

ON PAIRED DOUBLE DOMINATION NUMBER OF SOME SPECIAL GRAPHS

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In this paper, we continue the study of paired double domination in graphs. A paired – double dominating set of a graph G with no isolated vertex is a double dominating sets of vertices whose induced subgraph has a perfect matching. We establish Paired double domination number of Triangular Snake, Alternate Triangular Snake, Double Triangular Snake, Alternate Double Triangular Snake, diamond – necklace, diamond-bracelet, diamond-chain and double triangle-chain.

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Key words: Paired domination, Double domination number, Paired - double domination number.

1 INTRODUCTION

Let $G = (V, E)$ be graph with vertex set V and edge set E . We obtain with some terminology. For a vertex v of a graph G , the open neighborhood of a vertex $v \in V$ is $N(u) = \{ V/uv \in E \}$ and closed neighborhood of vertex $N[v] = N(v) \cup \{v\}$.

A subset $S \subseteq V$ is a dominating set of G , if for every vertex $v \in V$, $|N[v] \cap S| \geq 1$. The domination number is the minimum cardinality of a dominating set of G . A subset S of V is double dominating set of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 2$, that is v is in S and has at least one neighbor in S and v is in $V-S$ has at least two neighbors in S [3].

A set S is called paired – dominating set if it dominates V and $\langle S \rangle$ contains at least one perfect matching. A paired –dominating set S with matching M is a dominating set $S = \{v_1, v_2, v_3, \dots, v_{2t-1}, v_{2t}\}$ with independent edge set $M = \{e_1, e_2, \dots, e_t\}$ where each edge e_j joins two elements of S , that is M is perfect matching of $\langle S \rangle$. If $v_j v_k = e_i \in M$, we say that v_j and v_k are paired in S [5]. A set S is called a paired – double dominating set if it is a double dominating set and $\langle S \rangle$ contains at least one perfect matching. The double domination number $\gamma_{dd}(G)$ is the minimum cardinality of double dominating set of G , the paired –

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domination number $\gamma_{pr}(G)$ is the minimum cardinality of paired dominating set of G and paired double domination number $\gamma_{prdd}(G)$ is the minimum cardinality of a paired double dominating set of G . A Triangular Snake T_n is obtained from a path $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n-1$. That is every edge of a path is replaced by a triangle C_3 . An Alternate Triangular Snake $A(T_n)$ is obtained from a path $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ by joining u_i and u_{i+1} (alternatively) to a new vertex v_i . That is every alternatively edge of a path is replaced by a triangle C_3 . A Double Triangular Snake $D(T_n)$ is obtained from a path $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i , $1 \leq i \leq n-1$. That is, a double triangle snake consists of two triangular snakes that have a common path. A Double alternate Triangular Snake $A(D(T_n))$ consists of two alternate triangle snakes that have a common path. That is, double alternate Triangular Snake is obtained from a path $u_1, u_2, u_3, \dots, u_{n-1}, u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i . For $k \geq 2$ an integer, let N_k be the connected cubic graph constructed as follows. Take k disjoint copies $D_1, D_2, \dots, D_k$ of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in D_i . Let N_k be obtained from the disjoint union of these k diamonds by adding the edges $\{a_i b_{i+1} \mid i \in [k-1]\}$ adding the edge $a_k b_1$. We call N_k a diamond – necklace with k diamonds. For $k \geq 1$ a diamond-bracelet B_k with k diamonds is defined as follows. Using our earlier notation, let B_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, \dots, D_{k+1}$ by removing the diamond D_{k+1} and adding a triangle T with $V(T) = \{a, b, c\}$ and adding the edges bb_1 and aa_k . For $k \geq 1$, a diamond-chain L_k with k diamonds is defined as follows. Using our earlier notation, let L_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, \dots, D_{k+1}$ by removing the diamond D_{k+1} and adding two disjoint triangles T_1 and T_2 and adding an edge joining b_1 to a vertex of T_1 and adding an edge joining a_k to a vertex of T_2 . A double triangle-chain DT_2 is obtained from a diamond-chain L_2 with two diamonds by replacing the two diamonds with two triangles and adding two edges joining these two triangles.

Theorem 1.1. [7] For any path P_n , $\gamma_{prdd}(P_n) = \begin{cases} 2 & \text{if } n = 2 \\ \text{does not exist} & \text{if } n = 3 \\ 2\left\lceil \frac{n}{3} \right\rceil + 2 & \text{otherwise} \end{cases}$

Theorem 1.2. [7] For any cycle C_n $\gamma_{prdd}(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$

Theorem 1.3. [7] For any path P_n , $n \neq 3$, $\gamma_{dd}(P_n) \leq \gamma_{prdd}(P_n)$

Theorem 1.4. [7] For any cycle C_n , $\gamma_{dd}(C_n) \leq \gamma_{prdd}(C_n)$

Theorem 1.5. [7] If $n = 3k+2$ where $k \in \mathbb{N}$, then $\gamma_{prdd}(P_n) = \gamma_{prdd}(C_n)$.

