10.48047/jocaaa.2024.33.1A.66

# Stability of Three Species Syn-Ecosystem, Prey-Predator-Super Predator with Restricted Resources

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Article Received: December 21, 2023, Revised: March 13, 2024, Accepted: May 19, 2024

**Abstract:**

In this paper, we investigate a three-species syn ecological ecosystem consisting of a prey, a predator, and a super-predator, where the growth of the super-predator is assumed to be limited. Predator ($S_2$) surviving on the prey ($S_1$) and super predator ($S_3$) surviving on the predator ($S_2$). The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths $N_1$, $N_2$, $N_3$ of $S_1$, $S_2$, $S_3$ respectively. All five possible equilibrium points of the system are determined and analyzed. The system is said to be locally asymptotically stable if all characteristic roots of the corresponding Jacobian matrix are negative when they are real, or if they possess negative real parts when they are complex.

**Keywords:** Characteristic equation, Equilibrium State, Stable, Unstable.

## 1. Introduction

Theoretical ecology is concerned with the study of interactions between living organisms and their environment. The organisms primarily include plants and animals, while the environment consists of the physical, biological, and ecological surroundings that influence their survival and development. Thus, ecology relates to the scientific study of living beings in relation to their habits, habitats, and modes of interaction with one another and with nature. As a branch of evolutionary biology, theoretical ecology seeks to explain how populations are regulated in nature and to what extent biological systems maintain balance under natural constraints. A fundamental problem in theoretical ecology is population regulation, which addresses the factors that control population growth and persistence. Closely allied to this problem is the study of species distribution, which arises from various ecological interactions such as commensalism, predator–prey relationships, competition, mutualism, parasitism, and other interspecific associations. These interactions play a vital role in determining community

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structure, species coexistence, and ecosystem stability. Understanding such complex relationships requires a theoretical framework capable of capturing both biological realism and analytical tractability. Significant research contributions in the field of theoretical ecology have been discussed by Gillman [3] and Kot [4], who emphasized population dynamics, species interactions, and spatial distribution in ecological systems. Over the years, the combined efforts of ecologists and mathematicians have contributed substantially to the growth of this discipline, leading to the development of mathematical ecology as a powerful tool for ecological analysis.

Mathematical ecology may be broadly divided into two principal sub-disciplines: Autecology and Synecology. Autecology focuses on the study of individual species and their responses to environmental factors, whereas Synecology deals with interactions among multiple species within an ecological community. These classifications and foundational concepts are systematically described in the treatises of Anna Sher [1], Arumugam [2], and Sharma [21]. Together, these approaches provide a comprehensive framework for analyzing ecological systems at both species and community levels. Mathematical modeling plays a key role in providing insight into the mutual relationships—both positive and negative—among interacting species. By formulating ecological processes in mathematical terms, models help to identify governing principles, predict system behavior, and assess stability and persistence of populations. The general concepts of modeling in biological sciences have been introduced and developed by several authors, including Ma [6], Moghadas [7], Murray [8], and Sze-Bi Hsu [23]. Their works highlight the importance of differential equations, nonlinear dynamics, and stability analysis in understanding biological interactions.

Specific ecological systems have also been widely investigated through mathematical models. Srinivas [22] studied competitive ecosystems involving two and three species under conditions of limited and unlimited resources, revealing the influence of resource availability on coexistence and exclusion. Subsequently, Narayan [9] examined prey–predator models incorporating partial cover for the prey and alternate food sources for the predator, thereby extending classical predator–prey theory to more realistic ecological settings. Further, Kumar [5] investigated mathematical models of ecological commensalism, contributing to the understanding of asymmetric species interactions. In continuation of these developments, the present author Prasad [10–20] has investigated a variety of continuous and discrete mathematical models involving two-species, three-species, and four-species syn-ecosystems.

10.48047/jocaaa.2024.33.1A.66

These studies focus on qualitative analysis, equilibrium behavior, stability, and long-term dynamics of ecological systems under different interaction structures. Such investigations not only enrich theoretical ecology but also provide insights relevant to real-world ecological management and conservation.

