

Intersection graph of p -subgroups of a group

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Abstract

Let G be a group. Then Intersection graph of p - subgroups of a group is denoted by $P(G)$ whose vertex set is all the proper p -subgroup of G and two vertices are adjacent if and only if they have non-trivial intersection. For this graph, we classified all the finite groups whose $P(G)$ is planar, complete, connected and diameter.

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1 Introduction

Graphs have become an essential tool in studying algebraic structures. In research there are many graphs are defined for algebraic structures and the intersection graph of subgroups of a group G , denoted by $I(G)$ is a very famous graph in this area. In 1964, Bosak [1] introduced the concept of the intersection graphs of semigroups. After that B. Cs'ak'any, G. Poll'ak [2] has began the intersection graph of subgroups of finite group and Zelinka trained the new idea intersection of subgroups of a finite abelian group [10].

In present days, many researchers gave many results on intersection graph of subgroups of groups have been given in the literature. There are many authors studied the properties for the same for instance [6], [8], [9].

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In [4], P. Devi defined a new graph namely intersection graph of p -subgroups of a group. Now we have studied the intersection graph of p -subgroups and it is defined as follows. Let G be a group. The intersection graph of p -subgroups is denoted by $P(G)$ whose vertex set is all proper p -subgroups of G and two vertices are adjacent if and only if their intersection is non trivial. Here we have characterize the groups whose $P(G)$ is planar, connected, complete and diameter. Also studied some other properties. A *simple graph* $G = (V, E)$ consists of V , a nonempty set of *vertices* and E , a set of unordered pairs of distinct elements V called *edges*. A graph $G = (V, E)$ is said to be *connected*, if for any two vertices x, y there exist a path from x to y . A simple graph in which every pair of two vertices is adjacent is called a *complete graph*. K_n is the complete graph on n vertices. A *planar graph* is a graph that can be embedded in the plane such that no edges intersect except at their endpoints. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y ; such a partition (X, Y) is called a bipartition of the graph. If G is simple and every vertex in X is joined to every vertex in Y , then G is called a *complete bipartite graph* is denoted by $K_{m,n}$ with $|X| = m$ and $|Y| = n$. A *star graph* is complete bipartite graph $K_{1,n}$. A connected acyclic graph is called *tree*. The *girth* refers to the length of the shortest cycle present within a simple graph. The *complement* of a graph is denoted by G in which any two vertices are connected by an edge if does not have an edge between them in G . Here we have studied the main properties for the same.

2 Main Results

Theorem 2.1. *Let G be a finite group. Then $P(G) \cong I(G)$, if and only if $|G| = p^\alpha$, where p is a prime and $\alpha \geq 2$ or G does not have a subgroup with composite order.*

Proof. By the definition of intersection graph of p -subgroups of group, we consider only the p -subgroups of G , so $V(P(G)) = V(I(G))$ only when $|G| = p^\alpha$, or G does not have a subgroup with composite order. So proof follows. $\square$

Theorem 2.2. *Let G be a finite group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where p_i 's are distinct prime $\alpha_i \geq 1$. Then $P(G)$ contains $P(P_1) \cup P(P_2) \cup P(P_3) \cup \dots \cup P(P_k)$ as a subgraph, where P_i 's are the Sylow p_i -subgroup of G .*

Proof. For every $i = 1, 2, \dots, k$ G has a Sylow p_i -subgroup, let it be P_i respectively. Here each P_i are the p_i -subgroup of G . So $P(P_i)$ is a subgraph of $P(G)$ for every i , and its union is a subgraph of $P(G)$. $\square$

Theorem 2.3. *Let G be a finite group. Then $P(G)$ is complete if and only if $G \cong Z_{p^\alpha}$ or Q_{2^α} .*

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Proof. Suppose G has more than one prime factor, then G has atleast two subgroups of orders p^α, q^β let it be H_1, H_2 respectively. Here H_1 and H_2 are not intersect non-trivially, so they are non-adjacent in $P(G)$. So we have $|G| = p^\alpha$. Then by the above observation $P(G) \cong I(G)$ and so $P(G)$ is complete if and only if $I(G)$ is complete. By corollary 2.2 in [8], we have $G \cong Z_{p^\alpha}$ or Q_{2^α} . $\square$

Theorem 2.4. *Let G be a finite group. Then $P(G)$ is connected if and only if G is a p -group, where p is a prime. Also $\text{diam}(P(G)) \leq 3$.*

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_i 's are distinct primes and $\alpha_i \geq 1$.

