - u_n\| + \|u_n - y_n\|) \\ &\le a(\|w_n - z_n\| + \|u_n - y_n\|) \\ &\le a(\|y_n - x_n\| + \|u_n - y_n\|) \\ &\le a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|u_n - y_n\|)\end{aligned}$$

which implies that

$$\|x_{n+1} - y_n\| \le \frac{a}{1-a}(\|x_{n+1} - x_n\| + \|u_n - y_n\|).$$

Obviously, by (3.3) and Step 2, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \ \text{ as } n \to \infty. \tag{3.22}$$

By (3.2) and (3.22), we also have

$$\|x_n - u_n\| \le \|x_n - y_n\| + \|y_n - u_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and $\lim_{n\to\infty}\|y_n - z_n\| = 0$. To this end, let $p \in \Omega$. Since $Fix(T) = Fix(T_{r_n})$ by Lemma 2.6 (iii), from Lemma 2.2, we have

$$\begin{aligned}&\|y_n - p\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \alpha_n\mu F)T_{r_n}A_{\lambda_n}x_n - (I - \alpha_n\mu F)p\|^2 \\ &\le (\alpha_n\|\gamma V x_n - \mu F p\| + \|(I - \alpha_n\mu F)T_{r_n}z_n - (I - \alpha_n\mu F)T_{r_n}p\|)^2 \\ &\le \alpha_n\|\gamma V x_n - \mu F p\|^2 + (1 - \tau\alpha_n)\|z_n - p\|^2 \\ &\quad + 2\alpha_n(1 - \tau\alpha_n)\|\gamma V x_n - \mu F p\|\|z_n - p\|.\end{aligned} \tag{3.23}$$

Moreover, since $VI(C, A) = Fix(A_{\lambda_n})$ by Lemma 2.4 (iii), from Lemma 2.4 (ii), we obtain

$$\begin{aligned}\|z_n - p\|^2 &= \|A_{\lambda_n}x_n - p\|^2 \\ &\le \langle A_{\lambda_n}x_n - A_{\lambda_n}p, x_n - p\rangle^2 \\ &= \langle z_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|z_n - p\|^2 + \|x_n - p\|^2 - \|x_n - z_n\|^2),\end{aligned}$$

and hence

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2. \tag{3.24}$$

Therefore, from (3.23) and (3.24), we deduce

$$\begin{aligned}\|y_n - p\|^2 \le \alpha_n\|\gamma V x_n - \mu F p\|^2 + (1-\tau\alpha_n)(\|x_n - p\|^2 - \|x_n - z_n\|^2)\\ + 2\alpha_n(1-\tau\alpha_n)\|\gamma V x_n - \mu F p\|\|z_n - p\|,\end{aligned}$$

and hence

$$\begin{aligned}&(1-\tau\alpha_n)\|x_n - z_n\|^2\\ \le&\alpha_n\|\gamma V x_n - \mu F p\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|)\\ &+ 2\alpha_n\|\gamma V x_n - \mu F p\|\|z_n - p\|\\ \le& \alpha_n\|\gamma V x_n - \mu F p\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|\\ &+ 2\alpha_n\|\gamma V x_n - \mu F p\|\|z_n - p\|.\end{aligned}$$

Since $\alpha_n \to 0$ by condition (C1) and $\|x_n - y_n\| \to 0$ by (3.22), we get $\|x_n - z_n\| \to 0$. Also, from (3.22), it follows that

$$\|y_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\| \to 0 \ (n \to \infty). \tag{3.25}$$

Step 5. We show that $\lim_{n\to\infty}\|u_n - z_n\| = \|T_{r_n} z_n - z_n\| = 0$. Indeed, from (3.3) and (3.25), we get

$$\|u_n - z_n\| = \|T_{r_n} z_n - z_n\| \le \|u_n - y_n\| + \|y_n - z_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 6. We show that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F))q, y_n - q\rangle \le 0,$$

where q is the unique solution of the variational inequality (3.2). First of all, from (3.3) and Step 4, without of loss generality, we may assume that u_n, z_n in $\widehat{C}$ for all $n \ge 0$.