## 2. Notation

| | | |
|---|---|---|
| $N_i(t)$ | : | The population strength of $S_i$ at time $t$, $i=1,2,3$ |
| $t$ | : | Time instant |
| $a_i$ | : | Natural growth rate of $S_i$, $i=1,2,3$ |
| $a_{33}$ | : | Self inhibition coefficients of $S_3$ |
| $a_{23}, a_{32}$ | : | Interaction coefficients of $S_2$ due to $S_3$ |
| $a_{12}, a_{21}$ | : | Interaction coefficients of $S_1$ due to $S_2$ |
| $k_3 = \frac{a_3}{a_{11}}$ | : | Carrying capacity of $S_3$ |

Further the variables $N_1, N_2, N_3$ are non-negative and the model parameters $a_1, a_2, a_3, a_{33}$, $a_{12}, a_{21}, a_{23}, a_{32}$ are assumed to be non-negative constants.

## 3. Basic Equations

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = a_1 N_1 - a_{12} N_1 N_2 \tag{1}$$

$$\frac{dN_2}{dt} = a_2 N_2 + a_{21} N_1 N_2 - a_{23} N_2 N_3 \tag{2}$$

$$\frac{dN_3}{dt} = a_3 N_3 - a_{33} N_3^2 + a_{32} N_2 N_3 \tag{3}$$

The system under investigation has five equilibrium states given by $\frac{dN_i}{dt} = 0, i = 1, 2, 3$

E1: $\overline{N}_1 = 0$, $\overline{N}_2 = 0$, $\overline{N}_3 = 0$

E2: $\overline{N}_1 = 0$, $\overline{N}_2 = 0$, $\overline{N}_3 = k_3$

E3: $\overline{N}_1 = \frac{-a_2}{a_{21}}, \overline{N}_2 = \frac{a_1}{a_{12}}, \overline{N}_3 = 0$

E4: $\overline{N}_1 = 0$, $\overline{N}_2 = \theta_1$, $\overline{N}_3 = \frac{a_2}{a_{23}}$

$$\text{Where } \theta_1 = \frac{a_2 a_{33} - a_3 a_{32}}{a_{23} a_{32}} \tag{4}$$

10.48047/jocaaa.2024.33.1A.66

$E_5$: $\overline{N}_1 = \theta_3$ , $\overline{N}_2 = \frac{a_1}{a_{12}}$ , $\overline{N}_3 = \theta_2$

Where $\theta_2 = \frac{a_3a_{12}+a_1a_{32}}{a_{12}a_{33}} > 0$, $\theta_3 = \frac{\theta_2 a_{23}-a_2}{a_{21}}$ (5)

## 4. Stability of Equilibrium States

Let $N=\left(N_1,N_2,N_3\right)=\overline{N}+U$ (6)

Where $U=(u_1,u_2,u_3)^T$ is a small perturbation over the equlibrium state $\overline{N}=\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$.

The equations (1), (2), (3) are quasi-linearized to obtain the equation for the perturbed state as,

$$\frac{dU}{dt}=AU \tag{7}$$

$$\text{Where } A=\begin{bmatrix} a_1-a_{12}\overline{N}_2 & -a_{12}\overline{N}_1 & 0 \\ a_{21}\overline{N}_2 & a_2+a_{21}\overline{N}_1-a_{23}\overline{N}_3 & -a_{23}\overline{N}_2 \\ 0 & a_{32}\overline{N}_3 & a_3-2a_{33}\overline{N}_3+a_{32}\overline{N}_2 \end{bmatrix} \tag{8}$$

The Characteristic equation is $|A-\lambda I|=0$ (9)

The Equilibrium state is stable, if all the roots of equation (9) are negative.

### *4.1 Stability of* $E_1:\overline{N}_1=0,\overline{N}_2=0,\overline{N}_3=0$

In this case, we have $A=\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ (10)

The characteristic equation is $(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)=0$ (11)

The Characteristic roots are $a_1,a_2,a_3$. Since all the three roots are positive values.

Hence the fully washed out state is **unstable** and the solutions of the equations (7) are

$$u_i=u_{i0}e^{a_i.t};i=1,2,3 \tag{12}$$

where $u_{10},u_{20},u_{30}$ are the initial values of $u_1,u_2,u_3$ respectively.