Case 1: $k \geq 2$. Let H_1, H_2 be two subgroups of G of orders $p_1^{\alpha_1}, p_2^{\alpha_2}$ respectively. Suppose there is a path $P : H_1 - K_1 - K_2 - K_3 - \dots - K_n - H_2$ from H_1 to H_2 , where $K_1, K_2, K_3, \dots, K_n$ are p -subgroups of G . Here H_1 is adjacent with K_1 and so $|K_1| = p_1^{i_1}$; K_1 is adjacent with K_2 and so $|K_2| = p_1^{i_2}$; proceeding like this, we have K_{n-1} which is adjacent with K_n and so $|K_n| = p_1^{i_n}$; Now K_n is adjacent with H_2 and so $|H_2| = p_1^{i_t}$ but $|H_2| = p_2^{\alpha_2}$, which is a contradiction. Since p_1 and p_2 are distinct primes. Therefore $P(G)$ is disconnected.

Case 2: $k = 1$. So $|G| = p_1^{\alpha_1}$. Assume that $G \not\cong Z_{p_1^{\alpha_1}}$. Let H, K be two subgroups in G of order p_1^i, p_1^j respectively. Suppose $i + j > \alpha_1$, then $H \cap K \neq \{e\}$ and they are connected by an edge in $P(G)$. Otherwise there exists a two subgroups, say H_1, H_2 in G of orders $p_1^{\alpha_1-1}, p_1^{\alpha_1-1}$ respectively with $H \subset H_1, K \subset H_2$. Now we can have a path $H - H_1 - H_2 - K$ from H to K , since $H_1 \cap H_2 \neq \{e\}$, so $P(G)$ is connected and diameter is equal to 3. If $G \cong Z_{p_1^{\alpha_1}}$, then by Theorem 2.3, $P(G)$ is complete. Combining all the cases proof follows. $\square$

Theorem 2.5. *Let G be a finite cyclic group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Then $P(G)$ is planar if and only if $\alpha_1, \alpha_2, \dots, \alpha_k \leq 4$ if $k \geq 2$ or $\alpha_1 \leq 5$ if $k = 1$.*

Proof. If $G \cong Z_{p_1^{\alpha_1}}$ then by Theorem [2.3], $P(G) \cong K_{\alpha_1-1}$, which is planar only when $\alpha_1 \leq 5$. Any finite cyclic group has an unique Sylow p -subgroups. So $P(G) \cong K_{\alpha_1-1} \cup K_{\alpha_2-1} \cup \dots \cup K_{\alpha_k-1}$, which is planar only when $\alpha_1, \alpha_2, \dots, \alpha_k \leq 4$. $\square$

Theorem 2.6. *Let G be a finite non-cyclic abelian group. Then $P(G)$ is planar if and only if $G \cong Z_p \times Z_p, Z_4 \times Z_2, Z_{pq} \times Z_p, Z_{pq} \times Z_{pq}, Z_{p^\alpha q} \times Z_q, \alpha \leq 4$.*

Proof. If $G \cong Z_p \times Z_p$, then by Theorem 2.3, $P(G) \cong I(G) \cong \overline{K_{p+1}}$. If $G \cong Z_{p^2} \times Z_p$, then by Theorem 2.3, $P(G) \cong I(G) \cong K_1 + (\overline{K_p} \cup K_{p+1})$, which is planar only when $p = 2$. If $G \cong Z_{pq} \times Z_p$, then G has an unique subgroup of orders p^2 ; which is isomorphic to $Z_p \times Z_p$, $p + 1$ subgroups of order p and an unique

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subgroup of order q . These are the only p -subgroups of G . Therefore, $P(G) \cong K_{1,p+1} \cup K_1$, which is planar. If $G \cong Z_{pq} \times Z_{pq}$, then G has a unique Sylow p -subgroup of order p^2 . Which is isomorphic to $Z_p \times Z_p$ and an unique subgroup of order q^2 , which is isomorphic to $Z_q \times Z_q$. These are the only p -subgroups of G . By the above argument, $P(G) \cong K_{1,p+1} \cup K_{1,q+1}$, which is planar. If $G \cong Z_{p^\alpha q} \times Z_q$, then G has a unique Sylow p -subgroup of order p^α which is isomorphic to Z_{p^α} and an unique subgroup of order q^2 which is isomorphic to $Z_q \times Z_q$. These are the only p -subgroups of G . Therefore, $P(G) \cong K_\alpha \cup K_{1,q+1}$ which is planar only when $\alpha \leq 4$. If $G \cong Z_{p^2 q} \times Z_p$, then G has a Sylow p -subgroup of order p^3 which is isomorphic to $Z_{p^2} \times Z_p$. By the above argument $Z_{p^2} \times Z_p$ together with its subgroups forms K_5 as a subgraph of $P(G)$. If $G \cong Z_p \times Z_p \times Z_p$, then $\langle(1,0,0), (0,1,0)\rangle, \langle(1,0,0), (0,0,1)\rangle, \langle(1,0,0), (1,1,0)\rangle, \langle(1,0,0), (0,1,1)\rangle$ are proper subgroups of G which have $\langle(1,0,0)\rangle$ as their intersection. It follows that these five subgroups forms K_5 as a subgraph of $P(G)$. If $G \cong Z_{p^2 q} \times Z_{pq}$, then G has a Sylow p -subgroup of order p^3 which is isomorphic to $Z_{p^2} \times Z_p$. By the above argument $Z_{p^2} \times Z_p$ together with its subgroups forms K_5 as a subgraph of $P(G)$. If G is a finite non-cyclic abelian group, then G has one of the following as its subgroups $Z_{p^2 q} \times Z_p$, $Z_{p^2 q} \times Z_{pq}$, $Z_{p^\alpha q} \times Z_q$, $Z_p \times Z_p \times Z_p$. It follows that the prime intersection graph of subgroups of these groups are non-planar. Combining all the cases proof follows. $\square$

