A Novel Fractional Model for Advancing Urban Flood Prediction

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ABSTRACT
Urban flooding is a growing threat to cities, livelihoods, and ecosystems. The non-linear dynamics of flood waves, characterized by memory effects and interactions over large-scale regions, are often beyond the reach of conventional equation-based flood models. Modern advancements in the field of fractional calculus of fera solution to these issues. The Tanganna derivatives are integrated in a flood model for an extended assessment of historical data and spatial heterogeneity in floods in this paper we using the Fractional Spectral Method (FSM), This method helps to correctly solve the differential equations with complex boundary conditions, and thus, this method makes a quick hand about converting them into an easier form of algebraic equation. Boundary conditions are represented using Robin and Neumann types, with the initial condition given by a hyper bolictangent distribution. This is confirmed through numerical simulations [6, 38, 39].

Keywords: flood dynamics, Fractional Spectral Method, Predictor-corrector scheme, Water height Distribution, Water height at, Sensitivity analysis.

1. INTRODUCTION

Models of dynamics of flooding in various metropolitan environments assume uniform flow and consistency, which can result in mistakes when estimating the levels and extents of floods. New modeling methods propose fractional calculus as a solid basis for simulating processes featuring memory. The benefit of this component is especially useful in flood modeling situations, aspas trainfall events are an important factor that affects the current state of floods due to moisture contained .This capacity of the application to capture detailed and extreme time-dependent diffusion behavior resulted in a very proper representation for derivatives use into flood modeling, nowadays. For example, it cannot explain the fast movements of flood waters because they require a diffusion equation to solve. The fractional diffusion equation is very well in describing these dynamics, which makes it useful for forecasting. Therefore, the objective of this research is to apply Tanganna fractional derivatives in a two-flood model for urban flood prediction, which improves accuracy as compared with existing models. Assumptions of the model include: uniform, steady flow; a homogeneous flood plain (wet-flooding boundaries); and satisfied laws of water equation requirements. We pass from the partial differential equation (PDE) to a system of equations by using Fractional Spectral Method (FSM). For realistic flood dynamics simulations, these equations are numerically solved. A distribution that captures the statistical detail of how dry transitions to wet and thereby providing a realistic way of objectively simulating when flooding starts defines the initial state — realistic as possible. It is implemented the mathematical boundary conditions based on Robin and Neumann condition to accurately represent model-floodplain boundaries (walls, levees or open boundaries).This because even if it is not you can apply this to overall as its work efficiently and accurate in handling situations. It uses base functions to divide the space and makes it easier by converting this problem in a set of equations. If so, we bifurcate the time-related part in a different way to retain stability and precision within an answer. To improve the accuracy of results, we use a predictor-corrector scheme that presents fractional derivatives with high fidelity by combining implicit methods [30, 36]

2. Mathematical Preliminaries

We give the background material required for our study in the following section.

Definition 2.1. The Tanganna fractional derivative of order ηdefined as [21]
form is defined as

$T_t^{\eta} h(t) = \frac{1}{\Gamma(n-\eta)} \int_0^t K(t,\tau) \frac{h^n(\tau)}{(t-\tau)^{\eta-n+1}} d\tau\,,\ t > 0$ (1)

The kernel function $K(t,\tau)$ can be can be customized for uses offering a flexible approach, than standard fractional derivatives. A distinguishing feature of Tanganna derivatives is their capacity to represent systems with memory spans and spatial relationships based on the selection of the kernel function$K(t,\tau)$.

Definition 2.2.
The Tanganna fractional integral of a function $h(t\ )$of order η defined by [21]

$I_t^{\eta} h(t) \coloneqq \frac{1}{\Gamma(\eta)} \int_0^t \frac{h(t)}{(t-\tau)^{1-\eta}} d\tau,$ (2)

