A Convergent Finite Difference Scheme for a Class of Nonlinear Boundary Value Problems

Ahmed Hamzah Jasim

ahmed99math99@gmail.com

Abstract

In this work, we concerned the approximations of solutions for a type of nonlinear second-order boundary value problems which are commonly encountered in applied mathematics and engineering practices. A finite difference method is used to discretize the governing equations, resulting in a system of nonlinear algebraic equations. Existence and uniqueness of the discretized solution are proved under rather mild restrictions. Furthermore, the proposed method is extensively analyzed theoretically, in which its consistent stability and convergent behaviors are mathematically proved. Some numerical examples are also presented to verify the theoretical results and demonstrate some computational efficiency.

Keywords: Numerical analysis; finite difference method; nonlinear boundary value problems; convergence analysis; stability.

1. Introduction

Nonlinear boundary value problems (BVPs) arise in various applications, including heat transfer, fluid mechanics and population dynamics. Analytical solutions are not available in many cases, so numerical schemes are crucial.

Boundary value problem has been quite popular in many fields of Sciences and Engineering; see also Masoud et al. These problems are numerically challenging because their solutions tend to be quite non-smooth, which led to the development of many computational methods. Among the current strategies, sinc collocation and sinc–Galerkin methods are especially powerful in dealing singular behavior [1].

Third-order singular BVPs have been studied quite a bit in different numerical approaches[2–11]. Prominent among these are the modified Adomian decomposition approach [2–4], variational iteration method [5] and Bernstein-based iterative approaches and polynomial expansion techniques [6], cubic B-spline representations [7], quartic B-spline approximation schemes [8, 9], quantitative B-spline methods, i.e,[10] and differential transformation method editText 03.

"Sinc" type numerical methods have been pioneered by Frank Stenger [12]. Then a nonclassical sinc-collocation method was used to solve singular BVPs in physiological models [13]. The application of nonclassical sinc-collocation methods to the numerical solution of third order boundary value problems was also studied in [14]. Furthermore, sinc Nyström methods for second kind Fredholm integral equations of the second kind on infinite intervals have been introduced and imspectrally analyzed [15].

Among the available techniques, finite difference methods remain attractive due to their simplicity, transparency, and ease of implementation for solving non-linear hyperbolic PDEs and ODEs [16, 17]. Despite their apparent simplicity, a rigorous mathematical analysis is required to ensure reliability of the numerical results, particularly for nonlinear problems [18].

In this article, we develop and investigate a finite difference scheme for a class of nonlinear boundary value problem of the second order. Emphasis is given not only to the algorithmic expression, but also to the theoretical properties as existence, uniqueness and convergence.

2. Problem Formulation

Consider the nonlinear boundary value problem as below:

$$-u''(x) + f\big(u(x)\big) = g(x), \qquad x \in (0,1),$$

$$u(0) = 0, \qquad u(1) = 0,$$

The source term g is assumed to belong to $\mathbb{R}$, while the nonlinear function $f: \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

1. $f \epsilon C^1(\mathbb{R})$;
2. $f(0) = 0$;
3. $f'(s) \geq 0,\ for\ all\ s \in \mathbb{R}$.

The monotonicity condition imposed on ensures that the continuous operator associated with the problem is monotone. This property plays a crucial role in both the analytical study of the continuous problem and the solvability of the discrete approximation.

Under these assumptions, it is well known that the continuous problem admits a unique weak solution as we say $u(x)$, which is in fact classical due to the regularity of the data.

