Abstract Cauchy Problems in Two Variables and Tensor Product of Banach Spaces

[1]Roshdi Khalil, [2]Waseem Ghazi Alshanti, [3]Ma′mon Abu Hammad

[1]Department of Mathematics, the University of Jordan, Amman, Jordan,

[2,3]Department of Mathematics, Al Zaytoonah University of Jordan, Amman, Jordan,

[1]roshdi@ju.edu.jo, [2]w.alshanti@zuj.edu.jo, [3]m.abuhammad@zuj.edu.jo

Abstract

In this paper we study linear abstract Cauchy problem in two variables. Theory of two-parameter semigroups of linear operators and tensor product of Banach spaces is needed to study the solution of such equation.

Keywords: Two-parameter semigroup, Abstract Cauchy problem, Operator valued function, Banach space.

MSC 2010: 26A33

1 Introduction

Ordinary and partial differential equations in which the unknown function and its derivatives take values in some abstract space such as Hilbert space or Banach space are called abstract differential equations. One of the most powerful tools for solving linear abstract differential equations is the method of semigroups of linear operators on Banach spaces. The basics of this method was originated, independently, by both E. Hille in (1948) [1] and K. Yosida in (1948) [2]. The power of the semigroup approach became clear through contribution by W. Feller in (1952, 1954) [3]. One of the classical vector valued differential equations that can be handled via semigroups of operators is the so called the abstract Cauchy problem which has the form:

$$\frac{du}{dt} = Au(t), \qquad t \geq 0,$$
$$u(0) = x,$$

where $A : D(A) \subseteq X \to X$ a linear operator of an appropriate type, $x \in X$ is given and $u : [0, \infty) \to X$ is the unknown function. For both linear and nonlinear abstract Cauchy problems, there are many applications in engineering and applied sciences. For any abstract Cauchy problem, one can associate a family of bounded linear operators that is known as a semigroup of operators.

Let X be a Banach space, and $L(X, X)$ be the space of all bounded linear operators on X. A one-parameter semigroup is a family of linear operators, namely, $\{T(t)\}_{t\geq 0} \subseteq L(X, X)$ such that

$$(i)\ T(0) = I, \text{ the identity operator of } X,$$
$$(ii)\ T(s+t) = T(s)T(t) \text{ fore very } t, s \geq 0.$$

If, in addition, for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0+$, then the semigroup is called c_0-semigroup or strongly continuous semigroup.

The fact that every non-zero continuous real or complex function that satisfies the fact $g(s+t) = g(s)g(t)$ for every $t, s \geq 0$ has the form $g(t) = e^{ax}$, and that g is determined by the number $a = g'(0)$, reveals the association of an operator A to $\{T(t)\}_{t\geq 0}$ such that $Ax := \lim\limits_{t\to 0+} \frac{T(0+t)x - T(0)x}{t}$; $x \in D(A)$ and is called the infinitesimal generator of $\{T(t)\}_{t\geq 0}$.

In 2004, Khalil etal presented the definition of the infinitesimal generator for two parameter semigroups [4]. Recently, in 2019, M. Akkouchi et al.[5] carried out a theoretical framework for two-parameter semigroups of bounded linear operators on a Banach space. For more related works, we refer the reader to [6, 7, 8, 9].

The object of this paper is study the abstract Cauchy problem in two variables by considering two characteristics, namely, the concept of two-parameter semigroup of linear operators and the theory of tensor product of Banach spaces.

2 Preliminaries

Definition 1 *Let X be a Banach space, and $L(X,X)$ be the space of all bounded linear operators on X. A map defined by $T:[0,\infty)\times[0,\infty)\to L(X,X)$ is called a two-parameter semigroup or semigroup in two variables if*

(i) $T(0,0)=I$, *the identity operator of X,*

(ii) $T((s_1,t_1)+(s_2,t_2))=T(s_1,t_1)T(s_2,t_2)$ *fore very $t,s\geq 0$.*

Remark 1 *From the above definition, it follows that*

$$\begin{aligned} T(s,t) &= T((s,0)+(0,t)) \\ &= T(s,0)T(0,t). \end{aligned}$$

This implies that a semigroup in two variables is the product of two semigroups in one variable.

Definition 2 *The linear operator $L(1,1)$ defined by*

$$L(1,1)x = A_1x + A_2x,$$

where

$$\begin{aligned} A_1x &:= \lim_{s\to 0+}\tfrac{(T(s,0)-I)x}{s}, \\ A_2x &:= \lim_{t\to 0+}\tfrac{(T(0,t)-I)x}{t}. \end{aligned}$$

is the infinitesimal generator of the two-parameter semigroup $\{T(s,t)\}_{t,s\geq 0}$, A_1 and, A_2 are the generators of $T(s,0)$ and $T(0,t)$, respectively. We write L for $L(1,1)x$.