3. Model Formulation

Tanganna's fractional derivatives are advantageous for flood modeling because they can capture both local and non local interactions well. The ability to adapt is essential for accurately depicting the changing patterns of flood waves at various spatial scales, as they are influenced by past events and interactions. Including Tanganna derivatives in our flood model aims to achieve a comprehensive and accurate representation of flood dynamics. Consequently, this can improve our capacity to accurately forecast and efficiently control floods. In developing our flood model in two dimensions, we base our analysis on the following assumptions; The flow remains constant and consistent, unaffected, by factors like wind or additional water inputs beyond the flooding. The floodplain exhibits uniformity in both topography and soil properties. The water movement adheres to the principles outlined in water equations relevant for surface runoff in urban settings.
We derive the two-dimensional fractional diffusion equation for flood simulation using Tanganna fractional derivatives. The following represents the governing equation for the water depth $h(x,y,t)$:
$\frac{\partial h}{\partial t} = D_x^{\eta} \frac{\partial^2}{\partial x^2} + D_y^{\eta} \frac{\partial^2}{\partial y^2}$ (3)
where the Tanganna fractional derivatives in the x and y directions are, respectively, D_x^{η} and D_y^{η} [35].
Initial and Boundary Conditions
For initial conditions, a hyper bolictangent distribution, which mimics a realistic change from dry to flooded states, describes the initial condition of the flood model:
$h(x,y,0) = h_0 \tanh(\frac{x-x_0}{L}) \tanh(\frac{y-y_0}{L})$ (4)
The variables h_0 represent the initial water height , $x_0, y_0$are the center coordinates of the flood source, and L determines the steepness of the transition[15].
For boundary conditions, we utilize a combination of Neumann and Robin(mixed)boundary condition to represent practical physical interactions at the floodplain boundaries.
RobinBoundaryCondition:
$ah + b\frac{\partial h}{\partial n} = g(x,y,t)\ \ on\ \Gamma_1$ (5)
Where a and b are coefficients and $\frac{\partial h}{\partial n}$ is the normal derivatives and $g(x,y,t)$isaprescribedfunctionontheboundary Γ_1

Neumann Boundary Condition
$\frac{\partial h}{\partial n} = 0$ on Γ_2 (6)
suggesting that there is no flux across Γ_1[34].
These criteria guarantee that the model can effectively replicate how flood waves interact with different types of boundaries, like walls, levees, and open boundaries.

4. Numerical Scheme : Fractional Spectral Method(FSM)

To solve the diffusion problem, one uses the Fractional Spectral Method (FSM). By using the orthogonality of basis functions, the Fractional Spectral Method (FSM) transforms fractional partial differential equations into a system of equations. This approach is well-known for its accuracy and efficiency when solving complex boundary condition issues.[7].
In two dimensions, we can express the diffusion equation as follows:

$$\frac{\partial^\eta h}{\partial t^\eta} = D\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) \quad (7)$$

where$h(x, y, t)$ is the water height, $D$is the diffusion coefficient and η is the fractional order. The initial phase of solving this equation with the (FSM) method involves dividing down the time and space domains into segments. To discretize the domain, we use a method that approximates the solution using a series of functions. To ensure stability and accuracy in the solution, a finite difference scheme is applied to the time domain [10,25] .

Spatial Domain Discretization

With the spectral method, orthogonal basis functions $\phi_i(x)$and$\phi_j(y)$ are used to discretize the spatial domainas

$$h(x, y, t) \approx \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} H_{ij}(t)\, \phi_i(x)\phi_j(y) \quad (8)$$

where N_x andN_y are stand for respectively, the number of basis functions in the x and y directions [33].

Time Domain Discretization

The concept of difference allows us to partition time into discrete intervals approximation of $h(x_i, y_j, t_n)$ at the grid coordinates (x_i, y_j) and time t_n In order to accurately discretize the time derivative, we employ forward difference techniques such as:

$$\frac{\partial h}{\partial t} \approx \frac{{h_{ij}}^{n-1} - {h_{ij}}^{n}}{\Delta t} \quad (9)$$

where Δt represents the time step. When you use spectral spatial discretization and finite difference time discretization together, the numerical solution is stable and correct. This means that fractional partial differential equations can be solved with it [10, 18].

Predictor-Corrector Scheme

In order to improve the precision and consistency of the solution, we utilize a method known as a predictor-corrector strategy. This technique entails making a prediction of the solution at a given time using a method and subsequently refining it using an implicit method. By blending these methodologies, we guarantee that the fractional derivatives are depicted with accuracy in the solution, resulting in dependable flood forecasts.