### 4.2 Equilibrium state E2: $\overline{N}_1=0$, $\overline{N}_2=0$, $\overline{N}_3=k_3$

In this state, we have $A=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2-a_{23}k_3 & 0 \\ 0 & a_{32}k_3 & -a_3 \end{bmatrix}$

The characteristic equation is $(\lambda-a_1)[\lambda-(a_2-a_{23}k_3)](\lambda+a_3)=0$

$a_1$, $a_2-a_{23}k_3$, $-a_3$ are the characteristic roots. Since one of these three roots are positive values, hence the state is **unstable** and the equations (7) yield the solutions.

10.48047/jocaaa.2024.33.1A.66

$$u_1 = u_{10}e^{a_1 t},\ u_2 = u_{20}e^{(a_2 - a_{23}k_3)t},$$

$$u_3 = u_{30}e^{-a_3 t} + k_3 \frac{a_{32}u_{20}}{(a_2 + a_3 - a_{23}k_3)}\left[e^{(a_2 - a_{23}k_3)t} - e^{-a_3 t}\right]$$

**4.3 Equilibrium state E3:** $\bar{N}_1 = \frac{-a_2}{a_{21}}, \bar{N}_2 = \frac{a_1}{a_{12}}, \bar{N}_3 = 0$

In this state, we have $A = \begin{bmatrix} 0 & a_2 & 0 \\ \frac{a_1 a_{21}}{a_{12}} & 0 & -\frac{a_1 a_{23}}{a_{12}} \\ 0 & 0 & \frac{a_1 a_{32}}{a_{12}} \end{bmatrix}$

The characteristic equation is $(\lambda - b_1)(\lambda^2 - b_2) = 0$

The Characteristic roots are $\lambda = b_1 \lambda = \sqrt{b_2}, \lambda = -\sqrt{b_2}$

Where $b_1 = \frac{a_1 a_{32}}{a_{12}}, b_2 = \frac{a_1 a_2 a_{21}}{a_{12}}$

Since, two of these three roots are positive, hence the state is **unstable** and the equations (7) yield the solutions.

$$u_1 = \left(A_{10} - u_{10}\right)\cosh\sqrt{b_2}t + A_{20}\sinh\sqrt{b_2}t + A_{10}e^{b_1 t}$$

$$u_2 = B\left(A_{10} - u_{10}\right)\sinh\sqrt{b_2}t + BA_{20}\cosh\sqrt{b_2}t + b_1 A_{10}e^{b_1 t}$$

$$u_3 = u_{30}e^{b_1 t}$$

where

$$A_{10} = \frac{A_1}{b_1^2 - b_2}, A_{20} = \frac{a_2}{\sqrt{b_2}}\left(u_{20} - \frac{A_{10}b_1}{a_2}\right), B = \frac{\sqrt{b_2}}{a_2} > 0, A_1 = \frac{a_1 a_2 a_{23} u_{30}}{a_{12}}$$

**4.4 Equilibrium state E4 :** $\bar{N}_1 = 0, \bar{N}_2 = \theta_1, \bar{N}_3 = \frac{a_2}{a_{23}}$

In this state, we get $A = \begin{bmatrix} a_1 - a_{12}\theta_1 & 0 & 0 \\ a_{21}\theta_1 & 0 & -a_{23}\theta_1 \\ 0 & \frac{a_2 a_{32}}{a_{23}} & -\frac{a_2 a_{33}}{a_{23}} \end{bmatrix}$

The characteristic equation is $[\lambda - (a_1 - a_{12}\theta_1)]\left[\lambda^2 + \frac{a_2 a_{33}}{a_{23}}\lambda + (a_2 a_{32}\theta_1)\right] = 0$

Since, one root is $(a_1 - a_{12}\theta_1)$.

By Routh-Hurwitz criteria,

If $a_1 < a_{12}\theta_1$ then the state is state is stable, otherwise unstable.

Let $\lambda_1, \lambda_2$ be the zeros of the quadratic polynomial on the above equation.