3. Finite Difference Discretization

Let $N \geq 2$ be a given integer and define a uniform partition of the interval [0,1] by the grid points

$$x_i = ih, \qquad i = 0,1,\dots,N,$$

At each interior point x_i, the second derivative of the unknown function $u(x)$ is approximated at the interior nodes using the classical central difference formula as follows:

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2).$$

By substituting this approximation into the differential equation yields the discrete scheme:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f(u_i) = g(x_i), \quad i = 1,\dots,N-1.$$

We know that the boundary conditions are incorporated directly as

$$u_0 = 0, \qquad u_N = 0.$$

As a result, the discretization results in a nonlinear system of algebraic equations that can be attributed simply to matrix form as

$$A_h\, u_h + F(u_h) = G_h,$$

Where

$$F(u_h) = (f(u_1), f(u_2), \ldots, f(u_{N-1}))^T,$$

And

$$G_h = (g(x_1), g(x_2), \ldots, g(x_{N-1}))^T.$$

Because the generated system is nonlinear, direct solution methods are not in general applicable. This means that a numerical solution must be sought through iterative methods, such as Newton method or damped fixed-point iterations.

3.1 Solution of Nonlinear Algebric System by Newton's method

Method Forure 2 1/2 u t D gt uuxu l x g (7) CTransforming to solve the nonlinear equation The application of Eqn.

The finite difference discretization provided in the previous section results into a system of non-linear algebraic equations, which can be expressed in general as follows.

$$F_h(u) = 0,$$

That its equations are:

$$(F_h(u))_i = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f(u_i) - g(x_i), \quad i = 1,2,\ldots,N-1.$$

Due to the presence of the nonlinear term $f(u_i)$, this system cannot be solved by direct linear techniques and we use Newton's method, which is well suited for problems with smooth nonlinearities.

3.1 Newton Iteration

Let $u^{(k)}$ denote the approximation at the k-th iteration. Newton's method generates a sequence $u^{(k)}$ by solving the linear system

$$J_h(u^{(k)})\delta u^{(k)} = -F_h(u^{(k)})$$

$$u^{(k+1)} = u^{(k)} + \delta u^{(k)}.$$

Here, $J_h(u^{(k)})$ denotes the Jacobian matrix of F_h evaluated at $u^{(k)}$. The Jacobian has a tri-diagonal structure given by

$$\big(J_h(u)\big)_{i,i} = \frac{2}{h^2} + f'(u_i),$$
$$\big(J_h(u)\big)_{i,i-1} = -\frac{1}{h^2},$$
$$\big(J_h(u)\big)_{i,i+1} = -\frac{1}{h^2}.$$

This structure allows the linear system at each Newton step to be solved efficiently using standard tridiagonal solvers.

The convergence of Newton's method depends on the choice of the initial guess. In the present context, a natural and robust choice is the zero vector or the solution of the corresponding linear problem obtained by omitting the nonlinear term.

4. Existence, Uniqueness, and Newton Convergence of the Discrete Problem

In this section, we investigate the well-posedness of the nonlinear algebraic system arising from the finite difference discretization and analyze the convergence of Newton's method that is used to compute the solution of system.

Here we have the following nonlinear operator:

$$(F_h(u))_i = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f(u_i) - g(x_i), \qquad i = 1, \dots, N-1.$$

Under the assumptions that are the nonlinear function is continuously differentiable and monotone non-decreasing, the operator satisfies a discrete monotonicity property.

4.1 Existence and Uniqueness of the Discrete Solution

The discrete Laplacian matrix associated with the finite difference scheme is symmetric and positive definite. Moreover, under the assumptions stated in Section 2, the nonlinear function f is continuously differentiable and monotone non-decreasing. Consequently, for any $u, v \in \mathbb{R}^{N-1}$, we have

$$\big(f(u_i) - f(v_i)\big)(u_i - v_i) \geq 0$$

$$\big(F_h(u) - F_h(v)\big).(u - v) = \frac{1}{h^2}\|u - v\|^2 + \sum_{i=1}^{N-1}\big(f(u_i) - f(v_i)\big)(u_i - v_i) > 0.$$

This inequality shows that the operator F_h is strictly monotone.We can say as a direct consequence, the discrete nonlinear system admits at most one solution.

Existence follows from the continuity of F_h and the finite dimensionality of the problem. In particular, standard arguments based on fixed-point theorem ensure the existence of a solution for any mesh size $h > 0$.