The estimated outcome at the midway point in time is ${h_{ij}}^{n+\frac{1}{2}}$, while the corrected result at the time interval is ${h_{ij}}^{n+1}$. To compute the forecasting phase, we apply Eulers method:

$${h_{ij}}^{n+\frac{1}{2}} = {h_{ij}}^{n} + \Delta t\left(\frac{\partial h}{\partial t}\right)_{ij}^{n} \quad (10)$$

For the corrector step, we use an implicit backward Euler method, to correct the solution at the next time step $n + 1$

$${h_{ij}}^{n+\frac{1}{2}} = {h_{ij}}^{n} + \Delta t\left(\frac{\partial h}{\partial t}\right)_{ij}^{n+\frac{1}{2}} \quad (11)$$

The predictor-corrector system improves the accuracy of numerical solutions by incorporating both explicit and implicit steps. This results in more accurate forecasts of flood dynamics. In order to increase the precision and stability of numerical simulations, the predictor-corrector system is essential. Especially when dealing with fractional partial differential equation problems

These approaches yield a collection of equations for the coefficients $H_{ij}(t)$.let ${h^n}_{ij}$ be the water height at time t_n and grid point (x_i, y_j),and let ${H^n}_{ij}$ denote the appropriate coefficient for this specific coordinate and temporal interval. The initial phase of predication:

$${h_{ij}}^{n+1} = {h_{ij}}^{n} + \Delta t\left(D\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right)\right)_{ij}^{n+\frac{1}{2}} \quad (11)$$

Guides to the equation:

$${H_{ij}}^{n+1} = {H_{ij}}^{n} + \Delta t(D\left({K_x}^2 {H_{ij}}^{n+\frac{1}{2}} + {K_y}^2 {H_{ij}}^{n+\frac{1}{2}}\right)) \quad (12)$$

where the wave numbers in the x and y directions are denoted, respectively, by k_x and k_y. This system is a way to estimate the diffusion equation numerically by employing the predictor-corrector method [9].

5.Numerical simulation

water height at

The phrase "water height at" refers to the water level at certain points in the region at particular times

during the simulation. This usually entails tracking the temporal progression of the flood by monitoring the changes in water levels at various periods in time (such as grid points).
It provides insight into the flow of flood waters, their effects on different regions, and how they recede throughout the course of the simulation.

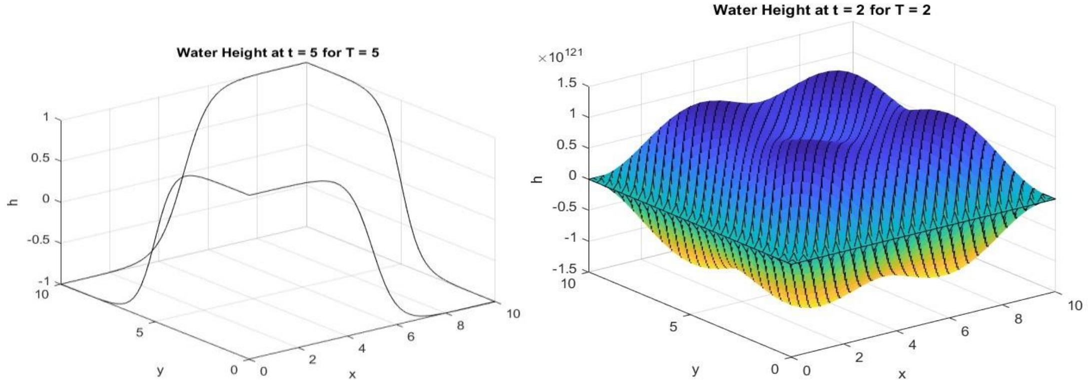
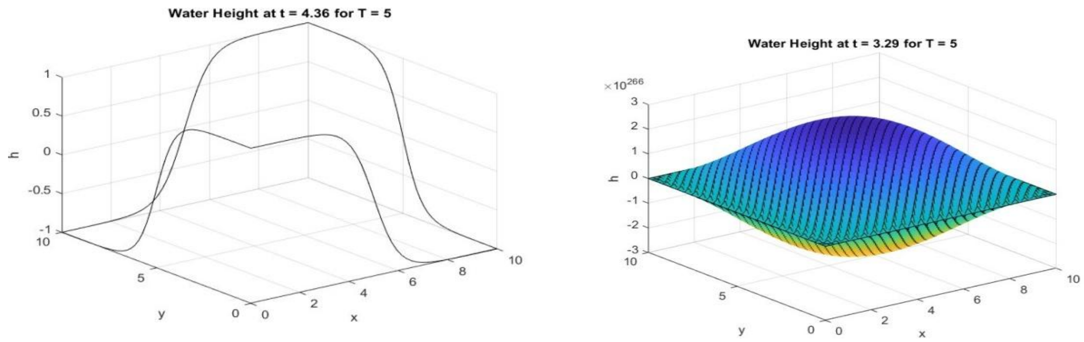


