Journal of Computational Analysis and Applications VOL. 34, NO. 12, 2025

10.48047/jocaaa.2025.34.12.61

CASCADE INDICES OF GRAPHS

Sana Aejaz[#1]

[#1] Assistant Professor, Dept. of Mathematics, Khaja Bandanawaz University

E-mail: sanakhadeer.glb@gmail.com

ABSTRACT

This paper introduces the first and second Cascade Graph indices and their respective polynomials. Furthermore, we compute these newly defined Cascade indices along with their corresponding polynomials for several standard graphs and notable nanostructures emerging in the domain of nanoscience.

Keywords: Cascade index, nanostructures, graphs.

Mathematics Subject Classification: 05C05, 05C12, 05C35

1. INTRODUCTION

Let $P(G)$ represent the vertex set and $Q(G)$ denote the edge set of a simple, connected graph G. The degree of a vertex v_1, denoted as $deg(v_1)$ indicates the count of vertices adjacent to v_1. For further graph-related terminologies and concepts, refer [1,2].

A graph featuring atoms at its vertices and bonds at its edges is termed a chemical graph. In the study of Chemical Sciences, Mathematical Chemistry provides significant assistance. A topological index is defined as a numerical parameter derived mathematically from the graph's structure. Numerous topological indices have been studied in theoretical chemistry, and many of these indices are based on the concept of vertex degree. The widely recognized degree-based graph indices in Chemical Graph Theory include the Zagreb, Nirmala, Sombor, and Revan delta indices; See [4].The applications of topological indices are evident across various fields in Science and Technology.

The first and second Cascade indices for a graph G are defined as:

$Ci_1(G) = \sum_{uv\in Q(G)}[deg(u) + deg(v_1) + deg(u)\,deg(v_1)]$

$$Ci_2(G) = \sum_{uv\in Q(G)} (deg(u) + deg(v_1))\,deg(u)\,deg(v_1)$$

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In this manuscript, G signifies a finite, simple, connected graph, while $P(G)$ and $Q(G)$ represent the vertex set and edge set of G, respectively. The degree $deg(v_1)$ of a vertex v_1 corresponds to the number of vertices adjacent to v_1. A $u-v$ path P_p in G comprises a sequence of vertices within G, commencing from u and concluding at v, with the stipulation that consecutive vertices in P_p are adjacent and no vertex is repeated. A path $\pi = v_1, v_2, v_3, \dots \dots v_{k+1}$ in G is classified as a descending path if for every $i, 1 \leq k$, it holds that $d_G(v_i) \geq d_G(v_{i+1})$. A vertex v is said to descendingly dominate vertex u if a descending path exists that originates from u to v. The descending neighborhood associated with vertex v is denoted as $N_{dn}(v)$ and is defined as: $N_{dn}(v) = \{u: v \text{ descendingly dominates } u\}$. The descending degree $d_{deg}(v)$ of vertex v is the quantity of descending neighbors of v.

The first and second descending indices were introduced in [50] and are defined as follows:

$$DDI_1(G) = \sum_{u \in P(G)} d_{deg}(u)^2$$

$$DDI_2(G) = \sum_{u \in P(G)} d_{deg}(u) d_{deg}(v).$$

Recently, several descending indices were examined in [7-8]. Inspired by the definition of the descending index and its extensive applications, we present the first and second Cascade descending indices of a graph as follows:

The first and second Cascade downhill indices for a graph G are defined as:

$$DDCi_1(G) = \sum_{uv \in Q(G)} [d_{deg}(u) + d_{deg}(v) + d_{deg}(u)\, d_{deg}(v)]$$

$$DDCi_2(G) = \sum_{uv \in Q(G)} (d_{deg}(u) + d_{deg}(v)) d_{deg}(u)\, d_{deg}(v)$$

Considering the first and second Cascade descending indices, we propose the first and second Cascade descending polynomials for a graph G defined as:

$$DDCi_1(G, x) = \sum_{uv \in P(G)} x^{[d_{\mathrm{deg}}(u) + d_{\mathrm{de}}(v) + d_{deg}(u) d_{deg}(v)]}$$

$$DDCi_2(G, x) = \sum_{uv \in P(G)} x^{[(d_{\mathrm{deg}}(u) + d_{deg}(v)) d_{deg}(u) d_{deg}(v)]}$$

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This paper determines the first and second Cascade descending indices and their associated polynomials for several standard graphs and particular nanostructures.

