Solution of the Second Order Cauchy Difference Equation On Free Monoid

M. Pradeep[1], S. Bala[2]

[1]PG and Research Department of Mathematics, Arignar Anna Government Arts College, Cheyyar-6044070, India, Email:pradeepmprnet12@gmail.com
[2]Department of Mathematics, S.I.V.E.T. College, Chennai-600073, Email:yesbala75@gmail.com

Received: 16.04.2024 Revised : 18.05.2024 Accepted: 23.05.2024

ABSTRACT
Let h:S → T be a function, where (S,.)is a monoid and (T,+) is an abelian group. In this paper, the following Second Order Cauchy difference of h : C(2)h(p1,p2,p3) = h(p1p2p3)−h(p1p2)−h(p1p3)−h(p2p3)+ h(p1)+h(p2)+h(p3)∀p1,p2,p3∈S is studied. We give some special solutions of C(2)h=0 on free Monoid.

Keywords: Cauchy difference equation, free Monoids

1. INTRODUCTION

It is well known from [1] that Jenson's functional equation

$h(x + y) + h(x - y) = 2h(x)$ (1.1)

with additional condition h(0)=0, is equivalent to Cauchy's equation

$h(x+y) = h(x)+h(y)$

on the real line. Let (S,.) be a monoid, (T,+) is an abelian group. Let e ∈ S and 0 ∈ T denote identity elements.

For a function h:S→T,its Cauchy difference equation $C^{(m)}h$, is defined by

$C^{(0)}h=h,$ (1.2)

$C^{(1)}h(p_1,p_2)=h(p_1p_2)-h(p_1)-h(p_2)$ (1.3)

$C^{(m+1)}h(p_1,p_2,\dots,p_{m+2})=C^{(m)}h(p_1,p_2,p_3\dots,p_{m+2})$
$-C^{(m)}h(p_1,p_3,\dots,p_{m+2})-C^{(m)}h(p_2,p_3,\dots,p_{m+2})$ (1.4)

The first order Cauchy difference equation $C^{(1)}$ h will be abbreviated as Ch. In [9], by using the reduction formulas and relations, as given in [2,3], the general solution of second order cauchy difference equation was provided on free monoids.

In this paper, we consider the following functional equation:

$h(p_1p_2p_3)-h(p_1p_2)-h(p_1p_3)-h(p_2p_3)$
$+h(p_1) +h(p_2)+h(p_3)=0 \quad \forall p_1,p_2,p_3\in S$ (1.5)

It follows from (1.4) that (1.5) is equivalent to the vanishing second order cauchy difference equation $C^{(2)}h=0$

The purpose of this paper is to determine the solutions of equation (1.5) on some given monoids. The solution of equation (1.5) will be denoted by

$KerC^{(2)}(S,T)=\{h:S\to T|h \text{satisfies}(1.5)\}$ (1.6)

Remark 1

1. $KerC^{(2)}(S,T)$ is an abelian group under the pointwise addition of functions;
2. $Hom(S,T)\le KerC^{(2)}(S,T)$

2. Properties of Solutions

Lemma1 Suppose that h ∈ $KerC^{(2)}$ (S,T) Then

$h(e)=0,$ (2.1)

$Ch(p, q) = 0$, when p = e or q = e (2.2)

Ch is a homomorphism with respect to each variable (2.3)

$h(p^n)=nh(p)+{}^{n}Ch(p,p)$ (2.4)

for all p,q ∈ S and n ∈ Z.

Proof: Putting p_1=e in(1.5)we get(2.1).

$h(p_2p_3)-h(p_2)-h(p_3)-h(p_2p_3)+h(e)+h(p_2)+h(p_3) =0$ therefore h(e)=0.

