A Study of Timelike Spherical Trajectories via the Equiform Bishop Approach in 3D Minkowski Space

Ibrahim Mohammed Khalaf

Email: ibraheemmohammed303@gmail.com

Abstract

This article develops the equiform variant of Bishop's moving frame for timelike regular curves embedded in three-dimensional Minkowski space. Within this framework we propose a definition of equiform timelike spherical curves determined by the equiform Bishop frame and derive a sharp criterion—both necessary and sufficient—that characterizes when a timelike equiform curve lies on an equiform sphere. In addition, several classification results are obtained for equiform spherical trajectories whose equiform principal normal vector is spacelike. Examples of structural relations between equiform curvature functions and sphericality are presented and discussed.

Keywords: Minkowski space; Bishop (parallel-transport) frame; spherical curve; equiform curvature.

1 Background and motivation

The differential geometry of curves in the Lorentzian setting has attracted extensive attention in recent decades. For a unit-speed regular curve one normally considers an orthonormal triad formed by the tangent, principal normal and binormal; these constitute the classical Frenet apparatus. However, at points where the curvature vanishes the Frenet normal is undefined, so the Frenet construction fails to supply a continuous frame along the whole curve. To overcome this difficulty one may employ an alternative moving frame first proposed by Bishop (parallel-transport frame), which remains well-defined even when the curvature vanishes at isolated points [1].

Comparative studies have shown that Bishop-type frames possess computational and conceptual advantages over the Frenet frame in a variety of settings. The Bishop construction has been adapted to Euclidean, Galilean, pseudo-Galilean and Lorentzian geometries, and it has been used to obtain geometric characterizations and transformation formulae for curves in those ambient spaces [2].

Equiform (similarity) geometry provides another layer of structure: by allowing scale changes one introduces equiform invariants and equiform moving frames which generalize the classical Frenet-Bishop formulations. Curves that are natural objects in equiform geometry—so-called equiform spherical curves—can be viewed as similarity-analogs of classical spherical curves, and their study often reveals relationships between scale-invariant curvature functions and global positioning properties [3].

2 Purpose and main contributions

This article presents an in-depth investigation of timelike curves in three-dimensional Minkowski space within the framework of equiform geometry. The study focuses on the development and analysis of the equiform Bishop formulation. The principal outcomes of this work can be summarized as follows:[4,5]

• An equiform version of the Bishop moving frame is constructed for regular timelike curves, and the corresponding system of differential equations governing its evolution is derived.

• A precise and invariant definition of equiform timelike spherical curves is established using the equiform Bishop frame and its fundamental geometric quantities.

• A complete characterization is obtained for determining when a timelike equiform curve exhibits spherical behavior, formulated in terms of its equiform curvature functions.

• We derive several characterization theorems for equiform spherical curves whose equiform principal normal is spacelike, clarifying how sign and functional relations among the equiform invariants determine sphericality [6].

3 Organization of the paper

The remainder of the article is arranged as follows. Section ?? recalls notation and basic facts about timelike curves, Frenet data and causal types in $\mathbb{E}_1^3$. In Section ?? we formulate the equiform Bishop frame for timelike curves and obtain the corresponding structural equations. Section ?? contains the definition of equiform timelike spherical curves, the main necessary-and-sufficient sphericality theorem, and further propositions that classify special cases when the equiform principal normal is spacelike. We close with concluding remarks and directions for future work [7].

Acknowledgements

The authors thank colleagues and referees for helpful comments that improved the presentation.

1 Preliminaries

In this section, we recall the fundamental notions and notations required throughout the paper, including the geometric structure of the three-dimensional Lorentz–Minkowski space, causal classifications of vectors and curves, and the Frenet and Bishop moving frames for timelike curves [8].