Figure 1. water height

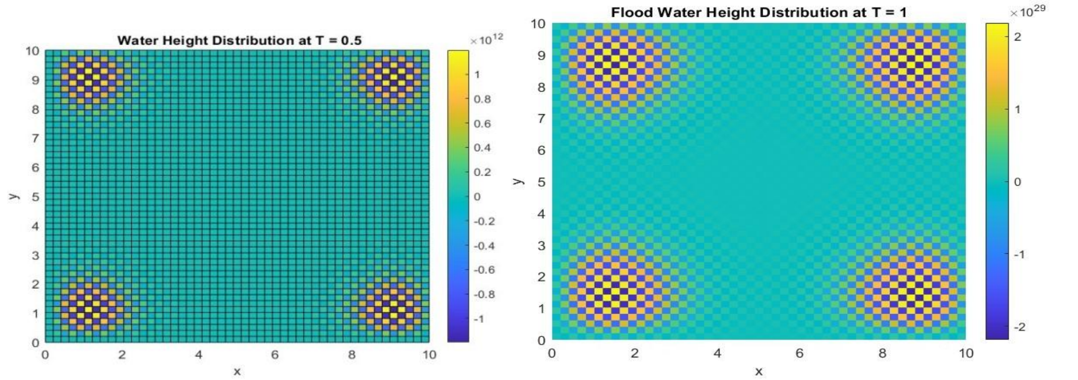


(a) (b)

Figure 2. water height

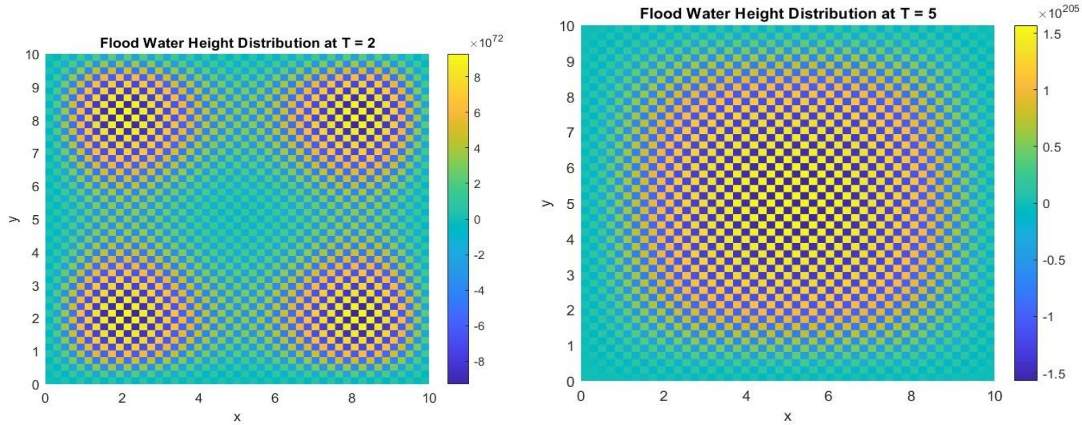
Flood Water Height Distribution

Distribution of flood water levels refers to how the water levels are arranged or laid out across the area at a specific moment in time. Rather than focusing on specific areas, it provides an overview of how flood waters disperse throughout the entire region at a given time. This distribution makes flood characteristics easier to visualize by highlighting the extent of the flooding and identifying specific zones. Although both ideas entail determining the water level during a flooding event, the latter offers a comprehensive picture of the water level throughout the entire domain at a given time, while the former concentrates on individual places or locations at different times. Their complementary perspectives on the dynamics and effects of flooding—the former providing temporal insights, the later providing spatial insights—allow them to better comprehend each other.