The solutions are given by

$$u_1 = u_{10}e^{(a_1 - a_{12}\theta_1)t}$$

10.48047/jocaaa.2024.33.1A.66

$$u_2 = X_{10}e^{\lambda_1 t} + X_{20}e^{\lambda_2 t} + X_{30}e^{(a_1 - a_{12}\theta_1)t}$$

$$u_3 = Y_{30}e^{(a_1 - a_{12}\theta_1)t} - Y_{10}Y_1e^{\lambda_1 t} - Y_{20}Y_2e^{\lambda_2 t}$$

Where

$$X_{10} = \frac{(u_{30} - Y_1) - Y_3(X_{30} - u_{20})}{Y_3 - Y_2}, \quad X_{20} = \frac{Y_2(X_{30} - u_{20}) - (u_{30} - Y_1)}{Y_3 - Y_2}$$

$$X_{30} = \frac{(a_1 - a_{12}\theta_1)a_{21}\theta_1 u_{10}a_{23} + a_{21}\theta_1 u_{10}a_2a_{33}}{a_{23}(a_1 - a_{12}\theta_1)^2 + a_2a_{33}(a_1 - a_{12}\theta_1) + a_2a_{32}a_{23}\theta_1}$$

$$Y_1 = \frac{\lambda_1}{a_{23}\theta_1}, Y_2 = \frac{\lambda_2}{a_{23}\theta_1}, Y_3 = \frac{(a_1 - a_{12}\theta_1)X_{30}}{a_{23}\theta_1}$$

**4.5 Equilibrium state E5 :** $\overline{N}_1 = \theta_3$, $\overline{N}_2 = \frac{a_1}{a_{12}}$, $\overline{N}_3 = \theta_2$

In this state, the matrix A = $\begin{bmatrix} 0 & -a_{12}\theta_3 & 0 \\ \frac{a_1a_{21}}{a_{12}} & 0 & -\frac{a_1a_{23}}{a_{12}} \\ 0 & a_{32}\theta_2 & -a_3 - \frac{a_1a_{32}}{a_{12}} \end{bmatrix}$

The characteristic equation is

$$\lambda^3 + \theta\lambda^2 + [\theta_2\theta a_{23} + a_1a_{21}\theta_3]\lambda + a_1a_{21}\theta_3\,\theta = 0$$

Where $\theta = \frac{a_1a_{32}}{a_{12}}$

Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of above equation.

If $\lambda_1, \lambda_2, \lambda_3$ noted to be negative, hence the state is **stable** and the equations (7) yield the solutions.

$$u_1 = Z_{10}e^{\lambda_1 t} + Z_{20}e^{\lambda_2 t} + Z_{30}e^{\lambda_3 t}$$

$$u_2 = X_1Z_{10}e^{\lambda_1 t} + X_2Z_{20}e^{\lambda_2 t} + X_3Z_{30}e^{\lambda_3 t}$$

$$u_3 = Z_1Z_{10}e^{\lambda_1 t} + Z_2Z_{20}e^{\lambda_2 t} + Z_3Z_{30}e^{\lambda_3 t}$$

Where

$$Z_{10} = \frac{D_1}{D}, \quad Z_{20} = \frac{D_2}{D}, \quad Z_{30} = \frac{D_3}{D}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix}, \; D_1 = \begin{vmatrix} u_{10} & 1 & 1 \\ u_{20} & X_2 & X_3 \\ u_{30} & Z_2 & Z_3 \end{vmatrix}, D = \begin{vmatrix} 1 & u_{10} & 1 \\ X_1 & u_{20} & X_3 \\ Z_1 & u_{30} & Z_3 \end{vmatrix}, D = \begin{vmatrix} 1 & 1 & u_{10} \\ X_1 & X_2 & u_{20} \\ Z_1 & Z_2 & u_{30} \end{vmatrix}$$

10.48047/jocaaa.2024.33.1A.66

$$X_1 = \frac{\lambda_1}{a_{12}\theta_3},\ X_2 = \frac{\lambda_2}{a_{12}\theta_3},\ X_3 = \frac{\lambda_3}{a_{12}\theta_3}$$

$$\mathrm{Z}_1 = \frac{a_{12}\lambda_1 X_1 - a_1 a_{21}}{a_1 a_{23}},\ \mathrm{Z}_2 = \frac{a_{12}\lambda_2 X_2 - a_1 a_{21}}{a_1 a_{23}},\ \mathrm{Z}_3 = \frac{a_{12}\lambda_3 X_3 - a_1 a_{21}}{a_1 a_{23}}$$

## 5. Conclusion

This paper deals with an investigation on the stability on three species syn ecology consisting of a prey - predator and a super predator with limited resources for the super prey. The model equations constitute a set of three first order non-linear coupled differential equations. All possible equilibrium states of the model are identified and criteria for their stability is discussed. It is observed that, in all states, only the equilibrium states $E_4$ and $E_5$ are **stable.**

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