2. RESULTS FOR SOME STANDARD GRAPHS

Proposition 1: If G is an n-regular graph with p vertices and $p \geq 2$, then

$$DDCi_1(G) = \frac{pn(p^2-1)}{2}$$

Proof: Assume G is an n -regular graph comprising p vertices with $p \geq 2$ and $\frac{np}{2}$ edges. Thus, for every vertex v in G, we have $d_{deg}(v) = p-1$. Consequently, we find:

$$DDCi_1(G) = \sum_{uv\in P(G)} [d_{deg}(u) + d_{deg}(v) + d_{deg}(u)d_{deg}(v)]$$

$$= \frac{np}{2}[p-1+p-1+(p-1)(p-1)] = \frac{np(p^2-1)}{2}$$

Corollary 1.1: If C_p denotes a cycle with $p \geq 3$ vertices, then:

$DDCi_1(C_p) = p(p^2-1)$

Corollary 1.2: If K_p indicates a complete graph with $p \geq 3$ vertices, then:

$$DDCi_1(K_p) = \frac{p(p-1)(p^2-1)}{2}$$

Proposition 2: If P_p represents a path with $p \geq 3$ vertices, then:

$$DDCi_1(P_p) = (p-1)(p^2-2p-1)$$

Proof: For P_p as a path with $p \geq 3$ vertices, $P = P_p$ contains two types of edges based on the descending degree of the end vertices of each edge as follows:

$Q_1 = \{uv \in Q(P) | d_{deg}(u) = 0, d_{deg}(v) = p-1\}$, $|Q_1| = 2$.

$Q_2 = \{uv \in Q(P) | d_{deg}(u) = d_{deg}(v) = p-1\}$, $|Q_2| = p-3$.

Then:

$$DDCi_1(P_p) = \sum_{uv\in Q(P_p)} [d_{deg}(u) + d_{deg}(v) + d_{deg}(u)d_{deg}(v)]$$

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$= 2\ [0 + (p-1) + 0(p-1)] + (p-3)[P-1) + (p-1)^2]$

$=(p-1)(p^2-2p-1).$

Proposition 3: If $K_{p,q}$ is a complete bipartite graph with $p < q$, then:

$$DDCi_1\left(K_{p,q}\right) = pq^2$$

Proof: Let $K_{p,q}$ be a complete bipartite graph with $p < q$ There are $p+q$ vertices and pq edges. Clearly, $K_{p,q}$ comprises one type of edge based on the descending degree of the end vertices of each edge as follows:

$Q_1 = \{uv \in QK_{p,q}\ ) | d_{deg}(u) = 0, d_{deg}(v) = p\},\qquad |Q_1| = pq.$

Then:

$$DDCi_1\left(K_{p,q}\right) = \sum_{uv\in Q(K_{p,q})} [d_{\deg}(u) + d_{\deg}(v) + d_{\deg}(u)d_{deg}(v)]$$

$$=pq[0 + p + 0p] = pq^2.$$

Proposition 4: If G is an n-regular graph with p vertices and $p \geq 2$, then:

$$DDCi_2(G) = pn(p-1)^3$$

Proof: Assume G is an n-regular graph with p vertices and $p \geq 2$ with $\frac{pn}{2}$ edges. Then $d_{deg}(v) = p-1$ for every vertex v in G. We obtain:

$$DDCi_2(G) = \sum_{uv\in Q(G)} \left(d_{deg}(u) + d_{deg}(v)\right) d_{deg}(u)d_{deg}(v)$$

$$=\frac{pn}{2}[(p-1+p-1)(p-1)(p-1)] = pn(p-1)^3.$$

Corollary 4.1 If C_p is a cycle with $p \geq 3$ vertices, then

$$DDCi_2\left(C_p\right) = 2p(p-1)^3$$

Corollary 4.2 If K_p denotes a complete graph with $p \geq 3$ vertices, then:

$$DDCi_2\left(P_p\right) = 2(p-3)(p-1)^3$$

Proposition 5: If P_p is a path with $p \geq 3$ vertices, then:

$$DDCi_2\left(P_p\right) = 2(p-3)(p-1)^3$$

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Proof: Let P_p be a path with $p \geq 3$ vertices. Clearly, $P = P_p$ contains two types of edges based on the descending degree of the end vertices of each edge as follows:

$Q_1 = \{uv \in Q(P) | d_{deg}(u) = 0, d_{deg}(v) = p-1\}$, $|Q_1| = 2$.

$Q_2 = \{uv \in Q(P) | d_{deg}(u) = d_{deg}(v) = p-1\}$, $|Q_2| = p-3$.

Then:

$$DDCi_2(P_p) = \sum_{uv \in Q(P_p)} \left(d_{\deg}(u) + d_{deg}(v)\right) d_{deg}(u) d_{deg}(v)$$

$$=2[(0+p-1)0(p-1)] + (p-3)[(p-1+p-1)(p-1)^2]$$

$$=2(p-3)(p-1)^3.$$

Proposition 6: If $K_{p,q}$ is a complete bipartite graph with $p < q$, then:

$$DDCi_2(K_{p,q}) = 0$$

Proof: If $K_{p,q}$ is a complete bipartite graph with $p < q$, it contains $p+q$ vertices and pq edges. Clearly, Km,n has one type of edges based on the downhill degree of end vertices of each edge:

$Q_1 = \{uv \in Q(K_{p,q}) | d_{deg}(u) = 0, d_{deg}(v) = p\}$, $|Q_1| = pq$..

Then

$$DDCi_2(K_{p,q}) = \sum_{uv \in Q(K_{m,n})} \left(d_{\deg}(u) + d_{deg}(v)\right) d_{deg}(u) d_{deg}(v) = pq(0+q)0q = 0.$$

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