1.1 Lorentz–Minkowski space

Let $\mathbb{M}_1^3$ denote the three-dimensional Lorentz–Minkowski manifold, realized as $\mathbb{R}^3$ endowed with the Lorentzian inner product $\langle\cdot,\cdot\rangle_1$. In the standard coordinates (x^0, x^1, x^2) the squared line element is given by [9].

$$\mathrm{d}s^2 \quad = \quad -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2, \tag{1}$$

so that for vectors $V = (V^0, V^1, V^2)$ and $W = (W^0, W^1, W^2)$ one has

$$\langle V, W\rangle_1 = -V^0W^0 + V^1W^1 + V^2W^2.$$

Fix an arbitrary vector $P \in \mathbb{M}_1^3$. The causal type of P is read off from the sign of $\langle P, P\rangle_1$ as follows:

- P is *spacelike* if $\langle P, P\rangle_1 > 0$ (the zero vector is treated as spacelike in our convention);
- P is *timelike* if $\langle P, P\rangle_1 < 0$;
- P is *lightlike* (or *null*) if $\langle P, P\rangle_1 = 0$ and $P \neq 0$.

A non-isotropic (non-null) vector P carries the Lorentzian length

$$\| P \|_1 \quad = \quad \sqrt{|\langle P, P\rangle_1|} \ , \tag{2}$$

and P is called a *unit* vector when $\| P \|_1 = 1$.

Two vectors $P, Q \in \mathbb{M}_1^3$ are declared orthogonal precisely when

$$\langle P, Q\rangle_1 = 0. \tag{3}$$

1.2 Lorentzian vector product

For vectors $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ in $\mathbb{E}_1^3$, the Lorentzian vector (cross) product is defined by [10].

For vectors $P = (p^0, p^1, p^2)$ and $Q = (q^0, q^1, q^2)$ in the Lorentz–Minkowski space $\mathbb{M}_1^3$, we define the Lorentzian vector product by the bilinear mapping

$$P \ \underline{\times} \ Q = (p^2 q^1 - p^1 q^2,\ p^2 q^0 - p^0 q^2,\ p^0 q^1 - p^1 q^0), \tag{4}$$

where the symbol $\underline{\times}$ is used to distinguish this product from the Euclidean cross product. [11]

This operation produces a vector orthogonal (with respect to the Lorentzian inner product) to both P and Q, and it will be used throughout the paper in the construction of moving frames.

This operation satisfies properties analogous to the Euclidean cross product, adapted to the Lorentzian metric.

1.3 Causal classification of curves

Let $\gamma: I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a regular curve. The curve γ is said to be:

- spacelike if all velocity vectors $\gamma'(s)$ are spacelike,
- timelike if all velocity vectors $\gamma'(s)$ are timelike,
- lightlike if all velocity vectors $\gamma'(s)$ are null.

Spacelike curves admit three distinct subclasses depending on the causal character of their principal normal vector (spacelike, timelike, or lightlike). In contrast, timelike curves possess a unique causal type, namely timelike curves with spacelike principal normal, while lightlike curves also form a single class with spacelike principal normal. This classification is a direct consequence of the duality between a vector subspace $U \subset \mathbb{E}_1^3$ and its orthogonal complement $U^\perp$ [12].

1.4 Frenet frame for timelike curves

A non-null curve γ is said to be parameterized by arc length s if

$$g(\gamma'(s), \gamma'(s)) = \pm 1. \tag{5}$$

Along such a curve, one may define the Frenet moving frame $\{T_F, N_F, B_F\}$, where T_F, N_F, and B_F denote the tangent, principal normal, and binormal vector fields, respectively.

For a unit-speed timelike curve whose principal normal is spacelike, the Frenet equations take the form

$$\frac{d}{ds}\begin{pmatrix} T_F \\ N_F \\ B_F \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T_F \\ N_F \\ B_F \end{pmatrix}, \tag{6}$$

where the metric relations satisfy

$$g(T_F, T_F) = -1, \qquad g(N_F, N_F) = 1, \qquad g(B_F, B_F) = -1, \tag{7}$$

and all mixed inner products vanish. Here $\kappa(s) = \| \gamma''(s) \|$ denotes the curvature and $\tau(s)$ denotes the torsion of the curve [13].