2. MAIN RESULTS

Theorem:2.1 For Triangular Snake T_n $n \geq 3$, $\gamma_{prdd}(T_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$

Proof:

Let $V(T_n) = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n, v_1, v_2, v_3, \dots, v_{n-1}\}$ be the vertex set of T_n and S be the γ_{prdd} set of T_n .

Let $S_1 = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n\}$ and $S_2 = \{v_{n-1}\}$.

Case (i) If n is even, then $S = S_1$. Thus $\langle S \rangle$ contains a P_n graph. Hence $\gamma_{prdd}(T_n) = n$.

Case (ii) If n is odd, then $S = S_1 \cup S_2$. Thus $\langle S \rangle$ contains a P_{n+1} graph. Hence $\gamma_{prdd}(T_n) = n+1$.

Theorem:2.2 For Alternate Triangular Snake $A(T_n)$ $n \geq 4$, $\gamma_{prdd}(A(T_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$

Proof:

Let $V(A(T_n)) = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n, v_1, v_2, v_3, \dots, v_{\lfloor \frac{n}{2} \rfloor}\}$ be the vertex set of $A(T_n)$ and S be the γ_{prdd} set of $A(T_n)$. Let $S_1 = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n\}$ and $S_2 = \{v_{\lfloor \frac{n}{2} \rfloor}\}$.

Case (i) If n is even, then $S = S_1$. Thus $\langle S \rangle$ contains a P_n graph. Hence $\gamma_{prdd}(A(T_n)) = n$.

Case (ii) If n is odd, then $S = S_1 \cup S_2$. Thus $\langle S \rangle$ contains a P_{n+1} graph. Hence $\gamma_{prdd}(A(T_n)) = n+1$.

Theorem:2.3 For Double Triangular Snake $D(T_n)$ $n \geq 4$, $\gamma_{prdd}(D(T_n)) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$

Proof:

Let $V(D(T_n)) = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n, v_1, v_2, v_3, \dots, v_{n-1}, w_1, w_2, w_3, \dots, w_{n-1}\}$ be the vertex set of $D(T_n)$ and S be the γ_{prdd} set of $D(T_n)$. Let $S_1 = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n\}$ and $S_2 = \{w_{n-1}\}$.

Case (i) If n is even, then $S = S_1$. Thus $\langle S \rangle$ contains a P_n graph. Hence $\gamma_{prdd}(D(T_n)) = n$.

Case (ii) If n is odd, then $S = S_1 \cup S_2$. Thus $\langle S \rangle$ contains a P_{n+1} graph. Hence $\gamma_{prdd}(D(T_n)) = n + 1$.

Theorem:2.4 For Double alternate Triangular Snake $A(D(T_n))$ $n \geq 4$, $\gamma_{prdd}(A(D(T_n))) =$

$$\begin{cases} n & \text{if } n \text{ is even} \\ n + 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof:

Let $V(A(D(T_n))) = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n, v_1, v_2, v_3, \dots, v_{\lfloor \frac{n}{2} \rfloor}, w_1, w_2, w_3, \dots, w_{\lfloor \frac{n}{2} \rfloor}\}$ be the vertex set of $A(D(T_n))$ and S be the γ_{prdd} set of $A(D(T_n))$.

Let $S_1 = \{u_1, u_2, u_3, \dots, u_{n-1}, u_n\}$ and $S_2 = \{w_{\lfloor \frac{n}{2} \rfloor}\}$.

Case (i) If n is even, then $S = S_1$. Thus $\langle S \rangle$ contains a P_n graph.

Hence $\gamma_{prdd}(A(D(T_n))) = n$.

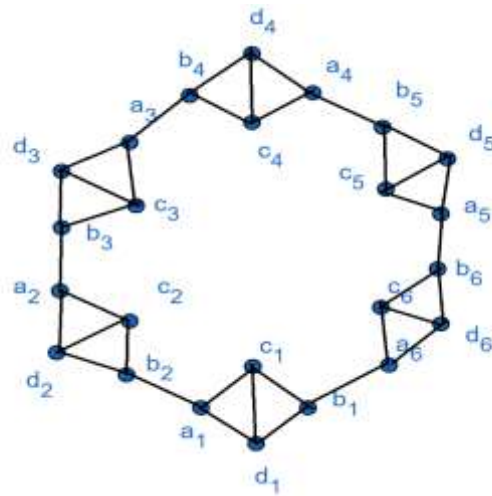
Case (ii) If n is odd, then $S = S_1 \cup S_2$. Thus $\langle S \rangle$ contains a P_{n+1} graph.

