Inertial hybrid and shrinking projection methods for sums of three monotone operators

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Abstract.

In this paper, we introduce two iterative algorithms for finding the solution of the sum of two monotone operators by using hybrid projection methods and shrinking projection methods. Under some suitable conditions, we prove strong convergence theorems of such sequences to the solution of the sum of an inverse-strongly monotone and a maximal monotone operator. Finally, we present a numerical result of our algorithm which defined by the hybrid method.

Keywords: Hybrid projection methods, Shrinking projection methods, Monotone operators and Resolvent.

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1 Introduction

In this work, we consider the problem is finding a zero point of the sum of three monotone operators that is,

$$\text{find } z \in H \text{ such that } 0 \in (A + B + C)z, \tag{1.1}$$

where A is a multi-valued maximal monotone operator and B, C are two single monotone operators. In 2017, Davis and Yin [5] shown that the problem (1.1) can be related to a convex optimization problem, that is,

$$\text{minimize}_{x \in H} F(x) + G(x) + M(x),$$

where $A = \partial R, B = \partial S$ and $C = \nabla P$ with ∂R and ∂S denote the subdiferentials of R and S, respectively. The convex optimization problem involves several specific problems that have emerged in material sciences, medical and image processing and signal and image processing (see more in [6, 7]). Moreover, the monotone inclusion problems (1.1) includes some special cases. For example, when $B = 0$, problem (1.1) becomes find $x \in H$, such that

$$0 \in Ax + Cx. \tag{1.2}$$

If $C = 0$, problem (1.1) reduces to find $x \in H$, such that

$$0 \in Ax + Bx. \tag{1.3}$$

If $B = 0$ and $C = 0$, problem (1.1) reduces to the simple monotone inclusion find $x \in H$ such that

$$0 \in Ax. \tag{1.4}$$

So, we have the problem (1.1) is very important. Many researcher study and develop algorithm methods to solve the solution. Davis and Yin [5] introduced the fixed-point equation for solving monotone inclusions with three operators. In 2018, Cevher et al. [8] extended the three-operator splitting algorithm [5] from the determinist setting to the stochastic setting for solving the problem (1.1). Similarly, Yurtsever et al. [9] introduced a stochastic three-composite minimization algorithm to solve the convex minimization of the sum of three convex functions. In addition, Yu et al. [10] introduced an outer reflected forward-backward splitting algorithm to solve this problem as

$$x_{n+1} = J_r^A(x_n - \lambda Bx_n - \lambda Cx_n) - r(Bx_n - Bx_{n-1}). \tag{1.5}$$

The sequence $\{x_n\}$ converges weakly to solution of the problem (1.1).

Motivated and inspired by all above contributions, in this work, we will introduce two iterative algorithms for finding the solution of the sum of three monotone operators by using hybrid projection method and shrinking projection method. Under some suitable conditions, we prove strong convergence theorems of such sequences to the solution of the sum of three monotone operators. Finally, we will present a numerical result of our algorithm which defined by the hybrid method and applied to image inpainting.

2 Preliminaries

Let H be a real Hilbert space and $\mathcal{C}$ be a nonempty closed convex subset of H. Denote that $\to$ and $\rightharpoonup$ are a weak and strong convergence, respectively. I denotes the identity operator on H. For a given sequence, let $\omega_w(x_n) := \{x : \exists x_{n_k} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Lemma 2.1. *Let $x \in H$ and $z \in \mathcal{C}$. Then we have*

(i) $z = P_{\mathcal{C}}(x)$ *if* $\langle x - z, z - y\rangle \geq 0$, *for all* $y \in \mathcal{C}$.

(ii) $\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\| \leq \|x - y\|$, *for all* $x, y \in H$

(iii) $\|x - P_{\mathcal{C}}(x)\|^2 \leq \|x - y\|^2 - \|y - P_{\mathcal{C}}(x)\|^2$ *for all* $y \in \mathcal{C}$.

Definition 2.2. *[1] Let $T : H \to H$ be a single-valued operator. Then*

(i) T *is said to be nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in H.$$

(ii) T *is said to be firmly nonexpansive if*

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \quad \text{for all } x, y \in H.$$

It is obvious that a firmly nonexpansive operator is nonexpansive.

(iii) T *is said to be L-Lipschitz continuous, for some* $L > 0$, *if*

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in H.$$

If $L = 1$, *then* T *is nonexpansive.*

(iv) T *is said to be c-cocoercive (or c-inverse strongly monotone), if*

$$\langle x - y, Tx - Ty \rangle \geq c\|Tx - Ty\|, \quad \text{for all } x, y \in H,$$

where $c > 0$.

