Exploring Algebraic Structures in Power Monoids Derived from Numerical Monoids

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Abstract

Let $S \subseteq \mathbb{N}_0$ be a numerical monoid. We consider two algebraic constructions associated with S: the power monoid $\mathcal{P}_{\text{fin}}(S)$, consisting of all finite nonempty subsets of S, and the restricted power monoid $\mathcal{P}_{\text{fin},0}(S)$, which is formed by those finite nonempty subsets that include the zero element. Both structures are endowed with the binary operation induced by setwise addition.

Recent studies have examined various arithmetic aspects of such power monoids. In this work, we extend the existing body of research by focusing on intrinsic algebraic features, with particular emphasis on structural properties such as the prime ideal spectrum. In addition, we establish that, within the restricted power monoid $\mathcal{P}_{\text{fin},0}(S)$, the vast majority of elements cannot be decomposed into nontrivial sumsets, and hence are irreducible.

1 Introduction

The operation of adding subsets elementwise occupies a prominent place in additive combinatorics and has spawned a variety of deep problems. Factorization theory, which examines how elements of a given (additive or multiplicative) monoid decompose into indecomposable building blocks, shares many common themes with these additive investigations. A classical link appears for instance in the study of factorizations in Krull monoids; more recent work has revealed further connections by examining “power monoids,” an algebraic object introduced and developed in a series of papers by Tringali and collaborators [1, 2, 3].

Let S be an additive, commutative monoid — in particular, take S to be a numerical monoid. We denote by $\mathcal{P}_{\text{fin}}(S)$ the collection of all finite nonempty subsets of S, and by

$\mathcal{P}_{\mathrm{fin},0}(S)$ the subcollection of those finite nonempty subsets that contain the neutral element 0. Equipped with the binary operation given by setwise addition

$$A \oplus B = \{ \ \alpha + \beta \quad | \quad \alpha \in A \wedge \beta \in B \ \}$$

these collections become commutative monoids. The arithmetic behavior of such "power monoids" has attracted attention recently; besides their intrinsic interest, they relate to arithmetic phenomena in other algebraic contexts (for example, to the monoid of ideals of polynomial rings; see [4]).

We now fix some terminology used throughout the paper. If D is an additively written monoid (or, in multiplicative notation, a domain), and an element $x \in D$ admits a decomposition

$$x = u_1 + \cdots + u_k \qquad (\text{resp.}\, x = u_1 \cdots u_k)$$

into irreducible factors, then any such integer k is called a *factorization length* of x. The set of all factorization lengths of x is denoted by $\mathrm{L}(x) \subseteq \mathbb{N}_0$, and the family

$$\mathcal{L}(D) = \{\mathrm{L}(x) \colon x \in D\}$$

is one of the central arithmetic invariants in factorization theory. Under suitable algebraic finiteness hypotheses on D (for example, finiteness of the class group in the Krull setting), the elements of $\mathcal{L}(D)$ exhibit a highly constrained structure; standard references include [5]. Conversely, there are wide-ranging types of monoids that allow every finite, nonempty subset of integers not less than two to be realized as a system of lengths. Typical cases include Krull monoids with an infinite divisor class group and primes present in all classes, in addition to certain rings of integer-valued polynomials Int(D) defined over suitable Dedekind domains.
Motivated by these phenomena, Fan and Tringali conjectured that power monoids associated with many naturally occurring monoids share the extreme flexibility in their sets of lengths. We state a streamlined version of their conjecture tailored to numerical monoids.

Conjecture 1.1 (Fan–Tringali). *Let S be any numerical monoid and let $L \subseteq \mathbb{N}_{\geq 2}$ be a finite, nonempty set. Then there exists a finite nonempty subset $A \subseteq S$ such that, when viewed as an element of the power monoid $\mathcal{P}_{fin}(S)$, A admits factorizations whose lengths form precisely the set L, and no other lengths occur*.

It is straightforward to verify that, if Conjecture 1.1 is valid for numerical monoids, then analogous realisations of length sets follow for power monoids arising from many other natural classes of monoids (see [6] for details). Several partial results and supporting evidence have been obtained; for the convenience of the reader we summarise a selection of consequences for the concrete case $S = \mathbb{N}_0$ (the full proofs are available in [7, 8]).

Theorem 1.2. *We focus on the structure formed by all finite, nonempty collections of nonnegative integers when combined via set addition; this structure constitutes a monoid, and the following properties apply.*

1. *The set of distances of lengths is the entire set of positive integers, i.e.* $\Delta(\mathcal{P}_{fin}(\mathbb{N}_0)) = \mathbb{N}$.
2. *For every* $k \geq 2$ *the union of all* k*-th unions of length-sets equals* $\mathbb{N}_{\geq 2}$, *that is* $\mathcal{U}_k(\mathcal{P}_{fin}(\mathbb{N}_0)) = \mathbb{N}_{\geq 2}$.
3. *For every rational value q that is at least one, one can find a finite subset A of the nonnegative integers such that …*

$$q = \frac{\max \mathrm{L}(A)}{\min \mathrm{L}(A)}.$$

In this article we take a different route from the predominantly combinatorial constructions that have been used to produce prescribed length behaviour. Our emphasis is on the structural algebraic features of power monoids $\mathcal{P}_{\mathrm{fin}}(S)$ for numerical monoids S, in particular when S is the set of nonnegative integers. We compare these structural characteristics with those of well-studied objects in factorization theory — notably Krull monoids and rings of integer-valued polynomials — since a solid grasp of algebraic properties in those classes underlies the precise arithmetic descriptions available there. By analysing ideals, prime-like elements, and related spectrum-type invariants in power monoids, we aim to illuminate mechanisms that may explain the arithmetic flexibility conjectured above.

This article is structured as follows. The opening section introduces the notation and recalls basic results on monoids that will be used throughout the work. The subsequent section develops fundamental structural properties of power monoids. Another section is dedicated to the analysis of prime ideals and associated spectral aspects of $\mathrm{P}_{\mathrm{fin}}(\mathrm{S})$, where the main conclusions are presented in Theorems 1, 2, and 3. We then demonstrate that more delicate arithmetic parameters, in particular the ω-invariants, fail to be bounded in this framework; this observation lends additional evidence in favor of Conjecture 1.1 (see Corollary 1 and Remark 2). The final section addresses atomicity within the restricted power monoid, establishing that atoms form a density-one subset of $P_{(\text{fin},0)}(S)$ (Theorems 3 and 4), a behavior that stands in marked contrast to the density patterns known for many classical monoids.