Hence $\gamma_{prdd}(A(D(T_n))) = n + 1$.

Theorem:2.5 For a diamond – necklace with k diamonds $\gamma_{prdd}(N_k) = 2k$

Proof:

Let $V(N_k) = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k\}$ be a vertex set of N_k . Consider the set $S = \{c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k\}$. Then S is a double dominating set and $\langle S \rangle$ has a perfect matching. Thus, S is a Paired double dominating set of N_k with cardinality $|S| = 2k$. Hence $\gamma_{prdd}(N_k) = 2k$.

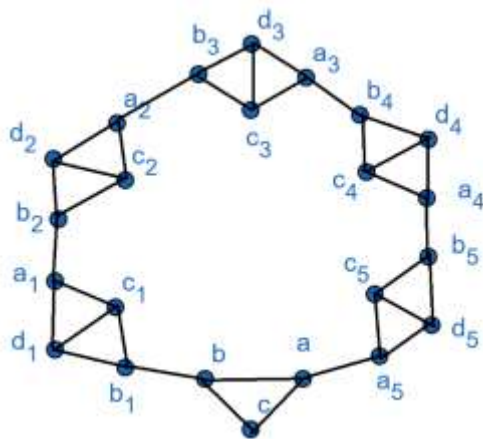
Illustration:2.2.1 N_6

If $k = 6$, then $S = \{c_1, c_2, c_3, c_4, c_5, c_6, d_1, d_2, d_3, d_4, d_5, d_6\}$. Thus $\langle S \rangle$ contains 6 P_2 graph. Hence $\gamma_{prdd}(N_6) = 2 \times 6 = 12$.

Theorem:2.6 For a diamond – bracelet $\gamma_{prdd}(B_k) = 2k + 2$

Proof:

Let $V(B_k) = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k, a, b, c\}$ be a vertex set of B_k . Consider the set $S = \{c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k, a, b\}$. Then S is a double dominating set and $\langle S \rangle$ has a perfect matching. Thus, S is a Paired double dominating set of B_k with cardinality $|S| = 2k + 2$. Hence $\gamma_{prdd}(B_k) = 2k + 2$.

Illustration:2.2.2 B_5

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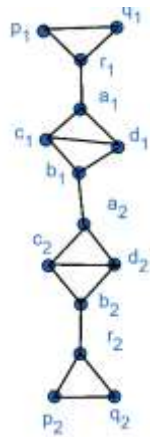
If $k = 5$, then $S = \{c_1, c_2, c_3, c_4, c_5, d_1, d_2, d_3, d_4, d_5, a, b\}$. Thus $\langle S \rangle$ contains 6 P_2 graph. Hence $\gamma_{prdd}(B_5) = 2 \times 6 = 12$.

Theorem:2.7 For a diamond – Chain $\gamma_{prdd}(L_k) = 2k + 4$

Proof:

Let $V(L_k) = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k, p_1, q_1, r_1, p_2, q_2, r_2\}$ be a vertex set of L_k . Consider the set $S = \{c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k, p_1, q_1, p_2, q_2\}$. Then S is a double dominating set and $\langle S \rangle$ has a perfect matching. Thus, S is a Paired double dominating set of L_k with cardinality $|S| = 2k + 4$. Hence $\gamma_{prdd}(L_k) = 2k + 4$.

Illustration:2.2.3



L_2

If $k = 2$, then $S = \{c_1, c_2, d_1, d_2, p_1, q_1, p_2, q_2\}$. Thus $\langle S \rangle$ contains 4 P_2 graph. Hence $\gamma_{prdd}(L_2) = 2 \times 4 = 8$.

Theorem:2.8 For A double triangle-chain $\gamma_{prdd}(DT_k) = 2k + 4$

Proof:

Let $V(DT_k) = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k, p_1, q_1, r_1, p_2, q_2, r_2\}$ be a vertex set of DT_k . Consider the set $S = \{c_1, c_2, \dots, c_k, d_1, d_2, \dots, d_k, p_1, q_1, p_2, q_2\}$. Then S is a double dominating set and $\langle S \rangle$ has a perfect matching. Thus, S is a Paired double dominating set of DT_k with cardinality $|S| = 2k + 4$. Hence $\gamma_{prdd}(DT_k) = 2k + 4$.

Illustration:2.2.4 DT_2

If $k = 2$, then $S = \{c_1, c_2, d_1, d_2, p_1, q_1, p_2, q_2\}$. Thus $\langle S \rangle$ contains $4P_2$ graph. Hence $\gamma_{prdd}(DT_2) = 2 \times 4 = 8$.

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