Then from (2.1) we obtain (2.2)
$Ch(p,e)=h(pe)-h(p)-h(e)$
$= h(p)-h(p)$
$=0$
Similarly we can obtain
$Ch(e,q) = 0,$
Furthermore, by the definition of Ch, we have
$Ch(p,qr) =h(pqr)-h(p)-h(qr)$
and
$Ch(p,q) + Ch(p,r) = h(pq) - h(p) - h(q) + h(pr) - h(p) - h(r)$ One can easily check that
$Ch(p,qr)-Ch(p,q)-Ch(p,r)=C^{(2)}h(p,q,r)=0$
$Ch(p,qr)-Ch(p,q)-Ch(p,r)$
$=h(pqr)-h(p)-h(qr)-h(pq)+h(p)+h(q)-h(pr)+h(p)+h(r)$
$=h(pqr)-h(pq)-h(pr)-h(qr)+h(p)+h(q)+h(r)$
$=C^{(2)}h(p, q,r)$
$=0$ by (1.5)
Hence, the above relation simply the Ch(p,.) is a homomorphism. Similarly, the fact is also true for Ch(.,q). This proves (2.3).
We now consider (2.4).Actually, it is trivial for n = 0,1 by (2.1) and by the definition of Ch. Suppose that (2.4) holds for all natural numbers smaller than $n\geq 3$, then
$h(p^n)=h(p^{n-1}\cdot p)$
$=h(p^{n-1})+h(p)+Ch(p^{n-1},p)$
$=[(n-1)h(p)+(n-1)C_2Ch(p,p)+h(p)+(n-1)Ch(p,p)]$
$=nh(p) +nC_2Ch(p,p)$
where the definition of Ch and (2.3) are used in the second equation. This gives (2.6) fora all $n\geq 0$.On the other hand, for any fixed integer $n>0$, by (1.4) and (2.1), we have
$Ch(p^n,p^{-n})=h(e)-h(p^n)-h(p^{-n})$

$$\Rightarrow h(p^{-n})=-h(p^n)-Ch(p^n,p^{-n})$$
$$=-nh(p)+\frac{n(n-1)}{2}Ch(p,p)-(-n^2)Ch(p,p)$$
$$=-nh(p)-\frac{n^2-n}{2}Ch(p,p)+n^2Ch(p,p)$$
$$=-nh(p)+\frac{-n(-n-1)}{2}Ch(p,p)$$

from (2.5) and the above claim for $n>0$. This confirms (2.6) for $n<0$.

Remark 2
For any function $h:S\rightarrow T$, the following statements are pairwise equivalent:
(i) The function $h \in KerC^{(2)}(S,T)$;
(ii) Ch(.,q) is a homomorphism;
(iii) Ch(p,.) is a homomorphism;
$\Longrightarrow$Ch is a homomormphism with respect to each variable.
Since T is abelian, (2.3) implies in particular that Ch can be factored through the abelianizedS^{ab}

Remark 3
Let $h : S \rightarrow T$ be any function. For any fixed $p \in S$, we may consider the function $i(x):=Ch(p,q)$.Taking the Cauchy difference of i once we get $Ci(p_1,p_2) = i(p_1p_2)-i(p_1)-i(p_2)$.Since $i = Ch(.,q)$ we may write that as
$CCh(.,q)(p_1,p_2)=Ch(p_1p_2,q)-Ch(p_1,q)-Ch(p_2,q)$
$=C^2(p_1,p_2,q)$
Continuing with the second order Cauchy difference of I we get
$C^2Ch(.,q)(p_1,p_2,p_3)=CCh(.,q)(p_1p_2,p_3)-CCh(.,q)(p_1,p_3)-CCh(.,q)(p_2,p_3)$
$=C^2(p_1p_2,p_3,q)-C^2(p_1,p_3,q)-C^2(p_2,p_3, q)$
$=C^3(p_1,p_2,p_3,q)$
$\Longrightarrow$ $C^{(m)}Ch(.,q)(p_1,p_2,...,p_{m+1})=C^{(m+1)}h(p_1,p_2,...,p_{m+1},q)$for all higher orders m.

Lemma 2

(Lemma 2.4 in [8]) The following identity is valid for any function h:S→T and l ∈ N;

$$h(p_1p_2 \dots p_t) = \sum_{m\leq t} \sum_{1\leq j_1<j_2<\cdots<j_m\leq t} D^{(m-1)}h(p_{j_1}, p_{j_2}, \dots, p_{j_m}) \qquad (2.5)$$

Proposition1

Suppose that h ∈ KerD⁽²⁾(S,T). Then

$$h(p_1^{n_1}p_2^{n_2}\dots p_l^{n_l}) = \sum_{1\leq i\leq l} n_i h(p_i) + \frac{n_i(n_i-1)}{2} Dh(p_i,p_i) + \sum_{1\leq i_1<i_2\leq l} n_{i_1}n_{i_2} Dh(p_{i_1}, p_{i_2}) \qquad (2.6)$$

for $n_i \in Z$ and all $p_i \in S, i=1,2,\dots l$ such that $p_j \neq p_{j+1}, j=1,2,\dots,l-1$