(a) (b)

Figure 3. Flood Water Height



(a) (b)

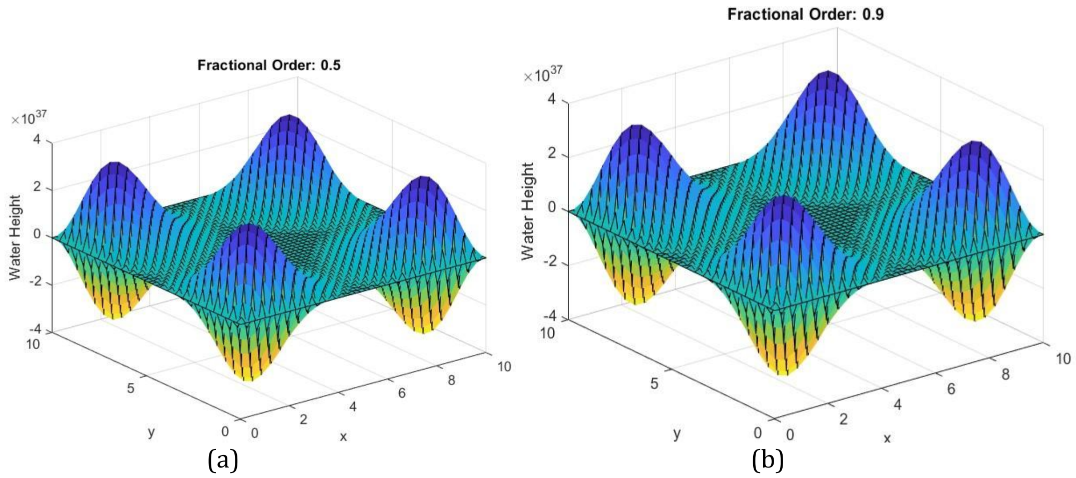
Figure 4. Flood Water Height Distribution

6. Sensitivity Analysis

We performed a sensitivity analysis to see how the fractional flood model responds to changes in significant elements in order to assess the model's robustness and dependability. This investigation closely examines the effects of changing factors, such as the diffusion coefficient and order on the models' predictions of flood behavior.

Fractional Order Sensitivity

Examining orders allows us to investigate the ways in which different degrees of fractional differentiation affect the memory effects and spatial differences that the model represents. Understanding how the model responds to changes in the diffusion coefficient variable helps us understand flood wave propagation. The outcomes suggest that higher fractional values lead to enhanced memory effects and spatial variation in flood behavior. However, elevated values may result in instability or computational challenges. Choosing the appropriate value is crucial in order to achieve a balance between the complexity of the model, the feasibility of computation, and the level of accuracy.

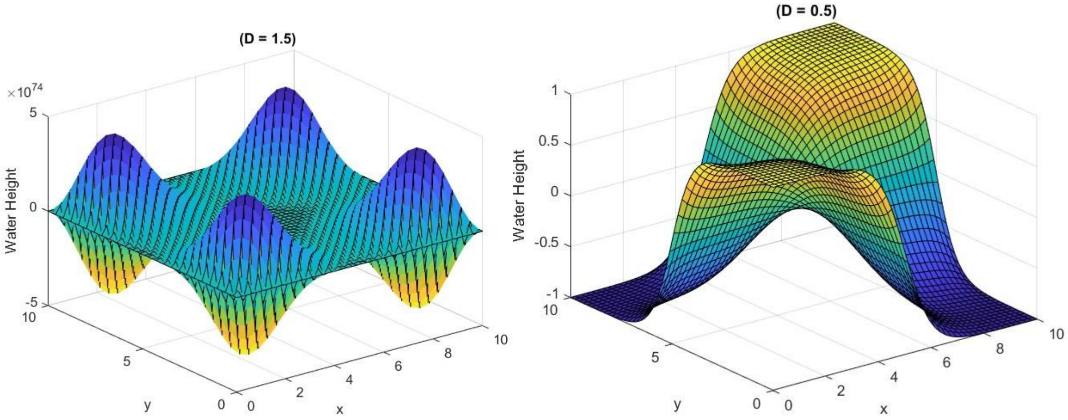


(a) (b)

Figure 5. Fractional Order Sensitivity.

Diffusion Coefficient Sensitivity

Understanding how the model responds to changes in the diffusion coefficient variable helps us understand flood wave propagation. A change in the diffusion coefficient can affect the way flood waters flow, including the are a flooded and the speed at which they spread. The variation in the diffusion coefficient demonstrates the model's response to changes in flood water spreading and dispersal. Higher diffusion coefficients cause flood waves to move faster. Disperse extensively. Understanding the relationship between the diffusion coefficient and flood dynamics can improve the accuracy of parameter estimates. Enhance the predictive capabilities of the models.