2 Preliminaries on Monoids and Related Structures

Throughout this paper, the symbol $\mathbb{N}$ is used to denote the collection of positive integers, and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For real numbers $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha \leq \beta$, we introduce the corresponding discrete interval [6].

$$[\alpha, \beta]_{\mathbb{Z}} = \{ \ x \in \mathbb{Z}: \alpha \leq x \leq \beta \ \}.$$

If X and Y are subsets of $\mathbb{Z}$, their *Minkowski sum* and *difference* are given by

$$X \oplus Y = \{x + y: x \in X, \ y \in Y\}, \qquad X \ominus Y = \{x - y: x \in X, \ y \in Y\}.$$

For a nonempty set $X \subseteq \mathbb{Z}$, we define its set of gaps by

$$\Delta(X) = \{ \ d \in \mathbb{N}: \exists \ \ x \in X \text{with} X \cap [x, x + d]_{\mathbb{Z}} = \{x, x + d\} \ \}.$$

For each $k \in \mathbb{N}$, we denote by $k \oplus X$ the k-fold sumset

$$k \oplus X = \underbrace{X \oplus \cdots \oplus X}_{k\text{times}},$$

and by $k \cdot X = \{kx: x \in X\}$ its dilation. By convention, if $k = 0$ then $k \oplus X = k \cdot X = \{0\}$.[7].

2.1 Monoids and Cancellation Properties

Let H be a commutative semigroup with identity element. Depending on the context, we use either additive or multiplicative notation; however, for the purposes of factorization theory, multiplicative notation will be adopted here. The group of invertible elements of H is denoted by $H^{\times}$. We say that H is *reduced* if $H^{\times} = \{1\}$, and we write [8].

$$H_{\text{red}} = H/H^{\times}$$

for the associated reduced monoid.

An element $a \in H$ is called

- *cancellative* if $ab = ac$ implies $b = c$ for all $b, c \in H$,
- *unit-cancellative* if $a = au$ implies $u \in H^{\times}$.

Every cancellative element is unit-cancellative. In this article, the term *monoid* always refers to a commutative, unit-cancellative semigroup with identity [9].

2.2 Free Abelian Monoids and Divisor Theory

Given a set P, let $\mathcal{F}(P)$ denote the free abelian monoid generated by P. Each element $x \in \mathcal{F}(P)$ admits a unique representation [10].

$$x = \prod_{p \in P} p^{\nu_p(x)},$$

where $\nu_p: \mathcal{F}(P) \to \mathbb{N}_0$ is the p-adic valuation. The *length* of x is defined as

$$|x| = \sum_{p \in P} \nu_p(x).$$

Suppose H is a monoid and E is a subset of H. We refer to E as an s-ideal whenever it is stable under right multiplication by elements of H, that is, $EH = E$.

The collection of all prime s-ideals of H is denoted by $\mathrm{s} - \mathrm{Spec}(H)$; note that the empty set is included. A subset $E \subseteq H$ is said to be *divisor-closed* if $b|a$ with $a \in E$ and $b \in H$ implies $b \in E$. The smallest divisor-closed submonoid containing E is written as $[[E]]$. For a single element $a \in H$,

$$[[a]] = \{\ b \in H: b \text{divides} a^n \text{forsome} n \in \mathbb{N}\ \}.$$

A monoid H is called

- *locally finitely generated* if $[[a]]_{\mathrm{red}}$ is finitely generated for every $a \in H$,
- *torsion-free* if $a^n = b^n$ for some $n \in \mathbb{N}$ implies $a = b$,
- a *Krull monoid* if it is cancellative, completely integrally closed, and satisfies the ascending chain condition on divisorial ideals [11].

2.3 Grothendieck Group

Associated with every monoid H is an abelian group $\mathrm{gp}(H)$ together with a monoid homomorphism [12].

$$\iota: H \longrightarrow \mathrm{gp}(H)$$

It satisfies the subsequent universal mapping property: for every abelian group G and each monoid morphism $\varphi: H \to G$, one can find a unique group morphism $\psi: gp(H) \to G$

making the diagram commute, namely $\varphi = \psi \circ \iota$. In the case where H is cancellative, the group $gp(H)$ is naturally identified with the quotient group of H [13].

2.4 Factorizations and Arithmetic Invariants

An element $u \in H$ is called an *atom* if $u \notin H^{\times}$ and every factorization $u = ab$ with $a, b \in H$ forces $a \in H^{\times}$ or $b \in H^{\times}$. An element $p \in H$ is called *prime* if $p \notin H^{\times}$ and $p|ab$ implies $p|a$ or $p|b$. Every prime element is an atom.

Let $\mathcal{A}(H)$ denote the set of atoms of H. The *factorization monoid* of H is the free abelian monoid [14].

$$\mathrm{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\mathrm{red}})),$$

together with the canonical homomorphism $\pi: \mathrm{Z}(H) \to H_{\mathrm{red}}$. For $a \in H$, we define

$$\mathrm{Z}_H(a) = \pi^{-1}(aH^{\times}),$$

$$\mathrm{L}_H(a) = \{\ |z|: z \in \mathrm{Z}_H(a)\ \},$$

$$\mathcal{L}(H) = \{\mathrm{L}_H(a): a \in H\}.$$

An atom u is said to be *strongly irreducible* if $|\mathrm{Z}(u^n)| = 1$ for all $n \in \mathbb{N}$. Prime cancellative elements enjoy this property automatically [15].

We further define the global distance set

$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L),$$

and for each $k \in \mathbb{N}$ the union

$$\mathcal{U}_k(H) = \bigcup_{\substack{L \in \mathcal{L}(H) \\ k \in L}} L.$$

A monoid is called *atomic* if every nonunit admits at least one factorization into atoms.

2.5 Transfer Homomorphisms

A homomorphism $\theta: H \to B$ between monoids is called a *transfer homomorphism* if

1. $B = \theta(H)B^{\times}$ and $\theta^{-1}(B^{\times}) = H^{\times}$,
2. whenever $\theta(a) = bc$ in B, there exist $x, y \in H$ such that $a = xy$ with $\theta(x) \in bB^{\times}$ and $\theta(y) \in cB^{\times}$.

Such homomorphisms preserve arithmetic invariants, in particular

$$\mathsf{L}_H(a) = \mathsf{L}_B(\theta(a)) \qquad \text{for all} a \in H.$$

A monoid is called *transfer Krull* if it admits a transfer homomorphism to a Krull monoid [16].

2.6 Numerical Monoids and Power Monoids

Every submonoid of the additive group $\mathbb{Z}$ must be either a group itself or lie entirely within $\mathbb{N}_0$ or its negative counterpart $-\mathbb{N}_0$. A submonoid S of $\mathbb{N}_0$ is called numerical precisely when $\gcd(S) = 1$, which is equivalent to requiring that only finitely many nonnegative integers are excluded from S. The greatest integer missing from S is written as $F(S)$ and is known as the Frobenius number. Such numerical monoids are finitely generated and have a uniquely determined minimal generating set $A(S)$ [17].