First we prove that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, u_n - q\rangle \le 0.$$

To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)q, u_n - q\rangle = \lim_{i\to\infty}\langle(\gamma V - \mu F)q, u_{n_i} - q\rangle.$$

Since $\{u_{n_i}\}$ is bounded, we can choose a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ and $z \in H$ such that $u_{n_{i_j}} \rightharpoonup z$. Without loss of generality, we may assume that $u_{n_i} \rightharpoonup z$. Since $\widehat{C}$ is closed and convex, it is weakly closed and hence $z \in \widehat{C}$. Since $u_n - z_n \to 0$ as $n \to \infty$ by Step 5, we have $z_{n_i} \rightharpoonup z$.

Now, we show that $z \in \Omega$. First we prove that $z \in Fix(T)$. In fact, from definition z_{n_i}, we have

$$\langle y - u_{n_i}, Tu_{n_i}\rangle - \frac{1}{r_{n_i}}\langle y - u_{n_i}, (1 + r_{n_i})u_{n_i} - z_{n_i}\rangle \le 0, \quad \forall y \in C. \tag{3.26}$$

Put $z_t = tv + (1 - t)z$ for all $t \in (0, 1]$ and $v \in C$. Then $z_t \in C$ and from (3.26) and pseudocontractivity of T, it follows that

$$\begin{aligned}\langle u_{n_i} - z_t, Tz_t\rangle \ge{}& \langle u_{n_i} - z_t, Tz_t\rangle + \langle z_t - u_{n_i}, Tu_{n_i}\rangle \\ &- \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, (1 + r_{n_i})u_{n_i} - z_{n_i}\rangle \\ ={}& -\langle z_t - u_{n_i}, Tz_t - Tu_{n_i}\rangle - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - z_{n_i}\rangle \\ &- \langle z_t - u_{n_i}, u_{n_i}\rangle \\ \ge{}& -\|z_t - u_{n_i}\|^2 - \frac{1}{r_{n_i}}\langle z_t - u_{n_i}, u_{n_i} - z_{n_i}\rangle \\ &- \langle z_t - u_{n_i}, u_{n_i}\rangle \\ ={}& -\langle z_t - u_{n_i}, z_t\rangle - \langle z_t - u_{n_i}, \frac{u_{n_i} - z_{n_i}}{r_{n_i}}\rangle.\end{aligned} \tag{3.27}$$

Since $u_n - z_n \to 0$ as $n \to \infty$ by Step 5 and $\liminf_{n\to\infty} r_n > 0$ by condition (C6), we have $\frac{u_{n_i} - z_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$. Therefore, as $i \to \infty$ in (3.27), it follows that

$$\langle z - z_t, Tz_t\rangle \ge \langle z - z_t, z_t\rangle,$$

and hence

$$-\langle v - z, Tz_t\rangle \ge -\langle v - z, z_t\rangle, \quad \forall v \in C.$$

Letting $t \to 0$ and using the fact that T is hemicontinuous, we have

$$-\langle v - z, Tz\rangle \ge -\langle v - z, z\rangle, \quad \forall v \in C.$$

Now, let $v = Tz$. Then we obtain that $z = Tz$ and so $z \in Fix(T)$.

Next, let us show that $z \in VI(C, A)$. From the definition of z_n, we get that

$$\langle y - z_{n_i}, Az_{n_i}\rangle + \langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \ge 0, \quad \forall y \in C. \tag{3.28}$$

Set $v_t = tv + (1 - t)z$ for all $t \in (0, 1]$ and $v \in C$. Then, it follows that $v_t \in C$. From (3.28), we have

$$\begin{aligned}\langle v_t - z_{n_i}, Av_t\rangle &\ge \langle v_t - z_{n_i}, Av_t\rangle - \langle v_t - z_{n_i}, Az_{n_i}\rangle - \langle v_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle v_t - z_{n_i}, Av_t - Az_{n_i}\rangle - \langle v_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