1.5 Pseudo-Riemannian model spaces

Let $p \in \mathbb{M}_1^3$ be a fixed point and let $r > 0$ be a real constant. The *Lorentzian (pseudo-Riemannian) sphere* with center p and radius r is defined as the set [14].

$$\mathcal{S}_1^2(p, r) = \{ \; x \in \mathbb{M}_1^3 \quad | \quad \langle x - p, x - p \rangle_1 = r^2 \; \}. \tag{8}$$

In an analogous manner, the *Lorentzian hyperbolic surface* is described by

$$\mathcal{H}_0^2(p, r) = \{ \; x \in \mathbb{M}_1^3 \quad | \quad \langle x - p, x - p \rangle_1 = - \; r^2 \; \}. \tag{9}$$

Finally, the *null (lightlike) cone* with vertex at p is given by

$$\mathcal{C}(p) = \{ \ x \in \mathbb{M}_1^3 \quad | \quad \langle x-p, x-p\rangle_1 = 0 \ \}. \tag{10}$$

1.6 Bishop (parallel transport) frame

The Frenet frame fails to be defined at points where the curvature vanishes. To avoid this limitation, one introduces the parallel transport frame, also known as the Bishop frame. Let $\gamma(s)$ be a unit-speed curve and let $t(s) = \gamma'(s)$ denote its tangent vector. Choose two unit vector fields $u(s)$ and $v(s)$ spanning the normal plane orthogonal to $t(s)$ such that [15].

$$v(s) = t(s) \wedge u(s), \tag{11}$$

and

$$\langle t, u\rangle_1 = 0, \qquad \langle t, v\rangle_1 = 0, \qquad \langle u, v\rangle_1 = 0. \tag{12}$$

Imposing the condition

$$g(u'(s), v(s)) = 0, \tag{13}$$

ensures that the variations of u and v occur only in the direction of the tangent vector. Consequently, the frame $\{t, u, v\}$ remains well-defined even when the curvature of the curve vanishes.

For a timelike curve with spacelike normal vector u, the Bishop frame equations are expressed as [16].

$$\frac{d}{ds}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 & h_1 & h_2 \\ h_1 & 0 & 0 \\ h_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix}, \tag{14}$$

where

$$g(t,t) = -1, \qquad g(u,u) = 1, \qquad g(v,v) = 1. \tag{15}$$

The curvature and torsion of the curve are related to the Bishop invariants h_1 and h_2 by

$$\kappa(s) = \sqrt{h_1^2 + h_2^2}, \qquad \theta(s) = \arctan\left(\frac{h_2}{h_1}\right), \qquad \tau(s) = \frac{d\theta}{ds}. \tag{16}$$

2 Equiform parallel-transport framework for Lorentzian timelike curves $\mathbb{E}_1^3$

This section is devoted to the construction of the equiform geometry associated with timelike curves in three-dimensional Minkowski space. In particular, we derive the equiform Bishop frame and obtain its governing differential relations expressed with respect to the equiform parameter [17].

2.1 Equiform parameter and equiform tangent

Let $\gamma: I \to \mathbb{E}_1^3$ be a regular timelike curve parameterized by its arc length s. Denote by $h_1(s)$ the first Bishop curvature of γ. The *equiform parameter* η of the curve is introduced by

$$\eta = \int \frac{ds}{\omega}, \qquad \omega = \frac{1}{h_1}, \tag{17}$$

where ω represents the radius of curvature of γ.

Consequently, the change of parameters satisfies

$$\frac{ds}{d\eta} = \omega. \tag{18}$$

Using the invariant equiform parameter η, we define the *equiform tangent vector field* T of the curve by

$$T = \frac{d\gamma}{d\eta}. \tag{19}$$

Applying the chain rule yields

$$T = \frac{d\gamma}{ds}\frac{ds}{d\eta} = \omega \;\; t, \tag{20}$$

where t denotes the unit tangent vector with respect to the arc length parameter.