Proof Replacing p_i in (2.7) by p^{n_i}, we have

$$h(p_1^{n_1}p_2^{n_2}\dots p_t^{n_t}) = \sum_{m\leq t} \sum_{1\leq j_1<j_2<\cdots<j_m\leq t} D^{(m-1)}h(p_{j_1}^{n_{j1}}, p_{j_2}^{n_{j2}}, \dots, p_{j_m}^{n_{jm}})$$

$D^{(m-1)}h = 0$ for $m\geq 3$ implies

$$h(p_1^{n_1}p_2^{n_2}\dots p_t^{n_t}) = \sum_{1\leq j\leq t} h(p_j^{n_j}) + \sum_{1\leq j_1<j_2\leq t} Dh(p_{j_1}^{n_{j1}}, p_{j_2}^{n_{j2}})$$

Therefore, by (2.6)and (2.5),we have

$$h\left(p_j^{n_j}\right) = n_j h(p_j) + \frac{n_j(n_j-1)}{2} Dh(p_j, p_j)$$

$$Dh\left(p_{j_1}^{n_{j1}}, p_{j_2}^{n_{j2}}\right) = n_{j_1}n_{j_2} Dh(p_{j_1}, p_{j_2})$$

which is (2.8). This completes proof.

Remark 4

In particular, if l=1, then Proposition 1 holds.

3. Solution on a free monoid

Since every free monoid can be embedded in a free group.

Let S be the free monoid and S*be the free group

In this section, we study the solutions on free monoid. We first solve(1.5)for the free monoid S on a single letter x.

Theorem1

Let S be the free monoid on one letter x. Then h ∈ KerC⁽²⁾(S,T) if f it is given by

$$h(x^n) = nh(x) + \frac{n(n-1)}{2} Ch(x,x) \forall n \in N \qquad (3.1)$$

Proof: Necessity. It can be obtained from (2.6) in Lemma 1.

Sufficiency. Taking (3.1) as the definition of h on S=< x >.By Remark 2, we only need to verify that Chis a homomorphism with respect to each variable and thus h belongs to KerC⁽²⁾(S,T).Let $a=x^m, b=x^n$ be any two elements of S.

Then it follows from (1.4) and (3.1)that

$$Ch(a,b)=Ch(x^m,x^n)$$
$$=h(x^{m+n})-h(x^m)-h(x^n)$$
$$=(m+n)h(x)+\frac{(m+n)(m+n-1)}{2}Ch(x,x)$$
$$-mh(x)+\frac{m(m-1)}{2}Ch(x,x)-nh(x)+\frac{n(n-1)}{2}Ch(x,x)$$

By a tedious calculation, we have
$Ch(x^m,x^n)=mnCh(x,x)$
which leads to the result that Ch is a homomorphism with respect to each variable
At the end of this section, for the free monoid on an alphabet ⟨A⟩ with
$|A|\geq 2$, we discuss some special solution of (1.5).
An element a ∈ A can be written in the form
$a=x^{n_1}x^{n_2}...x^{n_l}$,where $x_i \in A$, $n_i \in N$ (3.2)
For each fixed x ∈ A and fixed pair of distinct x, y ∈ A, define the functions V, V_2, V_3

$$V(a;x)=\sum_{x_i=x} n_i \qquad (3.3)$$
$$V_2(a;x,y)=\sum_{i<j,\, x_i=x,\, x_j=y} n_i n_j \qquad (3.4)$$
$$V_3(a;x,y)=\sum_{i>j,\, x_i=x,\, x_j=y} n_i n_j \qquad (3.5)$$

along with (3.2). Referring to [2,3],the above functions are well defined. Further more, they satisfy the following relations:
V is additive: $V(ab;x)=V(a;x)+V(b;x)$ (3.6)
$V(a,x)V(a,y)= V_2(a;x,y)+V_3(a;x,y)$ (3.7)
$V_2(a;x,y)=V_3(a;y,x)$ (3.8)