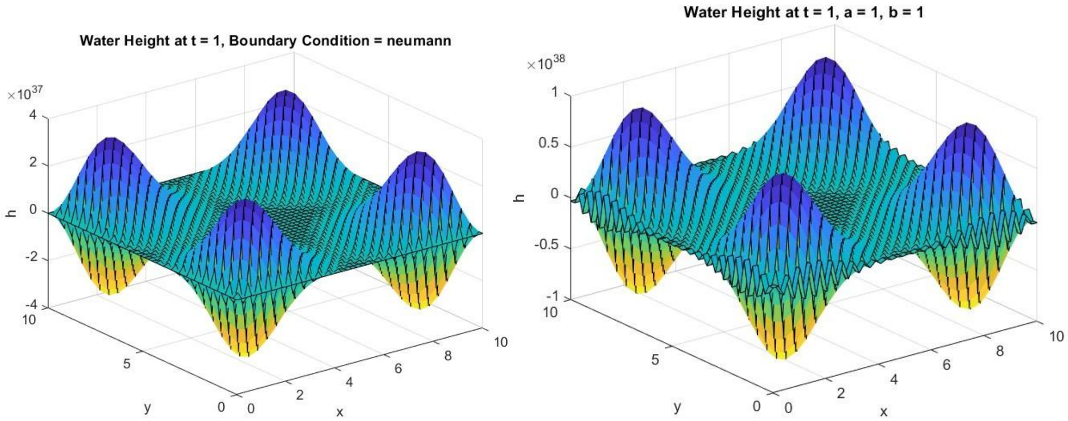
(a) (b)

Figure 6. Diffusion Coefficient Sensitivity.

Other Sensitivity Analysis

Further sensitivity analyses can be performed to observe the model's response to initial conditions, boundary conditions, and numerical parameters like grid size and time intervals. By identifying the variables that influence the model's predictions, these tests are helpful in fine-tuning the model's parameters.

The simulation's findings provide insight into the behavior of the fractional flood model in various situations. We can observe how different elements affect flooding patterns in different locations by tracking changes in height across time and space.

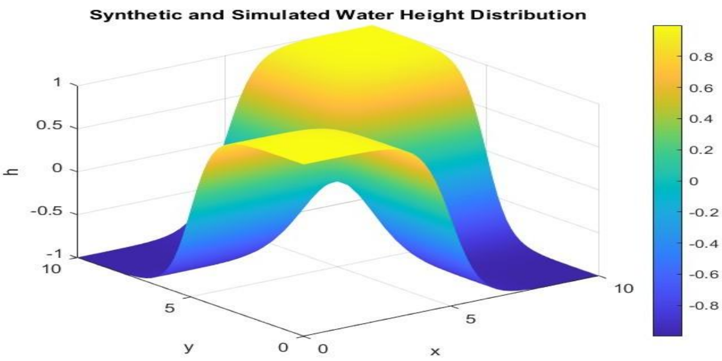


(a) (b)

Figure 7. Boundary conditions Sensitivity.

Model Robustness and Reliability

The sensitivity analysis generally demonstrates how effectively the fractional flood model can manage flood conditions and adjust to parameter changes. The model's adaptability to various environments and urban landscapes demonstrates its value in anticipating urban floods and evaluating hazards.



CONCLUSION

The fractional flood model has many advantages compared with conventional integer-order models. One, it yields better predictions regarding the propagation and attenuation of flood waves by correctly representing memory effects as well as non-local interactions inherent in flood dynamics [37].Moreover, the model can be adapted to temporal- specific flood characteristics by changing its fractional order and therefore widen its usage in different environmental conditions[8].Also, with the predictor-corrector approach, numerical simulation stability and reliability are improved, so that even in complex situations, consistent results can be produced[22].Enhanced flood forecasting and modeling skills enable emergency response planning and preventative flood management [23].The developed model provides decision-makers with critical tools for reducing the economic and environmental consequences of urban floods. This can be done by informing the design of resilient infrastructure and effective mitigation strategies. Furthermore, the ability to model different flood scenarios aids in the evaluation of various flood management measures, allowing for evidence-based decision-making and resource distribution[30] .In conclusion, the simulation findings show that the fractional flood model is good at describing detailed flood dynamics and has the potential to impact resilient urban planning and flood management strategies. The model's strengths include its ability to account for memory effects, adjust fractional ordering to meet different scenarios, and generate reliable forecasts using the predictor-corrector method. Future research should focus on improving the fractional flood model by incorporating additional variables such as infrastructure resilience, topographical features, and rainfall unpredictability.

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