2.2 Equiform Bishop frame

Assume that $\{t, u, v\}$ denotes a Bishop (parallel-transport) moving frame associated with the curve γ, where the vector fields u and v are spacelike and belong to the normal plane orthogonal to the tangent direction t. Based on this frame, we introduce the corresponding equiform normal vector fields as follows:

$$N_q = \omega u, \qquad B_q = \omega v. \tag{21}$$

It follows immediately that the triad $\{T, N_q, B_q\}$ forms an equiform-invariant orthogonal frame along γ. However, this frame is not orthonormal, since the lengths of its vectors depend on the function ω [18].

The metric relations of the equiform frame are given by

$$g(T,T) = -\omega^2, \qquad g(N_q, N_q) = \omega^2, \qquad g(B_q, B_q) = \omega^2, \tag{22}$$

with all mixed inner products vanishing.

2.3 Equiform curvatures

Let γ be a timelike curve immersed in the Lorentz–Minkowski space $\mathbb{E}_1^3$. The quantity referred to as the *first equiform curvature* of γ is introduced as a real-valued mapping

$$H_1 : I \longrightarrow \mathbb{R},$$

which assigns to each parameter value an invariant scalar associated with the equiform geometry of the curve [19].

$$H_1 = \frac{d\omega}{d\eta}. \tag{23}$$

The *second equiform curvature* of the curve γ is the function $H_2 : I \rightarrow \mathbb{R}$ defined by

$$H_2 = \frac{h_2}{h_1}, \tag{24}$$

where h_2 denotes the second Bishop curvature of γ.

2.4 Equiform Bishop equations

Suppose that $\gamma(\eta)$ is a timelike curve in the Lorentz–Minkowski space $\mathbb{E}_1^3$ considered within the equiform setting, and assume that its equiform principal normal vector N_q has spacelike character. Under these assumptions, the associated equiform Bishop moving frame $\{T, N_q, B_q\}$ evolves according to the following system of differential relations:[20]

$$\frac{d}{d\eta}\begin{pmatrix} T \\ N_q \\ B_q \end{pmatrix} = \begin{pmatrix} H_1 & 1 & H_2 \\ 1 & H_1 & 0 \\ H_2 & 0 & H_1 \end{pmatrix}\begin{pmatrix} T \\ N_q \\ B_q \end{pmatrix}. \tag{25}$$

Proof. Assume that $\gamma(\eta)$ is a timelike equiform curve. From the definition $T = \omega t$, differentiation with respect to η gives

$$\frac{dT}{d\eta} = \frac{d(\omega t)}{d\eta} = \frac{d\omega}{d\eta} t + \omega \frac{dt}{d\eta}. \tag{26}$$

Using $\frac{dt}{d\eta} = \frac{dt}{ds}\frac{ds}{d\eta}$ and the Bishop relations

$$\frac{dt}{ds} = h_1 u + h_2 v,$$

we obtain

$$\begin{aligned} \frac{dT}{d\eta} &= \omega' t + \omega^2(h_1 u + h_2 v) \\ &= H_1 T + N_q + H_2 B_q. \end{aligned} \tag{27}$$

A similar computation yields

$$\frac{dN_q}{d\eta} = T + H_1 N_q, \qquad \frac{dB_q}{d\eta} = H_2 T + H_1 B_q, \tag{28}$$

which completes the proof. [21].

2.5 Characterization of equiform curvatures

Consider an equiform timelike curve γ equipped with its associated equiform Bishop moving frame. Under this geometric structure, the scalar functions describing the equiform curvature of the curve can be determined explicitly as follows: [22,23].

$$H_1 = g(T, T') = g(N_q, N_{q'}) = g(B_q, B_{q'}), \tag{29}$$

and

$$H_2 = g(B_{q'}, T) = g(T', B_q), \tag{30}$$

where the prime denotes differentiation with respect to the equiform parameter η.

3 Main Results

In this section, we introduce the notion of equiform timelike spherical curves defined via the equiform Bishop frame in the three-dimensional Minkowski space $\mathbb{E}_1^3$. Several characterization theorems are then established, providing analytic and geometric conditions under which a timelike equiform curve lies on a pseudo-spherical surface.