Proposition 2
For any fixed x ∈ A and fixed pair of distinct x, y in A, the following assertions hold:
(i) V (.;x) belongs to $KerC^{(2)}$ (A,N);
(ii) V_2 (.;x) belongs to $KerC^{(2)}$(A,N);
(iii) V_3(.;x) belongs to $KerC^{(2)}$(A,N);
Proof: Claim (i) follows from the fact that x→V (a;x) is a morphism from ⟨A⟩ to N by (3.6).
Now we consider assertion (ii). Let a, b, c in the free monoid be written as

$$a=x_1^{r_1}x_2^{r_2}...x_l^{r_l},$$
$$b=y_1^{s_1}y_2^{s_2}...y_p^{s_p},$$
$$c=z_1^{t_1}z_2^{t_2}...z_q^{t_q},$$

Then

$$V_2(abc;x,y)=\sum_{i<j,\,x_i=a,\,x_j=y} r_i r_j+\sum_{i<j,\,y_i=x,\,y_j=y} s_i s_j+\sum_{i<j,\,z_i=x,\,z_j=y} t_i t_j$$
$$+\sum_{x_i=x,\,y_j=y} r_i s_j+\sum_{x_i=x,\,z_j=y} r_i t_j+\sum_{y_i=x,\,z_j=y} s_i t_j$$

$$V_2(ab;x,y)=\sum_{i<j,\,x_i=x,\,x_j=y} r_i r_j+\sum_{i<j,\,y_i=x,\,y_j=y} s_i s_j+\sum_{x_i=x,\,y_j=y} r_i s_j$$

$$V_2(ac;x,y)=\sum_{i<j,\,x_i=x,\,x_j=y} r_i r_j+\sum_{i<j,\,z_i=x,\,z_j=y} t_i t_j+\sum_{x_i=x,\,z_j=y} r_i t_j$$

$$V_2(bc;x,y)=\sum_{i<j,\,y_i=x,\,y_j=y} s_i s_j+\sum_{i<j,\,z_i=x,\,z_j=y} t_i t_j+\sum_{y_i=x,\,z_j=y} s_i t_j$$

$$V_2(a;x,y)=\sum_{i<j,\,x_i=x,\,x_j=y} r_i r_j$$

$$V_2(b;x,y)=\sum_{i<j,\,y_i=x,\,y_j=y} s_i s_j$$

$$V_2(c;x,y)=\sum_{i<j,\,z_i=x,\,z_j=y} t_i t_j$$

Hence, we have

$$V_2(abc;x,y)-V_2(ab;x,y)-V_2(ac;x,y)-V_2(bc;x,y)+V_2(a;x,y)$$
$$+V_2(b;x,y)+V_2(c;x,y)=0$$

This concludes assertion (ii).
Claims(iii) follows from(3.7) directly.
In order to present the solution of (1.5) on ⟨A⟩, we endow A with a linear order <.Each element a ∈ S can be written in the form

$$a=x_1^{m_{11}}x_2^{m_{12}}\cdots x_l^{m_{1l}}x_1^{m_{21}}x_2^{m_{22}}\cdots x_l^{m_{2l}}\cdots x_1^{m_{r1}}x_2^{m_{r2}}\cdots x_l^{m_{rl}} \tag{3.9}$$

where the letters in ascending order,

Theorem 2
Suppose that |A|>1.If h ∈ KerC²(⟨A⟩,T), then it has representation

$$h(a)=\sum_{n} V(a;x)h(x)+\sum_{n}\frac{V(a;x)(V(a;x)-1)}{2}Ch(x,x)+\sum_{x<y} V(a;x,y)Ch(x,y)$$
$$+\sum_{x<y} V_3(a;x,y)Ch(x,y) \tag{3.10}$$

The first two summations are over all letters x and the third and fourth summations are over ordered pairs of distinct letters x and y.
Conversely, let h: A→ T and Ch: AXA→ T be arbitrarily initiated functions. If we extend h to the free monoid ⟨A⟩ by taking (3.10) as its definition, then h belongs to KerC²(⟨A ⟩ , T)
Proof Suppose that h satisfies (1.5).For a written in the form (3.9), say, with l >1andr >1, let briefly $b_j:=x^{m_{j2}}\ldots x^{m_{jl}}$for j =1,2,...,r
From (2.3) and (3.3), in that order, we get

$$h(a)=h(x_1^{m_{11}}b_1x_1^{m_{21}}b_2\ldots x_1^{m_{r1}}b_r)$$

$$=h(x_1^{m_{11}}x_1^{m_{21}}b_1b_2\ldots x_1^{m_{r1}}b_r)+\sum_{1<q}Ch(x_q^{m_{1q}},x_1^{m_{21}})-Ch(x_1^{m_{21}},x_q^{m_{1q}})$$