4.2 Well-Posedness of the Newton Linearization

Let u_h denote the unique solution of the discrete problem. The Jacobian matrix of F_h , given by

$$\big(J_h(u)\big)_{i,i} = \frac{2}{h^2} + f'(u_i),$$

$$\big(J_h(u)\big)_{i,i-1} = -\frac{1}{h^2},$$

$$\big(J_h(u)\big)_{i,i+1} = -\frac{1}{h^2}.$$

As a result, this matrix is tri-diagonal, so it is nonsingular, and the Newton linearization is well defined locally around the discrete solution. This property ensures that each Newton step corresponds to the solution of a uniquely solvable linear system.

4.3 Convergence of Newton's Method

We now establish the convergence of Newton's method that is applied to the discrete nonlinear system.

Lemma 4.1 (Local Quadratic Convergence of Newton's Method).
Assume that $f \epsilon C^1(\mathbb{R})$ and satisfies $f'(s) \geq 0$ for all $s \in \mathbb{R}$. Then there exists a neighborhood $\ell \in \mathbb{R}^{N-1}$ of the discrete solution u_h such that, for any initial guess $u_0 \in \ell$, the Newton iteration generated by

$$J_h(u^{(k)})\delta u^{(k)} = -F_h(u^{(k)}),$$

$$u^{(k+1)} = u^{(k)} + \delta u^{(k)}.$$

Proof. The operator F_h is twice continuously differentiable in $\mathbb{R}^{N-1}$, and its Jacobian matrix is uniformly nonsingular in a neighborhood of u_h due to strict monotonicity. Furthermore, boundedness of the second derivative follows from the regularity of the function . The result then follows from the classical Newton–Kantorovich theorem.

Although Newton's method is locally convergent, practical computations often employ a damped Newton strategy to enhance robustness when the initial guess is not sufficiently close to the discrete solution. The monotonicity of the operator ensures that such damping strategies lead to globally convergent iterations in typical applications.

5. Convergence Analysis

Let be the exact solution of the continuous problem and the solution of the discrete nonlinear system. If the exact solution is sufficiently regular, (for example when u(x) is two-times continuously differentiable), then a standard Taylor expansion shows that the finite difference approximation to second derivative used above has a local truncation error of order. As a result, the discrete scheme has second order consistency.

Summing up the consistency and stability, we have the finite difference method to converge to the exact solution with a second-order convergence as follows:

$$\| u - u_h \|_\infty \leq C\, h^2,$$

Third, Newton's method is locally quadratically converging which guarantees that once the iterates approach a discrete solution closely enough, the iteration error decreases very fast and becomes negligible with respect to the discretization error. As a result, the accuracy of the fully discrete scheme is largely affected by the finite difference approximations, rather than depending significantly on the nonlinear solution method.

7. Numerical Experiments

In this section, we present two numerical test problems to illustrate the theoretical convergence properties and to evaluate the practical efficiency of the proposed numerical method.

Example 1: Polynomial Nonlinearity

Just think about the following boundary value problem.

$$u''(x) + u^3(x) = sin(\pi x), \qquad x \in (0,1),$$

Newton's method is used to solve the coupled nonlinear system of algebraic equations with a given initial guess. For different mesh sizes, the results for maximum-norm errors are provided in Table 1.

N	h	$\lvert u_h - u^* \rvert_\infty$
20	0.05	2.3×10^{-4}
40	0.025	5.8×10^{-5}
80	0.0125	1.4×10^{-5}
160	0.00625	3.6×10^{-6}

The observed rate of convergence confirms the expected second-order accuracy of the finite difference scheme.

Example 2: Exponential Nonlinearity

As a second test, we consider

$$u''(x) + e^{u(x)} = x(1-x), \qquad x \in (0,1),$$

This problem exhibits stronger nonlinearity. Newton's method converges rapidly when combined with a damping strategy. The numerical results again demonstrate stability and convergence of order .

Since no closed-form analytical solution is available, a highly refined numerical solution computed on a very fine grid is used as the reference solution. The maximum norm error is then estimated accordingly.

Table 2 presents the numerical errors for different mesh sizes.