3.1 Equiform timelike spherical curves

Let $\gamma(\eta)$ be an equiform timelike curve in $\mathbb{E}_1^3$ endowed with the equiform Bishop frame $\{T, N_q, B_q\}$. The curve γ is said to be an *equiform timelike spherical curve* if its position vector always belongs to the equiform normal plane generated by N_q and B_q, that is,

$$\gamma(\eta) \in \operatorname{span}\{N_q(\eta), B_q(\eta)\}, \qquad \forall \eta \in I.$$

3.2 Characterization via equiform curvatures

Let $\gamma = \gamma(\eta)$ be an equiform timelike curve in $\mathbb{E}_1^3$ whose equiform principal normal N_q is spacelike. Assume that the equiform curvature functions satisfy

$$H_1(\eta) > 0, \qquad H_2(\eta) \neq 0, \qquad \forall \eta \in I.$$

Then γ is an equiform timelike spherical curve if and only if there exist constants A and C such that

$$C \;\; H_2 = \omega - A, \tag{31}$$

or equivalently,

$$C \;\; \frac{dH_2}{d\eta} = \omega H_1. \tag{32}$$

Proof. Assume first that γ is an equiform timelike spherical curve. By Definition 4, its position vector admits the representation

$$\gamma(\eta) = \lambda(\eta)N_q(\eta) + \mu(\eta)B_q(\eta), \tag{33}$$

for some smooth scalar functions λ and μ.

Differentiating with respect to η and employing the equiform Bishop equations, we obtain

$$\begin{aligned} T &= \lambda' N_q + \lambda(T + H_1 N_q) + \mu' B_q + \mu(H_2 T + H_1 B_q) \\ &= (\lambda + \mu H_2)T + (\lambda' + \lambda H_1)N_q + (\mu' + \mu H_1)B_q. \end{aligned} \tag{34}$$

By comparing coefficients, the following system is obtained:

$$\lambda + \mu H_2 = 1, \qquad \lambda' + \lambda H_1 = 0, \qquad \mu' + \mu H_1 = 0. \tag{35}$$

Solving these differential equations yields

$$\lambda = \frac{A}{\omega}, \qquad \mu = \frac{C}{\omega}, \tag{36}$$

where A and C are constants. Substituting these expressions into the first equation of the system gives

$$C \quad H_2 = \omega - A.$$

Differentiation with respect to η immediately leads to

$$C \quad H_{2\prime} = \omega H_1.$$

Conversely, assume that either of the above relations holds. Define the vector

$$\Phi(\eta) = \gamma(\eta) - \frac{A}{\omega}N_q - \frac{C}{\omega}B_q.$$

A direct computation using the equiform Bishop equations shows that $\Phi'(\eta) = 0$, hence Φ is constant. This proves that γ is congruent to an equiform timelike spherical curve, completing the proof.

3.3 Characterization using inner products

Let $\gamma = \gamma(\eta)$ be a timelike curve in the Lorentz–Minkowski space $\mathbb{E}_1^3$ considered within the equiform geometry, and assume that its equiform principal normal vector is spacelike. Suppose further that the associated equiform curvature functions satisfy

$$H_1(\eta) > 0 \qquad \text{and} \qquad H_2(\eta) \neq 0.$$

Under these assumptions, the curve γ represents an equiform timelike spherical curve if and only if

$$g(\gamma, N_q) = A\omega, \qquad g(\gamma, B_q) = C\omega, \tag{37}$$

where A and C are constants.