Given an additive monoid $S \subseteq \mathbb{Z}$, we define its *power monoid*

$$\mathcal{P}_{\text{fin}}(S) = \{\ A \subseteq S : A \text{finite and nonempty}\ \},$$

equipped with setwise addition. The *restricted power monoid* $\mathcal{P}_{\text{fin},0}(S)$ consists of those elements of $\mathcal{P}_{\text{fin}}(S)$ that contain 0.

If S is numerical, then both $\mathcal{P}_{\text{fin}}(S)$ and $\mathcal{P}_{\text{fin},0}(S)$ are reduced, commutative monoids with zero element $\{0\}$. Each element admits only finitely many factorizations, and its associated set of lengths is finite [18].

While the arithmetic of numerical monoids is well understood and highly structured, the behavior of their power monoids is radically different. In particular, all arithmetic invariants investigated so far for power monoids turn out to be infinite, supporting the conjectural picture outlined in the introduction [19].

3 Elementary Algebraic Features of Power Monoids Associated with Numerical Monoids

In this section we examine foundational algebraic aspects of power monoids arising from numerical monoids. Our focus is on the identification of prime elements, cancellation phenomena, and direct-sum decompositions. More refined notions, such as strong irreducibility and cancellation properties, are postponed to later sections [21].

3.1 Prime elements in the power monoid of $\mathbb{N}_0$

We begin with the special case of the power monoid built over the nonnegative integers.

1. The singleton $\{1\}$ is a cancellative prime element of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$. Moreover,

$$\mathcal{P}_{\text{fin}}(\mathbb{N}_0) = \{\{k\}: k \in \mathbb{N}_0\} \oplus \mathcal{P}_{\text{fin},0}(\mathbb{N}_0).$$

No other element of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ is prime.

2. If $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ admits a decomposition

$$\mathcal{P}_{\text{fin},0}(\mathbb{N}_0) = H_1 \oplus H_2$$

into submonoids, then necessarily $H_1 = \{\{0\}\}$ or $H_2 = \{\{0\}\}$.

Proof. (1) Assume that $\{1\}$ divides a sumset $A \oplus B$ with $A, B \subseteq \mathbb{N}_0$ finite and nonempty. Then every element of $A \oplus B$ is at least 1, which forces either $\min A \geq 1$ or $\min B \geq 1$. Hence $\{1\}$ divides A or B, showing primality [22].

If $\{1\} \oplus A = \{1\} \oplus B$, then $A = B$, and thus $\{1\}$ is cancellative. The stated decomposition follows from the standard structural result for cancellative prime elements.

Now let $A \in \mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ be irreducible with $A \neq \{1\}$. Then $0 \in A$, so $A \in \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. Suppose A were prime. Observing that

$$A \oplus [0, \max A] = [0, 2\max A] = [0, \max A] \oplus [0, \max A],$$

we conclude that A divides $[0, \max A]$, forcing $A = [0, \max A]$. Since A is irreducible, this implies $A = \{0,1\}$. However, one checks that $\{0,1\}$ divides certain sumsets without dividing either summand, contradicting primality.

(2) Assume $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0) = H_1 \oplus H_2$. Both H_1 and H_2 must be divisor-closed. Since $\{0,1\}$ is an atom, it belongs to exactly one of them, say H_1. Then all intervals of the form $[0, k]$ lie in H_1. If $B = \{0\}$, then

$$B \oplus [0, \max B] = [0,0] \subseteq H_1,$$

and divisor-closedness implies $B \in H_1$, forcing $H_2 = \{\{0\}\}$.

3.2 Power monoids over proper numerical monoids

The situation changes substantially when one moves from $\mathbb{N}_0$ to a proper numerical monoid. Let S be a numerical monoid with $S \neq \mathbb{N}_0$. Then the following assertions hold:

1. The monoid $\mathcal{P}_{\text{fin}}(S)$ contains no prime elements [23].
2. If $\mathcal{P}_{\text{fin}}(S)$ admits a decomposition $\mathcal{P}_{\text{fin}}(S) = H_1 \oplus H_2$ into submonoids H_1 and H_2, then necessarily $H_1 = \{\{0\}\}$ or $H_2 = \{\{0\}\}$.
3. For numerical monoids S and T, if $\mathcal{P}_{\text{fin}}(S)$ and $\mathcal{P}_{\text{fin}}(T)$ are isomorphic as monoids, then $S = T$.

Proof. (1) The collection of singleton sets $\{\{s\}: s \in S\}$ forms a divisor-closed submonoid of $\mathcal{P}_{\text{fin}}(S)$ that is isomorphic to S. Since S itself contains no prime elements, the same holds for this submonoid. Now let $A \subseteq S$ with $|A| \geq 2$, and choose elements $a_1 < a_2$ in A. Setting $d = a_2 - a_1 > 0$, one can construct sufficiently large intervals contained in S whose sumsets are divisible by A, while neither summand is divisible by A. Consequently, A cannot be a prime element [24].

(2) Assume that $\mathcal{P}_{\text{fin}}(S) = H_1 \oplus H_2$. Let a be the smallest element of S such that $a - 1 \notin S$, and consider the set $A = \{a, a+1\}$. This set is an atom of $\mathcal{P}_{\text{fin}}(S)$. Without loss of generality, suppose $A \in H_1$. Since H_1 is divisor-closed, it follows that the divisor-closed submonoid generated by A, denoted $[[A]]$, is contained in H_1. A dilation argument shows that every element of $\mathcal{P}_{\text{fin}}(S)$ divides some multiple of A, and hence belongs to $[[A]]$. Therefore, $H_1 = \mathcal{P}_{\text{fin}}(S)$ and $H_2 = \{\{0\}\}$ [25].

(3) Let $\varphi: \mathcal{P}_{\text{fin}}(S) \to \mathcal{P}_{\text{fin}}(T)$ be a monoid isomorphism. Since $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ contains a prime element, whereas $\mathcal{P}_{\text{fin}}(S)$ does not unless $S = \mathbb{N}_0$, it follows that $S = T = \mathbb{N}_0$ in this case. Moreover, the submonoid consisting of singleton sets is the unique nontrivial cancellative divisor-closed submonoid, and hence must be preserved under φ, which forces $S = T$ [26].

3.3 Grothendieck groups of power monoids

espite the sensitivity of power monoids to the underlying numerical monoid, their Grothendieck groups exhibit uniform behavior [27].

Let S be a numerical monoid.

1. The Grothendieck group of $\mathcal{P}_{\text{fin}}(S)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
2. The Grothendieck group of $\mathcal{P}_{\text{fin},0}(S)$ is isomorphic to $\mathbb{Z}$.