Theorem 1 *Let $\{T(s,t)\}_{t,s\geq 0}$ be a two-parameter semigroup and L be its infinitesimal generator. Then*

$$DT(s,t)\begin{pmatrix}1\\1\end{pmatrix}x = (A_1+A_2)\,T(s,t)x.$$

3 Main Results

3.1 Abstract Cauchy Problems in Two Variables

In this section, we provide the solution of the non-homogeneous abstract Cauchy problems in two variables.

Consider the abstract Cauchy problem in two variables:

$$\frac{\partial u(s,t)}{\partial s} + \frac{\partial u(s,t)}{\partial t} = Lu(s,t) + f(s) + g(t), \tag{1}$$

where $L : X \to X$, closed linear operator with $Dom(L) \subseteq Rang(u)$. Let us assume the initial condition $u(0,0) = x_\circ$.

Procedure

(1) Consider the semigroup of operators

$$T(s,t)x = e^{(s+t)L}x, \ \ \forall s,t > 0 \text{ and } x \in X. \tag{2}$$

To make life easy, let us assume L to be of exponential order, in the sense:

(i) L is densely defined,
(ii) $\sum_{i=m}^{n} \frac{(s+t)^n}{n!} \left\| L^n x \right\| < \infty, \ \forall x \in Dom(L)$,
(iii) If $x \in Dom(L)$, then $C\left(x, L\right) = \left\{ x, \ Lx, \ L^2 x, \cdots \right\} \subseteq Dom(L)$.

(2) Let

$$\begin{aligned} u(s,t) &= T(s,0)x_\circ + T(0,t)x_\circ \\ &\quad + \int_0^s T(s-\theta,0) f(\theta)\, d\theta \\ &\quad + \int_0^t T(0,t-w) g(w)\, dw. \end{aligned} \tag{3}$$

The claim is such $u(s,t)$ is a solution of (1).

Indeed, using the form of $T(s-\theta,0)x = e^{sL}e^{-\theta L}x$ for any x in the domain of L, and similarly for$T(0,t-w)$ we get

$$\begin{aligned} \frac{\partial u}{\partial s} &= LT(s,0)x_\circ + L\int_0^s T(s-\theta,0) f(\theta)\, d\theta + f(s) \\ \frac{\partial u}{\partial t} &= LT(0,t)x_\circ + L\int_0^t T(0,t-w) g(w)\, dw + g(t). \end{aligned} \tag{4}$$

Hence,

$$\begin{aligned}\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} &= L\left[T(s,0)+T(0,t)\right]x_{\circ} \\ &\quad +L\int_0^s T(s-\theta,0)f(\theta)\,d\theta+f(s) \\ &\quad +L\int_0^t T(0,t-w)g(w)\,dw+g(t) \\ &= L\left[u(s,t)\right]+f(s)+g(t). \qquad (5)\end{aligned}$$

Thus, such $u(s,t)$ given in (ref A-3) satisfies equation (1).

3.2 Tensor Product Abstract Cauchy Problem in Two Variables

Definition 3 *Let X and Y be any two Banach spaces and X^* is the dual space of X. For $x \in X$ and $y \in Y$, the operator $T : X^* \to Y$, defined by*

$$T(x^*) = x^*(x)y = \langle x,\, x^*\rangle\, y,$$

is a bounded one rank linear operator. We write $x \otimes y$ for such T . Such operators are called atoms.

Atoms are used in theory of best approximation in Banach spaces [10] and they are considered among the fundamental ingredients in the theory of tensor product of Banach spaces. For more related work on tensor product of Banach spaces, we refer reader to [10, 11, 12, 17] .

Definition 4 *Let X and Y be Banach spaces and $A : Dom(A) \subseteq X \to Y$ be linear. The operator A is called of exponential order if:*

(i) *If $x \in Dom(A)$ then $\{x,\, Ax,\, A^2x,\, \cdots\} \subseteq Dom(A)$,*
(ii) $\sum_{n=1}^{\infty} \frac{t^n}{n!} A^n x < \infty$, $\forall x \in Dom(A)$.

Clearly every bounded linear operator is of exponential order.
We will write $e^{tA}x$ for $\sum_{n=1}^{\infty} \frac{t^n}{n!} \|A^n x\|$.
Now, consider the abstract Cauchy problem.

$$\begin{aligned}u'(s)\otimes v'(t) &= Au(s)\otimes Bv(t)+f(s)\otimes Bv(t) \\ &\quad +Au(s)\otimes g(t)+f(s)\otimes g(t) \qquad (6)\end{aligned}$$

5

where X and Y are Banach spaces, $u : [0, \infty) \to X$, $\upsilon : [0, \infty) \to Y$, $A : Dom(A) \subseteq X \to X$, $Rang(u) \subseteq Dom(A)$, $B : Dom(B) \subseteq Y \to Y$, $Rang(\upsilon) \subseteq Dom(B)$, A, B are closed operators of exponential order, and both $f : [0, \infty) \to X$, $g : [0, \infty) \to Y$ are given. Moreover, let us assume $u(0) = x_\circ$ and $\upsilon(0) = y_\circ$.