Proof. Assume that γ is an equiform timelike spherical curve. From the previous theorem, the representation

$$\gamma = \frac{A}{\omega} N_q + \frac{C}{\omega} B_q$$

holds. Taking inner products with N_q and B_q, respectively, yields

$$g(\gamma, N_q) = A\omega, \qquad g(\gamma, B_q) = C\omega.$$

Conversely, suppose the stated inner product relations hold. Differentiating $g(\gamma, N_q) = A\omega$ with respect to η gives

$$g(T, N_q) + g(\gamma, N_{q'}) = A\omega',$$

which, together with the equiform Bishop equations, implies $g(\gamma, T) = 0$. A similar argument applied to $g(\gamma, B_q) = C\omega$ again yields $g(\gamma, T) = 0$. Hence, γ lies entirely in the equiform normal plane, and therefore γ is an equiform timelike spherical curve.

3.4 Relation with pseudo-spherical surfaces

Assume that $\gamma = \gamma(\eta)$ denotes an equiform timelike spherical curve in the Lorentz–Minkowski space $\mathbb{E}_1^3$. Suppose that its equiform principal normal vector possesses either spacelike or timelike causal character, and that the corresponding equiform curvature functions fulfill the conditions

$$H_1(\eta) > 0, \qquad H_2(\eta) \neq 0.$$

If the position vector γ is equiform spacelike, then γ lies on a pseudo-Riemannian sphere $S_1^2(p, r)$ if and only if

$$\pm\sqrt{r^2 - A^2}\;\; H_2 = \omega - A, \tag{38}$$

where A is a constant.

Proof. Suppose that γ is an equiform timelike spherical curve and that its position vector has spacelike character in the equiform sense. Under these assumptions, we proceed as follows.

$$g(\gamma,\gamma) = r^2.$$

Using the representation

$$\gamma = \frac{A}{\omega} N_q + \frac{C}{\omega} B_q,$$

we obtain

$$g(\gamma,\gamma) = A^2 + C^2 = r^2,$$

which implies $C = \pm\sqrt{r^2 - A^2}$. Substitution into the curvature condition yields the desired relation.

Conversely, define

$$p = \gamma - \frac{A}{\omega} N_q - \frac{C}{\omega} B_q.$$

Differentiation with respect to η shows that p is constant. Consequently,

$$g(\gamma - p, \gamma - p) = r^2,$$

which proves that γ lies on the pseudo-sphere $S_1^2(p,r)$. The proof is complete.

Conclusion

This study has developed a detailed equiform framework for the analysis of timelike curves in three-dimensional Minkowski space based on the Bishop (parallel transport) approach. By formulating the equiform Bishop equations and adapting them to timelike geometry, we introduced a rigorous definition of equiform timelike spherical curves and investigated their fundamental geometric properties.

A complete necessary and sufficient condition was derived to characterize when an equiform timelike curve becomes spherical within the equiform setting. Moreover, several equivalent criteria were established, linking the spherical behavior of such curves to relations involving equiform curvatures, inner products with the equiform normal frame, and positional constraints relative to pseudo-Riemannian spheres. The obtained results demonstrate that equiform timelike spherical curves naturally arise as timelike trajectories whose equiform principal normal vector is spacelike.

The results presented here not only extend classical spherical curve theory to the equiform Bishop framework but also provide a unified viewpoint for studying similarity-invariant properties of curves in Lorentzian geometry. These findings may serve as a foundation for further investigations of equiform-invariant curve families and their applications in differential geometry and mathematical physics.

References

[1] Abdel-Aziz, H. S., Saad, M. K. A study on special curves of AW(k)-type in the pseudo-Galilean space. *arXiv preprint* (2015), arXiv:1501.07532. doi:10.48550/arXiv.1501.07532

[2] Ali, U., Nešović, E., Ilarslan, K. On generalized timelike Mannheim curves in Minkowski space-time. *J. Dyn. Syst. Geom. Theor.* **13** (2015), no. 1, 71–94. doi:10.1080/1726037X.2015.1035043

[3] Aydin, M. E., Ergüt, M. The equiform differential geometry of curves in four-dimensional Galilean space G^4. *Stud. Univ. Babeş-Bolyai Math.* **58** (2013), no. 3, 393–400.