Proof. Consider the additive monoid $H = \mathcal{P}_{\text{fin}}(S)$. Two pairs $(A, B), (C, D) \in H \times H$ represent the same element in the Grothendieck group if there exists $E \in H$ with

$$A \oplus D \oplus E = C \oplus B \oplus E.$$

This equality holds precisely when

$$\max A - \max B = \max C - \max D \qquad \text{and} \qquad \min A - \min B = \min C - \min D.$$

Hence the map

$$[(A, B)] \longmapsto (\max A - \max B, \min A - \min B)$$

yields an isomorphism onto $\mathbb{Z}^2$.

For $\mathcal{P}_{\text{fin},0}(S)$, the same argument applies with only maximal elements, leading to an isomorphism with $\mathbb{Z}$. [28].

3.4 Divisor-closed submonoids

The concluding remark of this section demonstrates that divisor theory in $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ can be reduced to the corresponding theory for the restricted power monoid. Let H be a divisor-closed submonoid of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$. Then there exists a divisor-closed submonoid $\widetilde{H}$ of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ such that either [29].

$$H = \widetilde{H} \qquad \text{or} \qquad H = \{\{k\}: k \in \mathbb{N}_0\} \oplus \widetilde{H}.$$

Proof. Define

$$\widetilde{H} := \{ \ B - \min B : B \in H \ \}.$$

By construction, $\widetilde{H}$ is divisor-closed and is contained in $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. If every element of H contains 0, then the equality $H = \widetilde{H}$ holds and there is nothing further to prove.

Otherwise, there exists an element $D \in H$ with $0 \notin D$. This implies that $\{1\}$ divides D, and hence $\{1\} \in H$. As a consequence, all singleton sets belong to H. Moreover, every element $B \in H$ admits a decomposition of the form

$$B = \{\min B\} \oplus (B - \min B),$$

which yields the stated structural description of H. [30].

4 Prime ideals and divisor-closed substructures in restricted power monoids

In this section we investigate prime-like ideals together with divisor-closed submonoids arising in the restricted power monoid of a numerical monoid. Let S be numerical. An s-ideal $p \subseteq \mathcal{P}_{\mathrm{fin},0}(S)$ is said to be prime precisely when the complement $\mathcal{P}_{\mathrm{fin},0}(S) \backslash p$ is divisor-closed. Hence, there is a one-to-one correspondence between maximal divisor-closed submonoids and minimal nonempty prime s-ideals, making it natural to describe the prime spectrum via divisor-closed submonoids.

Since inclusion of underlying numerical monoids $S_1 \subseteq S_2 \subseteq \mathbb{N}_0$ yields the obvious embedding $\mathcal{P}_{\mathrm{fin},0}(S_1) \subseteq \mathcal{P}_{\mathrm{fin},0}(S_2)$ as a divisor-closed submonoid, it suffices to classify divisor-closed submonoids of $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$: every restricted power monoid of a submonoid of $\mathbb{N}_0$ appears among these, but additional divisor-closed submonoids exist as well [31].

4.1 The reversal operator and its basic features

Given a set $B \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$, let $\max(B)$ denote its greatest element. We define the *reversal* of B as

$$Rev(B) := \max(B) - B = \{\ \max(B) - b | b \in B\ \},$$

which again belongs to $\mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$. The transformation

$$Rev: \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0) \to \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$$

enjoys the following basic properties for all $B, C \in \mathcal{P}_{\mathrm{fin},0}(\mathbb{N}_0)$:

1. $Rev(B \oplus C) = Rev(B) \oplus Rev(C)$.
2. B divides C with respect to sumset divisibility if and only if $Rev(B)$ divides $Rev(C)$.
3. Applying the reversal twice yields the original set, that is, $Rev(Rev(B)) = B$.

For a subset $H \subseteq \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, we define

$$Rev(H) := \{ \ Rev(B) | B \in H \ \}.$$

The properties listed above imply that H is a submonoid (respectively, a divisor-closed submonoid) if and only if $Rev(H)$ has the same property.

The reversal operation provides a flexible source of examples. Indeed, for every numerical submonoid $S \subseteq \mathbb{N}_0$, the set $Rev(\mathcal{P}_{\text{fin},0}(S))$ forms a divisor-closed submonoid of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. Furthermore, intersections of the form

$$\mathcal{P}_{\text{fin},0}(S) \cap Rev(\mathcal{P}_{\text{fin},0}(T))$$

are again divisor-closed.

The next example shows that not every divisor-closed submonoid arises from a numerical monoid in this way. Let $A = \{0,1,3\}$. A straightforward verification shows that the divisor-closed submonoid generated by A coincides with $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, although A itself does not belong to $\mathcal{P}_{\text{fin},0}(S)$ for any proper numerical submonoid $S \subsetneq \mathbb{N}_0$. In this case,

$$Rev(A) = \{0,2,3\},$$

and $Rev(A)$ generates the numerical monoid $\mathbb{N}_0 \backslash \{1\}$, which yields

$$[[A]] = Rev(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0 \backslash \{1\})).$$ [32].

4.2 Behavior of iterated sumsets

To analyse divisor-closed submonoids generated by a single set it is useful to control the structure of n-fold sumsets. For a finite nonempty $A \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}$ write

$$n \oplus A = \underbrace{A \oplus \cdots \oplus A}_{n\text{times}}.$$

The following stabilization statement is standard (we state it in a form convenient for our later use).

Let $A \in \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ with $0 \in A$. There exists an integer $n^*(A) \in \mathbb{N}$ such that for every $n \geq n^*(A)$

$$n \oplus A \quad = \quad A \quad \cap \quad (n\max A - Rev(A)).$$

Sketch of proof. For n large the structure of $n \oplus A$ is governed by the classical structure theorems for iterated sumsets of finite subsets of the integers: outside a small exceptional initial and terminal segment the n-fold sumset becomes a long interval whose endpoints are determined by $\min A$ and $\max A$, and the intersection description follows by comparing these intervals with the reversed pattern coming from $Rev(A)$. (See the literature on large-sumset structure for precise quantitative bounds; the lemma records the qualitative stabilization that we require.) [33].

An immediate consequence is the following description of cancellative elements.

The only cancellative elements in $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ are the singletons $\{k\}$ with $k \in \mathbb{N}_0$. In particular, every finite $A \subseteq \mathbb{N}_0$ with $|A| \geq 2$ fails to be cancellative already inside the divisor-closed submonoid $[[A]]$ that it generates. [34].

Proof. Singletons clearly are cancellative. Let A be finite with $|A| \geq 2$ and assume $0 \in A$ (translate otherwise). Take $n \geq n^*(A)$ so Lemma 4.2 applies. One checks that $(n+1) \oplus A$ can be written as $A \oplus B$ with $B \neq (n+1) \oplus A$ and also equals $n \oplus A \oplus A$, so two different decompositions of the same sumset witness the failure of cancellation for A inside $[[A]]$.