$$=h(x_1^{V(a;x_1)}b_1b_2\ldots b_r)+\sum\sum_{i<j\,1<q}Ch(x_q^{m_{iq}},x_1^{m_{j1}})-Ch(x_1^{m_{j1}},x_q^{m_{iq}})$$

$$=h(x_1^{V(a;x_1)}x_2^{V(a;x_2)}\ldots x_l^{V(a;x_l)})+\sum\sum_{i<j\,p<q}Ch(x_q^{m_{iq}},x_p^{m_{jp}})-Ch(x_p^{m_{jp}},x_q^{m_{iq}})$$

$$=h(x_1^{V(a;x_1)}x_2^{V(a;x_2)}\ldots x_l^{V(a;x_l)})-\sum\sum_{i<j\,p<q}m_{jp}m_{iq}[Ch(x_p,x_q)-Ch(x_q,x_p)]$$

$$=h(x_1^{V(a;x_1)}x_2^{V(a;x_2)}\ldots x_l^{V(a;x_l)})-\sum_{p<q}V_3(a;x_p,x_q)[Ch(x_p,x_q)-Ch(x_q,x_p)]$$

$$=\sum_p V(a;x_p)h(x_p)+\frac{V(a;x_p)(V(a;x_p)-1)}{2}Ch(x_p,x_p)$$

$$+\sum_{p<q}V(a;x_p)V(a;x_q)Ch(x_p,x_q)-\sum_{p<q}V_3(a;x_p,x_q)[Ch(x_p,x_q)-Ch(x_q,x_p)]$$

$$=\sum_p V(a;x_p)h(x_p)+\frac{V(a;x_p)(V(a;x_p)-1)}{2}Ch(x_p,x_p)+\sum_{p<q}V_2(a;x_p,x_q)Ch(x_p,x_q)$$

$$+\sum_{p<q}V_3(a;x_p,x_q)Ch(x_q,x_p)$$

This proves (3.10)where the variable s x_p and x_q are relabeled as x and y respectively.
We support the converse statement with the following observations:

1. For each fixed letter x, by (3.6), the map a→V (a;x) is a morphism from ⟨A⟩ to N, A morphism is certainly a solution of (1.5)
2. The map $n\to\frac{(n-1)n}{2}$ from N to N is a solution of (1.5). It is consequence of Theorem3.1.
3. The previous two observations combined lead to the fact that $a\to\frac{V(a;x)(V(a;x)-1)}{2}$ from ⟨A⟩ to N is a solution of (1.5)
4. Maps $a\to V_3$ (a;x,y)and $a\to V_2$ (a;y,x) from ⟨A⟩ to N are solution of (1.5), as stated Proposition 2
5. Each term occurring on the right hand side of (3.10) under summation represents a function which belongs to $KerC^2$(⟨A⟩ , T).

CONCLUSION

We studied Second Order Cauchy difference equation and solution has been found in Free Monoid generated by single and more than 2 character. We can extend this work to the other type Cauchy functional equations and also we can find the solution for any functional equations in Free Semi Group and Different type of Groups.

ACKNOWLEDGEMENT

We would like to sincerely thank Professor Dr.K Thirusangu, our Guru, for his invaluable advice and assistance during the research process. His knowledge and perceptions were really helpful in guiding our research and assisting us in overcoming obstacles.

REFERENCES

[1] Aczel,J: Lectures on Functional Equations and Their Applications. Academic Press New York (1966).
[2] Ng,CT: Jensen's functional equation on groups,Aequ.Math.39(1990), 85–99.
[3] Ng,CT: Jensen's functional equationon groupsII, Aequ.Math.58(1999), 311–320.

[4] Baron, K. Kannappan, P: On the Cauchy difference, Aequ. Math. 46 (1993),112–118.
[5] Brzdek,J:On the Cauchy difference on normed spaces, Abh.Math.semin. Univ.Hamb.66(1996)143–150
[6] E banks,B: Generalized Cauchy difference functional equations, Aequ.Math. 70 (2005), 154–176.
[7] E banks,B: Generalized Cauchy difference functional equations, IIProc.Am. Math. Soc. 136 (2008), 3911–3919.
[8] Ng, CT, Zhao,H: Kernel of the second order Cauchy difference on groups, Aequ.Math.86(2013),155–170.
[9] Guo and Li: Solutions of the third order Cauchy difference equationon groups, Advances in difference equations203 (2014).