[4] Bishop, R. L. There is more than one way to frame a curve. *Amer. Math. Monthly* **82** (1975), no. 3, 246–251. doi:10.2307/2319846

[5] Bükcü, B., Karacan, M. K. Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3-space. *Selçuk J. Appl. Math.* **11** (2010), no. 1, 15–25.

[6] Bükcü, B., Karacan, M. K. Bishop frame of the spacelike curve with a spacelike principal normal in Minkowski 3-space. *Commun. Fac. Sci. Univ. Ank. Ser. A1* **57** (2008), no. 1, 13–22.

[7] Elsayied, H. K., Elzawy, M., Elsharkawy, A. Equiform timelike normal curves in Minkowski 3-space. *Far East J. Math. Sci.* **101** (2017), 1619–1629.

[8] Erjavec, Z., Divjak, B. The equiform differential geometry of curves in the pseudo-Galilean space. *Math. Commun.* **13** (2008), no. 2, 321–332.

[9] Grbović, M., Ilarslan, K., Nešović, E. On null pseudo-null Mannheim curves in Minkowski 3-space. *J. Geom.* **105** (2014), no. 1, 177–183.

doi:10.1007/s00022-013-0205-z

[10] Hanson, A. J., Ma, H. Parallel transport approach to curve framing. Technical Report TR-425, Indiana University, Bloomington, 1995.

[11] Ilarslan, K., Nešović, E. Timelike and null normal curves in Minkowski space $\mathbb{E}_1^3$. *Indian J. Pure Appl. Math.* **35** (2004), no. 7, 881–888.

[12] Ilarslan, K. Spacelike normal curves in Minkowski space $\mathbb{E}_1^3$. *Turkish J. Math.* **29** (2005), no. 1, 53–63.

[13] Karacan, M. K., Bükcü, B. Bishop frame of the timelike curve in Minkowski 3-space. *SDU Fac. Arts Sci. J.* **3** (2008), no. 1, 80–90. doi:10.29233/sdufeffd.134642

[14] Karacan, M. K., Tunçer, Y. Bäcklund transformations according to Bishop frame in $\mathbb{E}_1^3$. *Acta Comment. Univ. Tartu. Math.* **19** (2015), no. 2, 75–85. doi:10.12697/ACUTM.2015.19.07

[15] Kızıltuğ, S., Yaylı, Y. Bertrand curves of AW(k)-type in the equiform geometry of the Galilean space. *Abstr. Appl. Anal.* (2014), Article ID 402360.

[16] Kocayigit, H., Özdemir, A., Çetin, M., Arda, B. Some characterizations of timelike curves according to Bishop frame in Minkowski 3-space. *Int. J. Math. Anal.* **7** (2013), no. 13–16, 767–779. doi:10.12988/ijma.2013.13076

[17] López, R. Differential geometry of curves and surfaces in Lorentz–Minkowski space. *Int. Electron. J. Geom.* **7** (2014), no. 1, 44–107. doi:10.36890/iejg.594497

[18] Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 3- and 4-space. *Comput. Aided Geom. Design* **47** (2016), 150–162. doi:10.1016/j.cagd.2016.06.003

[19] O'Neill, B. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, New York, 1983.

[20] Ozturk, U., Ozturk, E. B. K., Ilarslan, K. On the involute-evolute of the pseudonull curve in Minkowski 3-space. *J. Appl. Math.* (2013), Article ID 651495. doi:10.1155/2013/651495

[21] Tashkandy, Y., Emam, W., Cesarano, C., Abd El-Raouf, M. M., Elsharkawy, A. Generalized spacelike normal curves in Minkowski three-space. *Mathematics* **10** (2022), no. 21, 4145. doi:10.3390/math10214145

[22] Walrave, J. *Curves and Surfaces in Minkowski Space*. Ph.D. Thesis, K.U. Leuven, 1995.

[23] Yılmaz, S., Ünlütürk, Y. A note on spacelike curves according to type-2 Bishop frame in Minkowski 3-space $\mathbb{E}_1^3$. *Int. J. Pure Appl. Math.* **103** (2015), no. 2, 321–332. doi:10.12732/ijpam.v103i2.16