4.3 A structural classification of divisor-closed submonoids

The main structural result of this section expresses every divisor-closed submonoid of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ as an intersection of a restricted power monoid and a reversed restricted power monoid.

Let $H \subseteq \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ be a divisor-closed submonoid. Define the two numerical submonoids

$$S(H) := \bigcup_{B \in H} B \subseteq \mathbb{N}_0, \qquad T(H) := \bigcup_{B \in H} Rev(B) \subseteq \mathbb{N}_0.$$

Then

$$H = \mathcal{P}_{\text{fin},0}(S(H)) \cap Rev(\mathcal{P}_{\text{fin},0}(T(H))).$$

Moreover, there exists $A \in H$ with $\langle A \rangle = S(H)$ and $\langle Rev(A) \rangle = T(H)$ (here $\langle X \rangle$ denotes the submonoid of $\mathbb{N}_0$ generated by the finite set X), and for any such A one has $H = [[A]]$, the divisor-closed submonoid generated by A.

Proof. Set $S := S(H)$ and $T := T(H)$. By construction $H \subseteq \mathcal{P}_{\text{fin},0}(S) \cap Rev(\mathcal{P}_{\text{fin},0}(T))$.

Conversely, since H is divisor-closed, every element of H contains only elements of S and its reversal lies in T, so the right-hand side is contained in the divisor-closed closure of any $A \in H$ that generates S and T in the stated sense. The existence of such an A follows by an iterative finite-construction argument: begin with $\{0\}$ and successively add elements from members of H until the generated submonoid equals S; finiteness of the generators of numerical submonoids ensures termination. Finally, Lemma 4.2 (and the divisibility control developed below) yields that every B in the intersection divides a suitable multiple of A, whence $B \in [[A]]$, proving equality. [35].

The proof above relies on a technical divisibility lemma that we now isolate.

Let $A \in \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ satisfy $\gcd(A) = 1$. Put $S = \langle A \rangle$ and $T = \langle Rev(A) \rangle$. There exists $n^*(A)$ such that for every $B \in \mathcal{P}_{\text{fin},0}(S) \cap Rev(\mathcal{P}_{\text{fin},0}(T))$ and every integer $N \geq n^*(A)$ with

$$\max\Delta(B) \quad < \quad N\max A - F(S) - F(T) - \max B,$$

one has $B | N \oplus A$.

Proof idea. Write $a = \max A$ and choose $N \geq n^*(A)$ so that the description of $N \oplus A$ from Lemma 4.2 applies and produces a long middle interval in $N \oplus A$. Using that B sits inside S and that $Rev(B)$ sits inside T, one can translate appropriate blocks into that long interval and exhibit an explicit C with $B \oplus C = N \oplus A$, proving divisibility. The inequality involving $\max\Delta(B)$ guarantees that the local gaps of B do not obstruct the required covering [36].

4.4 Maximal divisor-closed submonoids and chains

The structural results established above allow one to describe the maximal divisor-closed submonoids of the restricted power monoid and to derive corresponding chain properties. The following statements hold.

1. For a set $A \in \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, the divisor-closed submonoid generated by A coincides with $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ if and only if the element 1 belongs to both A and its reversal $Rev(A)$.
2. The collection of maximal divisor-closed submonoids of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ consists precisely of

$$\mathcal{P}_{\text{fin},0}(\mathbb{N}_0\backslash\{1\}) \qquad \text{and} \qquad Rev(\mathcal{P}_{\text{fin},0}(\mathbb{N}_0\backslash\{1\})).$$

More generally, if $S_1, S_2 \subseteq \mathbb{N}_0$ are numerical monoids, then the maximal divisor-closed submonoids of

$$\mathcal{P}_{\text{fin},0}(S_1) \cap Rev(\mathcal{P}_{\text{fin},0}(S_2))$$

are obtained by replacing one of the monoids S_i $(i = 1,2)$ with a maximal proper numerical submonoid.

3. There exists an infinite strictly decreasing sequence of divisor-closed submonoids of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$.

4. Every increasing sequence of divisor-closed submonoids of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ eventually becomes constant.

Proof. (1) Suppose that $1 \in A \cap Rev(A)$. Then the numerical monoid generated by A equals $\mathbb{N}_0$, which implies that every finite subset of $\mathbb{N}_0$ belongs to the divisor-closed submonoid generated by A. Hence $[[A]] = \mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. Conversely, if $[[A]]$ coincides with $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, then in particular the singleton $\{1\}$ must lie in $[[A]]$. This forces the element 1 to occur both in A and in $Rev(A)$.

(2) and *(3)* follow from the explicit classification of divisor-closed submonoids together with the elementary order structure of numerical monoids. For *(3)*, consider the family

$$H_n := \mathcal{P}_{\text{fin},0}(\{0\} \cup \mathbb{N}_{\geq n}), \qquad n \geq 1,$$

which forms a strictly descending chain that does not stabilise.

(4) Let $\{H_i\}_{i\in I}$ be an ascending chain of divisor-closed submonoids of $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. The associated numerical monoids arising in the classification yield corresponding ascending chains. Since numerical submonoids of $\mathbb{N}_0$ are finitely generated, these chains must stabilise, and hence the original chain $\{H_i\}_{i\in I}$ stabilises as well. [37].

4.5 Finiteness and transfer obstructions

Divisor-closed submonoids of restricted power monoids exhibit several algebraic pathologies: they are typically very far from being Krull-like or even cancellative.

Let S be a numerical monoid and let $H \subseteq \mathcal{P}_{\text{fin},0}(S)$ be a nontrivial divisor-closed submonoid (so $\{\{0\}\} \subsetneq H$). Then H is not torsion-free, not locally finitely generated, and there exists no transfer homomorphism from H to any cancellative monoid. In particular, H is not transfer-Krull.

Idea of proof. Working inside $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ via the description from Theorem 4.3, one constructs an infinite family of atoms of H (for instance pairs of the form $\{0, dn\}$ for increasing n with a suitable common difference d), showing failure of finite generation. By arranging suitable decompositions among these sets one exhibits nontrivial torsion-type equalities, proving failure of torsion-freeness. Finally, any transfer homomorphism into a cancellative monoid would preserve certain atomicity relations and force impossible equalities there (a simple counting argument with the images of the constructed atoms), contradicting cancellativity; hence no transfer homomorphism exists. [38].

The preceding theorem motivates the following natural conjecture.

If S and T are numerical monoids and $\mathcal{P}_{\text{fin},0}(S) \cong \mathcal{P}_{\text{fin},0}(T)$ as monoids, then $S = T$.

Although the conjecture is open in full generality, the next results give partial invariants preserved under isomorphism.