From the fact that $\|z_n - x_n\| \to 0$ in Step 4 and $\liminf_{n\to\infty} \lambda_n > 0$ by condition (C5), it follows that $\frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \to 0$ as $i \to \infty$.. Since A is monotone, we also have $\langle v_t - z_{n_i}, Av_t - Az_{n_i}\rangle \geq 0$. Thus, it follows that

$$0 \leq \lim_{i\to\infty} \langle v_t - z_{n_i}, Av_t\rangle = \langle v_t - z, Av_t\rangle,$$

and hence

$$\langle v - z, Av_t\rangle \geq 0, \quad \forall v \in C.$$

It $t \to 0$, the hemicontinuity A yields that

$$\langle v - z, Az\rangle \geq 0, \quad \forall v \in C.$$

This implies that $z \in VI(C, A)$. Therefore, $z \in \Omega$.

Now, since q is the unique solution of the variational inequality (3.2), from Step 5, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \langle(\gamma V - \mu F)q, u_n - q\rangle \\
&= \lim_{i\to\infty} \langle(\gamma V - \mu F)q, u_{n_i} - z_{n_i}\rangle + \lim_{i\to\infty} \langle(\gamma V - \mu F)q, z_{n_i} - q\rangle \\
&\leq \lim_{i\to\infty} \|(\gamma V - \mu F)q\|\|u_{n_i} - z_{n_i}\| + \lim_{i\to\infty} \langle(\gamma V - \mu F)q, z_{n_i} - q\rangle \\
&= \langle(\gamma V - \mu F)q, z - q\rangle \leq 0.
\end{aligned} \tag{3.29}$$

By (3.3) and (3.29) , we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle(\gamma V - \mu F)q, y_n - q\rangle \\
&\leq \limsup_{n\to\infty} \langle(\gamma V - \mu F)q, y_n - u_n\rangle + \limsup_{n\to\infty} \langle(\gamma V - \mu F)q, u_n - q\rangle \\
&\leq \limsup_{n\to\infty} \|(\gamma V - \mu F)q\|\|y_n - u_n\| + \limsup_{n\to\infty} \langle(\gamma V - \mu F)q, u_n - q\rangle \leq 0.
\end{aligned}$$

Step 7. We show that $\lim_{n\to\infty} \|x_n - q\| = 0$, where q is the unique solution of the variational inequality (3.2). Indeed, from (3.1), Lemma 2.2, and lemma 2.7, we derive

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 \\
&= \|\alpha_n(\gamma V x_n - \mu F q) + (I - \alpha_n \mu F)T_{r_n} A_{\lambda_n} x_n - (I - \alpha_n \mu F)q\|^2 \\
&\leq \|(I - \alpha_n \mu F)T_{r_n} z_n - (I - \alpha_n \mu F)q\|^2 + 2\alpha_n \langle \gamma V x_n - \mu F q, y_n - q\rangle \\
&\leq (1 - \tau\alpha_n)^2 \|z_n - q\|^2 + 2\alpha_n \gamma \langle V x_n - V q, y_n - q\rangle \\
&\quad + 2\alpha_n \langle \gamma V q - \mu F q, y_n - q\rangle) \\
&\leq (1 - \tau\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \gamma l \|x_n - q\|\|y_n - q\| \\
&\quad + 2\alpha_n \langle(\gamma V - \mu F)q, y_n - q\rangle
\end{aligned}$$