Procedure

Let

$$u(s) = e^{sA} x_\circ + \int_0^s e^{(s-\theta)A} f(\theta)\, d\theta, \tag{7}$$

$$\upsilon(s) = e^{tB} y_\circ + \int_0^t e^{(t-\omega)B} g(\omega)\, d\omega. \tag{8}$$

Then, using the same technique as in section 1 for differentiating the integral we get:

$$\begin{aligned} u'(s) &= Ae^{sA} x_\circ + A\int_0^s e^{(s-\theta)A} f(\theta)\, d\theta + f(s) \\ &= Au(s) + f(s), \end{aligned} \tag{9}$$

$$\begin{aligned} \upsilon'(s) &= Be^{tB} y_\circ + B\int_0^t e^{(t-\omega)B} g(\omega)\, d\omega + g(t) \\ &= B\upsilon(s) + g(t). \end{aligned} \tag{10}$$

Compiling both (9) and (10) in tensor product form yields (6).

Remark 2 *Both $e^{sA} x_\circ$ and $e^{tB} y_\circ$ have no meaning unless A and B are, respectively, of exponential order. Thus, one can summarize the above result as follows:*

If A and B are of exponential order, then (6) *has a unique solution where $u(0) = x_\circ$ and $\upsilon(0) = y_\circ$.*

4 Conclusions

This paper has successfully introduced analytical methods for handling non-homogeneous abstract Cauchy problem in two variables. The first method is based on the new concept of two-parameter semigroup of linear operators and its infinitesimal generator. While the second method utilizes the theory of tensor product of Banach spaces coupled with the tensor product properties to formulate a solution to a general tensor version of nonhomogeneous abstract Cauchy problem. In both cases the obtained results seem to be very interesting and promising in the sense that they could be extended for further classes of abstract Cauchy problems.

References

[1] E. Hille and R. Phillips. Functional analysis and semi-groups. New York: American Mathematical Society. Vol. 31 (1948).

[2] K. Yosida. On the differentiability and the representation of one-parameter semi-group of linear operators. Journal of the Mathematical Society of Japan. 1.1 : 15-21 (1948).

[3] W. Feller. The general diffusion operator and positivity preserving semi-groups in one dimension. Annals of Mathematics, 417-436 (1954).

[4] Sh. Al-Sharif, R. Khalil. On the generator of two parameter semigroups. Applied mathematics and computation. 6;156(2):403-14 (2004).

[5] M. Akkouchi, M. Houimdi and M. Rhali. A theoretical framework for two-parameter semigroups. Gulf Journal of Mathematics 7.1 (2019).

[6] R. Khalil, R. Al-Mirbati and D. Drissi. Tensor product semigroups. European Journal of Pure and Applied Mathematics 3.5: 881-898 (2010).

[7] Z. Dahmani, A. Anber and I. Jebril. Solving Conformable Evolution Equations by an Extended Numerical Method. Jordan J. Math. Stat 15 : 363-380 (2022).

[8] Sh.Al-Sharif, M. Al Horani and R. Khalil. The Hille Yosida theorem for conformable fractional semi-groups of operators. Missouri Journal of Mathematical Sciences. 33(1):18-26 (2021).

[9] I. Batiha, Z. Chebana, T-E. Oussaeif, A. Ouannas, I. Jebril. On a Weak Solution of a Fractional-order Temporal Equation. Mathematics and Statistics, Vol. 10, Issue 5, pp. 1116-1120, (2022).

[10] R. Khalil. Best approximation in tensor products. Numerical Functional Analysis and Optimization. 8.3-4: 347-356 (1986).

[11] E. Abu-Sirhan and R. Khalil. Simultaneous Approximation In Operator And Tensor Product Spaces. Journal of Applied Functional Analysis 4.1 (2009).

[12] O. Yasin and R. Khalil. Tensor product C-semigroups of operators. J. Semigroup Theory Appl. 2014. (2014).

[13] G. Barbatis, E. Davies & J. Erdos. Cours on Operator Theory , King's College London.

[14] A. Pazy. Semigroup of Linear Operator and Applications in Partial Differential Equations, Springer-Verlag New York. Inc, (1983).

[15] E. Popescu. On The Fractional Cauchy Problem Associated With a Feller Semigroup, Romanian Academy, Mathematical Reports, Vol. 12(62), No. 2, pp. 181–188, (2010).

[16] K. Engel & R. Nagel. One Parameter Semigroups For Linear Evolution Equations, Graduate Texts in Math., Springer-Verlag, Berlin-Heidelgerg-New York, (2000).

[17] R. Khalil, R. Al-Mirbati & D. Drissi. Tensor product semigroups. European Journal of Pure and Applied Mathematics, 3(5), 881-898, (2010).

[18] L. Rabhi, M. Al Horani & R. Khalil. Inhomogeneous conformable abstract Cauchy problem. Open Mathematics, 19(1), 690-705, (2021).