4.6 Counting divisors and an asymptotic lemma

Let S be a numerical monoid. For a finite subset $A \subseteq S$, denote by $\tau_S(A)$ the number of divisors of A within $\mathcal{P}_{\text{fin}}(S)$. In the special case where $A \in \mathcal{P}_{\text{fin},0}(S)$, this quantity coincides with the number of divisors computed inside the restricted power monoid.

The following lemma describes the asymptotic behavior of the divisor-counting function $\tau_S(n \oplus A)$ as the parameter n tends to infinity.[39].

Assume that S is a numerical monoid and let $A \in \mathcal{P}_{\text{fin},0}(S)$ satisfy $\min A = 0$. Set $d := \gcd(A)$ and let $a_{\max} := \max A$. Then, as $n \to \infty$,

$$\log_2(\tau_S(n \oplus A)) \quad \sim \quad \frac{n \ a_{\max}}{d}.$$

Outline of proof. One obtains upper bounds by noting that every divisor of $n \oplus A$ is a subset of $n \oplus A$, hence $\tau_S(n \oplus A) \leq 2^{na_{\max}}$. For the lower bound one constructs many divisors by selecting subsets inside a long central interval of $n \oplus A$ guaranteed by Lemma 4.2; counting those choices and taking logarithms yields the stated asymptotic after dividing by n and normalising by d. [40].

4.7 Recovering numerical data from restricted power monoids

The asymptotic growth of divisor-counting functions can be exploited to reconstruct certain arithmetic features of numerical monoids from their associated restricted power monoids. The following result makes this principle precise.

Let S and T be numerical monoids. [41].

1. If the restricted power monoids $\mathcal{P}_{\text{fin},0}(S)$ and $\mathcal{P}_{\text{fin},0}(T)$ are isomorphic as monoids, then the numerical monoids S and T have the same number of atoms, that is,

$$|A(S)| = |A(T)|,$$

and their corresponding invariants $m(S)$ and $m(T)$ coincide. Here $A(\cdot)$ denotes the minimal generating set of a numerical monoid, while $m(\cdot)$ is defined as the smallest integer $m \geq \max A(\cdot)$ such that both m and $m-1$ belong to the monoid.

2. For a fixed numerical monoid T, there exist only finitely many numerical monoids S for which $\mathcal{P}_{\text{fin},0}(S)$ is isomorphic to $\mathcal{P}_{\text{fin},0}(T)$.

Sketch of proof. The first claim follows from the observation that the number of maximal divisor-closed submonoids of $\mathcal{P}_{\text{fin},0}(S)$ is equal to $|A(S)| + 1$. Since monoid isomorphisms preserve divisor-closed submonoids and their maximal elements, this count is invariant under isomorphism, yielding the equality $|A(S)| = |A(T)|$.

To compare the invariants $m(S)$ and $m(T)$, one considers a carefully chosen finite subset $A \in \mathcal{P}_{\text{fin},0}(S)$ whose divisor-closed hull coincides with the entire restricted power monoid if and only if $\max A \geq m(S)$. By applying the asymptotic estimates for divisor counts established earlier and using the fact that divisor-counting functions are preserved under monoid isomorphisms, one deduces that $m(S) = m(T)$.

The finiteness statement in *(2)* follows from the fact that there are only finitely many numerical monoids whose atoms are bounded above by a fixed constant. Since the previous part fixes both the number of atoms and their maximal size, only finitely many possibilities for S remain. [42].

As a direct consequence, the restricted power monoid $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ cannot be isomorphic to $\mathcal{P}_{\text{fin},0}(S)$ for any proper numerical submonoid $S \subsetneq \mathbb{N}_0$. More generally, distinguishing non-isomorphic restricted power monoids associated with different numerical monoids may require a finer investigation of the lattice of maximal divisor-closed submonoids and, in certain cases, an iterative analysis of the corresponding counting invariants [43].

4.8 Absolutely irreducible elements

We close the section by discussing *absolute irreducibility* (also called strong atoms) in power monoids.

Let S be any numerical monoid. The only absolutely irreducible element occurring in any of the monoids $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$, $\mathcal{P}_{\text{fin}}(S)$ or $\mathcal{P}_{\text{fin},0}(S)$ is the singleton $\{1\} \in \mathcal{P}_{\text{fin}}(\mathbb{N}_0)$. In particular, aside from $\{1\}$ (which is cancellative and prime in $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$), no element is absolutely irreducible [44].

Proof. The singleton $\{1\}$ is cancellative and prime in $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ and hence trivially strongly irreducible.

Let $A \in \mathcal{P}_{\text{fin},0}(S)$ be irreducible with $d = \gcd(A)$. By Lemma 4.6, the number $\tau_S(n \oplus A)$ grows exponentially in n with exponent proportional to $n \max A/d$. Therefore for large n the n-fold power $n \oplus A$ admits divisors other than the trivial dilates $k \oplus A$ $(0 \le k \le n)$, so $|\mathrm{Z}((n \oplus A))| > 1$ and A fails to be absolutely irreducible.

The same counting argument applies mutatis mutandis to irreducible singletons $\{k\}$ with $k \in \mathcal{A}(S)$ when $S \neq \mathbb{N}_0$: one produces other divisors in high powers showing $\{k\}$ is not absolutely irreducible. Thus no new absolutely irreducible elements occur [45].

5 Catenary Degrees and ω-Invariants in Power Monoids

This section is devoted to refined arithmetic invariants of power monoids, namely the *catenary degree* and the *ω-invariant*. Unlike sets of lengths, which record only the number of atoms in a factorization, these invariants probe the internal structure of factorizations themselves. Nevertheless, they exert strong control over the possible shapes of sets of lengths, as will be discussed at the end of the section. [46].

5.1 The ω-invariant

Let H be an atomic commutative monoid written multiplicatively. For an element $a \in H$, we define $\omega(H, a)$ as the smallest integer $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

Whenever $n \in \mathbb{N}$ and $a_1, \dots, a_n \in H$ satisfy $a | a_1 \cdots a_n$, there exists a subset $\Omega \subseteq \{1, \dots, n\}$ with $|\Omega| \leq N$ such that

$$a \mid \prod_{i \in \Omega} a_i.$$

Thus, $\omega(H, a)$ measures how far a is from behaving like a prime. Indeed, a is prime if and only if $\omega(H, a) = 1$. For every $a \in H$ one has the basic inequality

$$\sup \mathsf{L}(a) \leq \omega(H, a),$$

and consequently, if $\omega(H, a) < \infty$ for all $a \in H$, then every set of lengths in H is finite. We define the global ω-invariant of H by

$$\omega(H) = \sup\{\omega(H, u) : u \in \mathcal{A}(H)\}.$$

5.2 Distances between factorizations and catenary degrees

Let $\mathsf{Z}(H)$ denote the factorization monoid associated with a monoid H. Consider two factorizations $z, z' \in \mathsf{Z}(H)$ of the same element. Assume that they admit representations of the form

$$z = u_1 \cdots u_k \ \ v_1 \cdots v_r, \qquad z' = u_1 \cdots u_k \ \ w_1 \cdots w_s,$$

where all factors u_i, v_j, and w_ℓ are atoms of H, and where the atoms v_j and w_ℓ are pairwise distinct. The *distance* between z and z' is defined as

$$\mathrm{d}(z, z') := \max\{r, s\}.$$

By construction, $\mathrm{d}(z, z') = 0$ holds if and only if the two factorizations coincide.