$$\begin{aligned}
&\le (1-\tau\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\gamma l\|x_n-q\|(\|y_n-x_n\|+\|x_n-q\|)\\
&\quad + 2\alpha_n\langle(\gamma V-\mu F)q, y_n-q\rangle\\
&= (1-2(\tau-\gamma l)\alpha_n)\|x_n-q\|^2\\
&\quad + \alpha_n^2\tau^2\|x_n-q\|^2 + 2\alpha_n\gamma l\|x_n-q\|\|y_n-x_n\|\\
&\quad + 2\alpha_n\langle(\gamma V-\mu F)q, y_n-q\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le (1-2(\tau-\gamma l)\alpha_n)\|x_n-q\|^2 + \alpha_n^2\tau^2M_5^2 + 2\alpha_n\gamma l\|y_n-x_n\|M_5\\
&\quad + 2\alpha_n\langle(\gamma V-\mu F)q, y_n-q\rangle\\
&= (1-\overline{\alpha_n})\|x_n-q\|^2 + \overline{\beta_n},
\end{aligned}$$

where $M_5 = \sup\{\|x_n-q\| : n\ge 1\}$, $\overline{\alpha_n} = 2(\tau-\gamma l)\alpha_n$ and

$$\overline{\beta_n} = \alpha_n[\alpha_n\tau^2M_5^2 + 2\gamma l\|y_n-x_n\|M_5 + 2\langle(\gamma V-\overline{F})q, y_n-q\rangle].$$

From the conditions (C1) and (C2), $\|y_n-x_n\|\to 0$ in Step 3, and Step 6, it is easily seen that $\overline{\alpha_n}\to 0$, $\sum_{n=1}^{\infty}\overline{\alpha_n}=\infty$, and $\limsup_{n\to\infty}\frac{\overline{\beta_n}}{\overline{\alpha_n}}\le 0$. Hence, by Lemma 2.8, we conclude $x_n\to q$ as $n\to\infty$. This completes the proof. □

By taking $F\equiv I$, $V\equiv 0$, $\mu=1$, $\tau=1$, and $l=0$ in Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let H, C, A, T, T_{r_n} and A_{λ_n} be as in Theorem 3.1. Let $x_0\in\Omega := Fix(T)\cap VI(C,A)$ be chosen arbitrarily and let $\widehat{C} = S(x_0,\|x_0\|)\cap C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = (1-\alpha_n)T_{r_n}A_{\lambda_n}x_n,\\ x_{n+1} = (1-\beta_n)y_n + \beta_nT_{r_n}A_{\lambda_n}y_n, \quad \forall n\ge 0,\end{cases} \tag{3.30}$$

where $\{\alpha_n\}$, $\{\beta_n\}\subset[0,1)$ and $\{\lambda_n\}$, $\{r_n\}\subset(0,\infty)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the conditions (C1) *–* (C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q\in\Omega$, which solves the following minimum-norm problem: find $x^*\in\Omega$ such that*

$$\|x^*\| = \min_{x\in\Omega}\|x\|. \tag{3.31}$$

Proof. Take $F\equiv I$, $V\equiv 0$, $\mu=1$, $\tau=1$, and $l=0$ in Theorem 3.1. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q, q-p\rangle\le 0, \quad \forall p\in\Omega.$$

This obviously implies that

$$\|q\|^2\le\langle q,p\rangle\le\|q\|\|p\|, \quad \forall p\in\Omega.$$

18

It turns out that $\|q\| \leq \|p\|$ for all $p \in \Omega$. Therefore q is the minimum-norm point of Ω. $\square$

Taking $\beta_n = 0$ for $n \geq 0$ in Theorem 3.1 and Corollary 3.1, respectively, we derive the following results.

Corollary 3.2. *Let H, C, $\widehat{C}$, A, T, T_{r_n}, A_{λ_n}, F, V, γ, τ, $\widehat{\kappa}$, η, l and μ be as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by $x_0 \in \Omega$ and*

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} A_{\lambda_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\lambda_n\}, \{r_n\} \subset (0, \infty)$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the conditions (C1), (C2), (C3), (C5) *and* (C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in \Omega$, which is the unique solution of the variational inequality* (3.2).