Theorem 2.7. Let G be a finite non-abelian p -group. Then $P(G)$ is planar if and only if $G \cong Q_8$ or M_8 .

Proof. By the corollary 2.2 in [8], $P(G) \cong I(G)$. $\square$

Theorem 2.8. Let G be a finite non-abelian group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where p_i 's are distinct primes and $\alpha_i \geq 1, k \geq 2$. Then $P(G)$ is planar if and only if following conditions are hold.

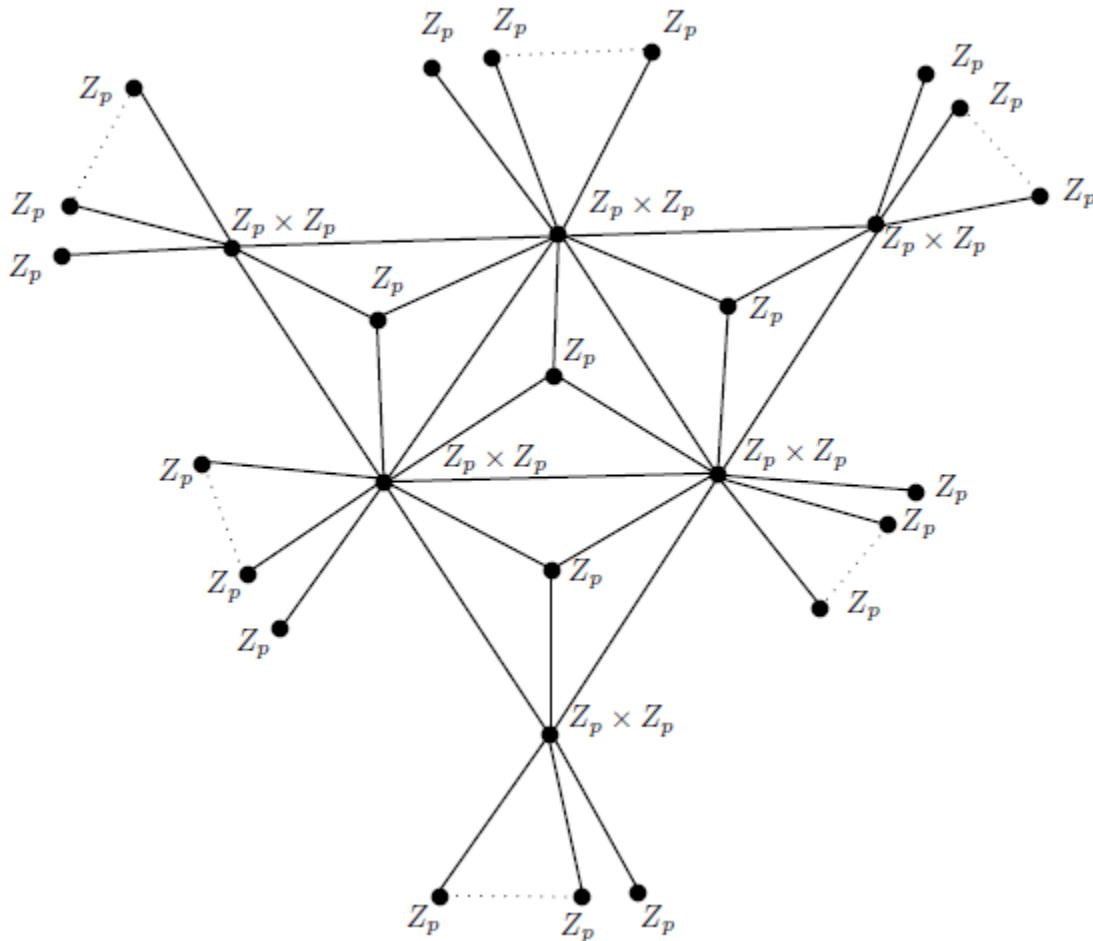
1. Every Sylow p -subgroups are isomorphic to Z_{p^2} or $Z_p \times Z_p$ with atmost three are intersect non-trivially.
2. Every Sylow p -subgroups are isomorphic to Z_{p^3} or $Z_p \times Z_p$ with atmost two of them are intersect non-trivially.
3. Every Sylow p -subgroups are isomorphic to Z_{p^4} or $Z_p \times Z_p$ with none of them are intersect non-trivially.