Let $a \in H$ and let $M \in \mathbb{N}_0$. A finite sequence of factorizations

$$z_0, z_1, \dots, z_t \in \mathrm{Z}(a)$$

is called an M-*chain* if

$$\mathrm{d}(z_{i-1}, z_i) \le M \qquad \text{forall} i \in \{1, \dots, t\}.$$

The *catenary degree* of the element a is defined as the smallest value

$$\mathrm{c}(a) \in \mathbb{N}_0 \cup \{\infty\}$$

for which any two factorizations of a can be connected by an $\mathrm{c}(a)$-chain. The *catenary degree of the monoid* H is then given by

$$\mathrm{c}(H) := \sup\{\ \mathrm{c}(a) : a \in H\ \}.$$

In general, one always has the inequality

$$\mathrm{c}(a) \le \sup \mathrm{L}(a),$$

where $\mathrm{L}(a)$ denotes the set of factorization lengths of a. Furthermore, if the set of distances $\Delta(H)$ is nonempty, then

$$1 + \sup \Delta(H) \le \mathrm{c}(H).$$

If, in addition, the monoid H is cancellative, this estimate can be sharpened to

$$2 + \sup \Delta(H) \le \mathrm{c}(H) \le \omega(H),$$

where $\omega(H)$ denotes the ω-invariant of H. [47].

5.3 Infinite $\boldsymbol{\omega}$-invariants in power monoids

If H is finitely generated, then $\omega(H) < \infty$. However, power monoids behave in a fundamentally different manner.

Let S be a numerical monoid, let $a \in S$, and set $A = \{0, a\}$. Then

$$\omega(\mathcal{P}_{\mathrm{fin}}(S), A) = \omega(\mathcal{P}_{\mathrm{fin},0}(S), A) = \infty.$$

Proof. Fix $n \in \mathbb{N}$ and $m \in \{1, \dots, n\}$. It suffices to show that $\omega(\mathcal{P}_{\mathrm{fin},0}(S), A) \geq n + 2$. Since $\mathcal{P}_{\mathrm{fin},0}(S)$ is divisor-closed in $\mathcal{P}_{\mathrm{fin}}(S)$, the same bound then holds in the full power monoid.

The sets

$$\{0,2a\}, \qquad \{0,2a,3a\}, \qquad \{0,a,(2n+5)a\}$$

are atoms of $\mathcal{P}_{\mathrm{fin},0}(S)$. Consider

$$A_{m,n} = m\{0,2a\} + \{0,2a,3a\} + \{0,a,(2n+5)a\}.$$

A direct computation yields

$$A_{m,n} = a([0,2m+4] \cup \{2n+5\} \cup [2n+7, 2m+2n+8]).$$

For $m = n$ one checks that

$$A_{n,n} = A + a([0,2n+4] \cup [2n+7, 4n+7]),$$

showing that A divides $A_{n,n}$, which is a sum of $n+2$ atoms. On the other hand, if $m < n$, then A does not divide $A_{m,n}$. Hence no bound smaller than $n+2$ satisfies the defining property of $\omega(\mathcal{P}_{\mathrm{fin},0}(S), A)$, proving the claim.

In sharp contrast, the catenary degree of each individual element in $\mathcal{P}_{\mathrm{fin},0}(S)$ is finite. In fact, the collection of all catenary degrees satisfies

$$\{\mathrm{c}(B) : B \in \mathcal{P}_{\mathrm{fin},0}(S)\} = \{\mathrm{c}(B) : B \in \mathcal{P}_{\mathrm{fin}}(S)\} = \mathbb{N}.$$

5.4 Consequences for ideal-theoretic finiteness

Atomic monoids for which every set of factorization lengths is finite are known to satisfy the ascending chain condition on principal ideals. In particular, this property holds for power monoids arising from numerical monoids. The picture changes significantly, however, when one considers more general systems of ideals [48].

1. Let S be a numerical monoid, and let r be a weak ideal system on either $\mathcal{P}_{\mathrm{fin},0}(S)$ or $\mathcal{P}_{\mathrm{fin}}(S)$ with the property that every principal ideal is an r-ideal. Then the ascending chain condition on r-ideals does not hold.
2. Let D be a Dedekind domain possessing infinitely many maximal ideals of finite

residue field. Then

$$\omega(Int(D), X) = \infty.$$

Moreover, for any weak ideal system r on $Int(D)$ for which all principal ideals are r-ideals, the ring $Int(D)$ fails to satisfy the ascending chain condition on r-ideals.

Proof. (1) Suppose that a monoid satisfies the ascending chain condition on r-ideals. Then, for every element a of the monoid, the associated invariant $\omega(H, a)$ must be finite. In the present setting, this contradicts previously established results showing that the ω-invariant is unbounded for power monoids of numerical monoids. Hence, the ascending chain condition on r-ideals cannot hold.

(2) Let K denote the quotient field of D. For each integer $n \geq 1$, one can construct irreducible polynomials

$$h, g_1, \ldots, g_{n+1} \in Int(D)$$

such that

$$Xh = g_1 \cdots g_{n+1}.$$

This identity implies that $\omega(Int(D), X) \geq n$ for every $n \in \mathbb{N}$, and therefore the ω-invariant of X in $Int(D)$ is infinite. The failure of the ascending chain condition on r-ideals then follows by the same reasoning as in part *(1)*. [49].

5.5 Implications for sets of lengths

The preceding results place power monoids of numerical monoids firmly at the "extreme" end of factorization theory.

Two broad paradigms govern the known structure theory of sets of lengths. On one side, strong algebraic finiteness assumptions (for instance, Krull monoids with finite class group) enforce rigid, almost periodic behavior of sets of lengths; such monoids necessarily have finite catenary degree and finite ω-invariant. On the opposite side lie classes of monoids for which *every* finite subset of $\mathbb{N}_{\geq 2}$ occurs as a set of lengths. All known examples of this second type—such as Krull monoids with infinite class group and full distribution of prime divisors, rings of integer-valued polynomials, and various weakly Krull or primary monoids—share the feature that both the catenary degree and the ω-invariant are infinite. Theorem 5.3 therefore provides strong additional evidence in favor of Conjecture 1.1. [51].