Corollary 3.3. *Let H, C, A, T, T_{r_n} and A_{λ_n} be as in Theorem 3.1. Let $x_0 \in \Omega$ be chosen arbitrarily and let $\widehat{C} = S(x_0, \|x_0\|) \cap C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = (1 - \alpha_n) T_{r_n} A_{r_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$. Let $\{\alpha_n\}$ and $\{\lambda_n\}$, $\{r_n\} \subset (0, \infty)$ satisfy the conditions (C1), (C2), (C3), (C5) *and* C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves the following minimum-norm problem* (3.31).

As direct consequences of Theorem 3.1 along with $\beta_n = 0$ for $n \geq 0$, we also have the following results. First, if, in Theorem 3.1, we take that $A \equiv I$, the identity mapping on C, then we obtain the following corollary.

Corollary 3.4. *Let H, C, $\widehat{C}$, A, T, T_{r_n}, F, V, γ, τ, $\widehat{\kappa}$, η, l and μ be as in Theorem 3.1. Let $x_0 \in Fix(T)$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_{r_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{r_n\} \subset (0, \infty)$. Let $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions (C1), (C2), (C3) *and* (C6) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in Fix(T)$, which is the unique solution of the variational inequality*

$$\langle (\gamma V - \mu F) q, q - p \rangle \geq 0, \quad \forall p \in Fix(T).$$

Next, if, in Theorem 3.1, $T \equiv I$ is the identity mapping on C along with $\beta_n = 0$ for $n \geq 0$, then we have the following corollary.

Corollary 3.5. *Let H, C, $\widehat{C}$, A, A_{λ_n}, F, V, γ, τ, $\widehat{\kappa}$, η, l and μ be as in Theorem 3.1. Let $x_0 \in VI(C, A)$ be chosen arbitrarily, and let $\widehat{C} = S(x_0, \frac{\gamma\|Vx_0\| + \mu\|Fx_0\|}{\tau - \gamma l}) \cap C$.*

Let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) A_{\lambda_n} x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset (0,\infty)$. Let $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the conditions (C1)*,* (C2)*,* (C3) *and* (C5) *in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in VI(C,A)$, which is the unique solution of the variational inequality*

$$\langle (\gamma V - \mu F)q, q - p \rangle \geq 0, \quad \forall p \in VI(C,A).$$

Remark 3.1.

1) Our results extend and unify most of the results that have been established for these important classes of nonlinear mappings. In particular, Theorem 3.1 and Corollary 3.2 improve Theorem 3.1 of Jung [12] and Theorem 3.1 of Wangkeeree and Nammanee [22] and Theorem 3.1 of Zegeye and Shahzad [26], respectively, in the sense that our convergence is for more general classes of nonlinear mappings such as hemicintinuos monotone mappings, hemicontinuous pseudocontractive mappings, boundedly Lipschitzian and strongly monotone mappings, and Lipschizian mappings.
2) It is worth pointing out that the variable parameters λ_n and r_n in our iterative algorithms are used in comparison with the corresponding iterative algorithms in [22,25,26].
3) Corollary 3.2 also includes Proposition 3.1 of Chen et al. [6], Theorem 3.1 of Iiduka and Takahashi [8] and Corollary 3.2 of Su et al. [16] in the convergence sense for more general classes of nonlinear mappings mentioned in 1).
4) Corollary 3.1 and Corollary 3.3 are new results for finding the minimum-norm point of $Fix(T) \cap VI(C,A)$.
5) Corollary 3.4 and Corollary 3.5 also improve the corresponding results of Chen et al. [5], Tian [21], Wangkeeree and Nammanee [22] and Zegeye and Shahzad [26] in the sense that our results are for more general classes of nonlinear mappings.
6) As in Corollary 3.1, if we take $F \equiv I$, $V \equiv 0$, $\mu = 1$, $\tau = 1$, and $l = 0$ in Corollary 3.4 and Corollary 3.5, then we can find the minimum-norm point of $Fix(T)$ and $VI(C,A)$, respectively.

Acknowledgments

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2021R1I1A3040289).

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, 2009.

[2] E. Blum and W. Oettli, From optimization and variationl inequalities, *Math. Student* 63, 113–146, (1994)

[3] F. E. Browder and W. V. Petryshn, Construction of fixed points of nonlinear mappings Hilbert space, *J. Math. Anal. Appl.* 20, 197–228 (1967).