Proof. Let $P_1, P_2, \dots, P_k$ be the Sylow $p_1, p_2, \dots, p_k$ subgroups of G . Sylow p_i -subgroups are need not be unique so let $P_1^1, P_1^2, \dots$ be the Sylow p_1 -subgroups of G . Similarly we can define Sylow $p_2, p_3, \dots, p_k$ subgroups of G . If $P(P_i)$ is non-planar, then $P(G)$ is non-planar, So by the Theorems [2.6, 2.7], $P_i \cong Z_{p^\alpha}, \alpha \leq 5, Z_p \times$

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$Z_p, Z_4 \times Z_2, Q_8, M_8$. If $P_i \cong Z_{p^5}, Z_4 \times Z_2, Q_8, M_8$ then P_i together with its subgroups forms K_5 as a subgraph and so $P(P_i)$ is non-planar. Now we have $P_i \cong Z_{p^2}, Z_{p^3}, Z_{p^4}, Z_p \times Z_p$.

Case 1: $P_i \cong Z_{p^2}$, for all i . If atleast four P_i 's are intersect non-trivially or $P_i^1 \cap P_i^2 \cap P_i^3 \cap P_i^4 \neq \{e\}$ then $P_i^1, P_i^2, P_i^3, P_i^4$ together with their intersection forms K_5 . Otherwise $P(G)$ is planar which is the union of K_4 and its subgraphs.



Case 2: $P_i \cong Z_{p^3}$, for all i . If atleast three P_i 's are intersect non-trivially or $P_i^1 \cap P_i^2 \cap P_i^3 \neq \{e\}$ then $P_i^1, P_i^2, P_i^3, H_1, H_2$ forms K_5 , where $H_1 = P_i^1 \cap P_i^2 \cap P_i^3$ and $H_2 \subset H_1$. Otherwise $P(G)$ is planar which is the union of K_4 and its subgraphs.

Case 3: $P_i \cong Z_{p^4}$, for every i . If atleast two P_i 's are intersect non-trivially or $P_i^1 \cap P_i^2 \neq \{e\}$ then $P_i^1, P_i^2, H_1, H_2, H_3$ forms K_5 where $H_1 = P_i^1 \cap P_i^2$ and $H_3 \subset H_2 \subset H_1$. Otherwise $P(G)$ is planar which is the union of K_4 and its subgraphs.

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Case 4: $P_i \cong Z_p \times Z_p$ for every i , $P_i^1 \cap P_i^2 \cap P_i^3 \cap P_i^4 \neq \{e\} = H$ then $P_i^1, P_i^2, P_i^3, P_i^4, H$ forms K_5 as a subgraph. Otherwise $P(G)$ is planar which is the union of and its subgraph.

Case 5: $P_i \cong Z_p \times Z_p$ for $i = 1$ to n , $P_i \cong Z_{p^2}$, for $j = n + 1$ to k , $P_i^1 \cap P_i^2 \cap P_j^1 \cap P_j^2 = H$, then $P_i^1, P_i^2, P_j^1, P_j^2, H$ forms K_5 . Otherwise $P(G)$ is planar which is the union of proper subgraph of K_5 .

Case 6: $P_i \cong Z_p \times Z_p$ for $i = 1$ to n , $P_i \cong Z_{p^3}$, for $j = n + 1$ to k , $P_i^1 \cap P_i^2 \cap P_j^1 = H$, then $P_i^1, P_i^2, P_j^1, P_j^2, H, K$ forms K_5 where $H \subset K \subset P_j^1, P_j^2$. Otherwise $P(G)$ is planar which is the union of proper subgraph of K_5 .

Case 7: $P_i \cong Z_p \times Z_p$ for $i = 1$ to n , $P_i \cong Z_{p^4}$, for $j = n + 1$ to k , $P_i^1 \cap P_j^1 = H$ then $P_i^1, P_j^1, H, K_1, K_2$ forms K_5 where $H \subset K_1 \subset K_2 \subset P_j^1$. Otherwise $P(G)$ is planar which is the union of $K_{1,3}$ and K_4 . $\square$

3 Conclusion

In this paper, we analyzed the intersection graph of p -subgroups of a group and established key graph-theoretic properties such as connectivity, completeness, planarity, and diameter. These results enhance the understanding of the structural features of subgroup intersection graphs. Future work may include the study of graph indices, chromatic number, and domination-related parameters. Such extensions are expected to further strengthen the connection between group theory and graph theory.

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