6 Density of Atoms in Power Monoids

Quantitative questions concerning the distribution of arithmetic properties in atomic monoids have played a central role in factorization theory since its inception. A classical line of research considers an atomic monoid H equipped with a suitable size or norm function $N: H \to \mathbb{N}$, allowing one to count elements with prescribed arithmetic behaviour. Typical examples arise from rings of algebraic integers endowed with the usual field norm.

For a fixed integer $k \in \mathbb{N}$, one studies counting functions of the form

$$P_k(x) = \#\{a \in H: N(a) \leq x,\ a \text{satisfiesproperty} \mathcal{P}_k\},$$

where $\mathcal{P}_k$ may refer to conditions such as

$$\max \mathrm{L}(a) \leq k, \qquad |\mathrm{Z}(a)| \leq k, \qquad |\mathrm{L}(a)| \leq k.$$

Recall that $\max \mathrm{L}(a) = 1$ if and only if $\mathrm{L}(a) = \{1\}$, which in turn is equivalent to a being an atom. Consequently, such counting problems encode information about the density of irreducible elements.

In all previously investigated classes of monoids, atoms form a sparse subset: their density (with respect to the chosen notion of size) is zero. The situation for restricted power monoids of numerical monoids is strikingly different. Here, almost every element is irreducible: the proportion of non-atoms tends to zero, and atoms have density one [52].

6.1 Asymptotic density of atoms

We quantify this phenomenon by ordering elements according to their maximal entry.

Let S be a numerical monoid.

1. If $H = \mathcal{P}_{\mathrm{fin},0}(S)$, then

$$\lim_{x \to \infty} \frac{\#\{A \in \mathcal{A}(H): \max A \leq x\}}{\#\{A \in H: \max A \leq x\}} = 1. \tag{1}$$

2. If $H = \mathcal{P}_{\mathrm{fin}}(S)$, then

$$\lim_{x \to \infty} \frac{\#\{A \in \mathcal{A}(H): \max A \leq x\}}{\#\{A \in H: \max A \leq x\}} \in [1/2, 1) \cap \mathbb{Q}. \tag{2}$$

Moreover, this limit equals $1/2$ if and only if $S = \mathbb{N}_0$.

The first statement can already be extracted from earlier work in a different formal language. Our approach, however, yields substantially sharper decay rates for the set of non-atoms and relies on probabilistic ideas rather than on algebraic normal forms. We interpret finite subsets of intervals as random objects and show that, with overwhelming probability, they fail to admit a nontrivial additive decomposition.

6.2 A probabilistic model

Fix $N \in \mathbb{N}$. We view a subset $A \subseteq [0, N]$ as a random set generated by a sequence $(\xi_n)_{0 \le n \le N}$ of independent Bernoulli random variables with

$$\mathbb{P}(\xi_n = 0) = \mathbb{P}(\xi_n = 1) = \frac{1}{2}, \qquad A = \{n \in [0, N]: \xi_n = 1\}.$$

All probabilities below are taken with respect to this product measure.

We say that a subset $A \subseteq [0, N]$ is a *genuine sumset* if there exist subsets $B, C \subseteq [0, N]$ such that

$$A = B + C \qquad \text{and} \qquad \min\{|B|, |C|\} \ge 2.$$

Denote by $\mathrm{Dec}(N)$ the family of all such sets.

Almost no subset of $[0, N]$ is a genuine sumset. More precisely,

$$\mathbb{P}(A \in \mathrm{Dec}(N)) = \exp(-\Omega(N)).$$

Consequently, the total number of genuine sumsets contained in $[0, N]$ is at most c^N for some constant $c < 2$. Earlier bounds of the form $2^{N-o(N)}$ are considerably weaker. On the other hand, trivial constructions show that $c > \sqrt{2}$, and determining the optimal value of c appears to be a delicate problem.

6.3 Auxiliary probabilistic estimates

The proof of Theorem 6.2 rests on large deviation bounds for random sequences. Let X be a finite alphabet with $|X| = g \ge 2$ and let $(a_1, \dots, a_N) \in X^N$ be uniformly random.

For $z \in X$, denote by

$$A(z) = \#\{n \in [1, N]: a_n = z\}$$

the number of occurrences of z.

Let $\varepsilon \in (0,1/g)$. There exists a constant $c(g,\varepsilon) \in (0,1)$ such that for every $z \in X$,

$$\mathbb{P}\left(\left|\frac{A(z)}{N} - \frac{1}{g}\right| > \varepsilon\right) \leq N^{1/2} \quad c(g,\varepsilon)^N.$$

Thus, for fixed g and ε, deviations from uniform frequency decay exponentially fast.

Next, consider patterns in Bernoulli sequences. Let $B = \{b_1 < \cdots < b_r\} \subseteq \mathbb{N}_0$ and $z = (z_1, \dots, z_r) \in \{0,1\}^r$. Define $A(z; B)$ to be the number of indices $n \leq N - b_r$ such that

$$(\xi_{n+b_1}, \dots, \xi_{n+b_r}) = z.$$

Let $r \in \mathbb{N}$ and $\delta \in (0,1)$. There exists $c > 0$, depending only on r and δ, such that

$$\mathbb{P}(|A(z; B) - (N - b_r)2^{-r}| > \delta N) \leq \exp(-cN).$$

The proof partitions $[0, N]$ into finitely many classes so that the translated blocks $n + B$ are disjoint inside each class, reducing the claim to Lemma 6.3. [44].

6.4 Proof of Theorem 6.2

We decompose $\mathrm{Dec}(N)$ into three families according to the sizes of the summands:

$$\mathrm{Dec}(N) = \mathrm{Dec}_1 \cup \mathrm{Dec}_2 \cup \mathrm{Dec}_3,$$

where $A = B + C$ belongs to

- Dec_1 if $|B| + |C| < N/5$,
- Dec_2 if $\min\{|B|, |C|\} \in \{2,3\}$,
- Dec_3 if $|B| + |C| \geq N/5$ and $\min\{|B|, |C|\} \geq 4$.

Elementary counting shows that $\mathbb{P}(A \in \mathrm{Dec}_1) = O(\alpha^N)$ for some $\alpha < 1$. The cases Dec_2 and Dec_3 are handled using Lemma 6.3: in both regimes, the existence of many forced patterns contradicts the expected random behaviour with probability $\exp(-\Omega(N))$. Summing the three contributions yields the claim.

6.5 Derivation of density statements

We now deduce Theorem 6.1.

Proof of Theorem~\refthm:density-atoms. (1) Let $H = \mathcal{P}_{\mathrm{fin},0}(S)$. Non-atoms in H with $\max A \leq x$ correspond to subsets of $[0, x]$ that decompose as genuine sumsets. By Theorem 6.2, such sets form an exponentially small fraction of all subsets, proving (1).

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