[4] R. E. Bruck, On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space, *J. Math. Anal. Appl.* 61, 159–164 (1977).

[5] L. C. Ceng, Q. H. Ansari and J. C. Yao, Some iterative methods for finding fixed points and for solving constrained convex minimization problems, *Nonlinear Anal.* 74, 5286–5302 (2011).

[6] J. Chen, L. Zhang and T. Fan, Viscosity approximation methods for nonexpansive mappings and monotone mappings, *J. Math. Anal. Appl.* 334, 1450–1461 (2007).

[7] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.* 6, 117–136, (2005).

[8] H. Iiduka and W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, *Nonlinear Anal.* 61, 341–350 (2005).

[9] H. Iiduka, W. Takahashi and M. Toyoda, Approximation of solutions of variational inequalities for monotone mappings, *PanAmer. Math. J.* 14, 49–61 (2004).

[10] S. He and X. L. Liang, Hybrid steepest-descent methods for solving variational inequalities governed by boundedly Lipschitzian and strongly monotone operators, *Fixed Point Theory Appl.* 2010, Article ID 673932, 16 pages, doi:10.1155/2010/673932, (2010).

[11] S. He and H. K. Xu, Variational inequalites governed by boundedly Lipschitzian and strongly monotone operators, *Fixed Point Theory*, 10, 245–258, (2009).

[12] J. S. Jung, A new iteration method for nonexpansive mappings and monotone mappings in Hilbert spaces, *J. Inequal. Appl.* 2010, Article ID 251761, 16 pages, doi:10.1155/2010/251761,(2010).

[13] J. S. Jung, Strong convergence of iterative methods for k-strictly pseudo-contractive mappings in Hilbert spaces, *Applied Math. Comput.* 215, 3746–3753 (2010).

[14] F. Liu and M. Z. Nashed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, *Set-Valued Anal.* 6, 313–344 (1998).

[15] P. L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* 20, 493–517 (1967).

[16] Y. Su, M. Shang, and X. Qin, An irerative method of solution for equilibrium and optimization problems, *Nonlinear Anal.* 69, 2709–2719 (2008).

[17] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama (2000).

[18] W. Takahashi and K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, *Nonlinear Anal.* 10, 45–57 (2009).

[19] W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 118, 417–428 (2003).

[20] Y. Tang, Strong convergence of viscosity approximation methods for the fixed point of pseudo-contractive and monotone mappings, *Fixed Point Theory Appl.* 2013, doi:10.1186/1687-1812-2013-273 (2013).

[21] M. Tian, A general iterative method based on the hybrid steepest descent scheme for nonexpansive mappings in Hilbert spaces, In 2010 International Conefrence on Computational Intelligence and Soft ware Engineering, CiSE 2010, art. no. 5677064, (2010).

[22] R. Wangkeeree and K. Nammanee, New iterative methods for a common solution of fixed points for pseudo-contractive mappings and variational inequalities, *Fixed Point Theory Appl.* 2013, doi:10.1186/1687-1812-2013-233, (2013)

[23] H. K. Xu, An iterative algorithm for nonlinear operator, *J. London Math. Soc.* 66, 240–256 (2002).

[24] I. Yamada, The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings, in D. Butnariu, Y. Censor, S. Reich (Eds), *Inherently Parallel Algorithm for Feasibility and Optimization, and Their Applications,* Kluwer Academic Publishers, Dordrecht, Holland, pp. 473–504, (2001).

[25] H. Zegeye, An iterative approximation method for a common fixed point of two pseudocontractive mappings, *Interational Scholarly Reserach Network ISRN Math. Anal.* 2011 Article ID 621901, 14 pages.

[26] H. Zegeye and N. Shahzad, Strong convergence of an iterative method for pseudo-contrcative and monotone mappings, *J. Glob. Optim.* 54, 173–184 (2012).