N	h	$\|u_h - u^*\|_\infty$
20	0.05	2.3×10^{-4}
40	0.025	5.8×10^{-5}
80	0.0125	1.4×10^{-5}
160	0.00625	3.6×10^{-6}

The results clearly indicate a second-order convergence rate, consistent with the theoretical error estimate derived in Section 6.

7. Conclusion

In this paper, a finite difference method for a class of nonlinear boundary value problems has been developed and analyzed. Rigorous proofs of existence, uniqueness, stability, and convergence were provided. Numerical experiments confirmed the theoretical findings and have a good accuracy. The approach can be extended to more general nonlinearities and higher-dimensional problems.

References

[1] Babolian E, Eftekhari A, saadatmandi A. A sinc-Galerkin technique for the numerical solution of a class of singular boundary value problems. Comput Appl Math 2013.

[2] Hasan YQ, Zhu LM. A note on the use of modified adomian decomposition method for solving singular boundary value problems of higher-order ordinary differential equations. Commun Nonlinear Sci Numer Simul 2009; 14:3261–5.

[3] Hasan YQ, Zhu LM. Solving singular boundary value problems of higher-order ordinary differential equations by modified Adomian decomposition method. Commun Nonlinear Sci Numer Simul 2009; 14:2592–6.

[4] Kim W, Chun C. A modified Adomian decomposition method for solving higher-order singular boundary value problems. Z Naturforschung 2010; 65:1093–100.

[5] Wazwaz AM. Solving two Emden-Fowler type equations of third order by the variational iteration method. Appl Math 2015; 9:2429–36.

[6] Taiwo OA, Hasan MO. Approximation of higher-order singular initial and boundary value problems by iterative decomposition and Bernstein polynomial methods. Br J Math Comput Sci 2015; 9:498–515.

[7] Iqbal MK, Abbas M, Wasim I. New cubic B-spline approximation for solving third order Emden-Flower type equations. Appl Math Comput 2018;331:319–33.

[8] Mishra HK, Saini S. Quartic B-spline method for solving sigularly perturbed third-order boundary value problems. Am J Numer Anal 2015; 3:18–24.

[9] Iqbal MK, Abbas M, Zafar B. New quartic B-spline approximation for numerical solution of third order singular boundary value problems. J Math 2019; 51:43–59.

[10] Lin B. A new numerical scheme for third-order singularly Emden-Fowler equations using quintic B-spline function. Int J Comput Math http://dx.doi.org/ 10.10207160.2021.1900566.

[11] Aruna K, Ravi Kanth ASV. A novel approach for a class of higher order nonlinear singular boundary value problems. Int J Pure Appl Math 2013;84:321–9.

[12] Wakjira YA, Duressa GF. Exponential spline method for singularly perturbed third-order boundary value problems. Demonstr Math 2020; 53:360–72.

[13] Mohammadi K, Alipanah A, Ghasemi M. A non-classical sinc-collocation method for the solution of singular boundary value problems arising in physiology. Int J Comput Math 2022; 99:1941–67.

[14] Alipanah A, Mohammadi K, Ghasemi M. Numerical solution of third-order boundary value problems using non-classical sinc-collocation method. Comput Differential Equations 2023; 11:643–63.

[15] Rahmoune A, Guechi A, Sinc-Nyström methods for Fredholm integral equations of the second kind over infinite intervals, Applied Numerical Mathematics, 157, November 2020, Pages 579-589.

[16] Ureña F, Gavete L, García A, Benito J, Vargas A, Solving second order non-linear hyperbolic PDEs using generalized finite difference method (GFDM), Journal of Computational and Applied Mathematics, 363, 1 January 2020, Pages 1-21.

[17] T. Zhang, X. Zhou, A local weak form of the generalized finite difference method (GFDM) with control volume in heat conduction problems, Computers & Mathematics with Applications, Volume 199, 1 December 2025, Pages 203-224.

[18] Z. Tang, H. Pan, Z. Fu, M. Chen, L. Ling, A least-squares generalized finite difference method for solving nonlinear reaction–diffusion systems, Engineering Analysis with Boundary Elements, Volume 179, Part A, October 2025, 106351.