(v) T *is said to be monotone if*

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \text{for all } x, y \in H.$$

Remark 2.3. *If* C *is c-cocoercive, then* C *is* $1/c$*-Lipschitz continuous and monotone. By using the* L*-Lipschitz continuity of* B*, we obtain that* $B + C$ *is* $(L + 1/c)$*-Lipschitz continuous. Moreover, since* C *is c-cocoercive, we have* C *is monotone.*

Definition 2.4. *Let* $A : H \to 2^H$ *be a set-valued operator and the domain of* A *be* $D(A) = \{x \in H : Ax \neq \emptyset\}$*. The graph of* A *is denoted by* $Graph(A) = \{(x, u) \in H \times H : u \in Ax\}$*. Then the operator* A *is monotone if* $\langle x_1 - x_2, z_1 - z_2 \rangle \geq 0$ *whenever* $z_1 \in Ax_1$ *and* $z_2 \in Ax_2$.

A monotone operator A *is maximal if for any* $(x, z) \in H \times H$ *such that*

$$\langle x - y, z - w \rangle \geq 0$$

for all $(y, w) \in Graph(A)$ *implies* $z \in Ax$.

Let A be a maximal monotone operator and $r > 0$. Then we can define the resolvent $J_r : R(I + rA) \to D(A)$ by

$$J_r^A = (I + rA)^{-1}$$

where $D(A)$ is the domain of A. We know that J_r^A is nonexpensive and we can study the other properties in references [12, 11, 13].

Lemma 2.5. *[4] Let* $A : H \to 2^H$ *be a maximal monotone mapping and let* $B : H \to H$ *be a Lipschitz continuous and monotone mapping. Then* $A + B$ *is maximally monotone.*

Lemma 2.6. [2] *Let* $\mathcal{C}$ *be a closed convex subset of a real Hilbert space* H*,* $x \in H$ *and* $z = P_{\mathcal{C}}x$*. If* $\{x_n\}$ *is a sequence in* $\mathcal{C}$ *such that* $\omega_w(x_n) \subset \mathcal{C}$ *and*

$$\|x_n - x\| \leq \|x - z\|,$$

for all $n \geq 1$*, then the sequence* $\{x_n\}$ *converges strongly to a point* z.

Lemma 2.7. *[3] Let* $\mathcal{C}$ *be a closed convex subset a real Hilbert space* H*, and* $x, y, z \in H$*. Then, for given* $a \in \mathbb{R}$*, the set*

$$U = \{v \in \mathcal{C} : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex and closed.

3 Hybrid Projection Methods

In this section, we introduce a intertial hybrid projection method and prove a strong convergence theorem.

(A1) $A : H \to 2^H$ is maximal monotone.

(A2) $B : H \to H$ is monotone and L-Lipchitz continuous, for some $L > 0$.

(A3) $C : H \to H$ is c-cocoercive.

(A4) $\Omega := (A+B+C)^{-1}(0) \neq \emptyset$.

The method is of the following form.

Algorithm 3.1 : Inertial hybrid projection algorithm (IHP Algorithm) Initialization : Choose $x_0, x_1 \in H, \alpha_n \in [0,1)$.
Iterative step : Compute x_{n+1} via

$$\begin{cases} w_n = x_n + \alpha_n(x_n + x_{n-1}), \\ y_n = J^A_{r_n}(w_n - r_nBw_n - r_nCw_n), \\ z_n = y_n - r_n(By_n - Bw_n), \\ C_n = \{z \in H : \|z_n - z\|^2 \leq \|w_n - z\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{3.1}$$

where

$$0 < r_n < \min\{c, \frac{1}{2L}\} \quad \text{and} \quad \lim_{n\to\infty} r_n = 0.$$

Lemma 3.1. *Let $\{z_n\}$ be a sequence generated by IHP Algorithm. If conditions $(A1)-(A4)$ hold, we have*

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2, \quad \textit{for all } u \in \Omega. \tag{3.2}$$

Proof. Let $a_n = r_n^2\|By_n - Bw_n\|^2 - 2r_n\langle y_n - u, By_n - Bw_n\rangle$. Thus

$$\begin{aligned} \|z_n - u\|^2 &= \|y_n - r_n(By_n - Bw_n) - u\|^2 \\ &= \|y_n - u\|^2 - 2r_n\langle y_n - u, By_n - Bw_n\rangle + r_n^2\|By_n - Bw_n\|^2 \\ &= \|w_n - u\|^2 + \|y_n - w_n\|^2 + 2\langle w_n - u, y_n - w_n\rangle + a_n \\ &= \|w_n - u\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - u\rangle + a_n \\ &= \|w_n - u\|^2 - \|y_n - w_n\|^2 - 2\langle y_n - u, w_n - y_n + r_n(By_n - Bw_n)\rangle \\ &\quad + r_n^2\|By_n - Bw_n\|^2. \end{aligned} \tag{3.3}$$

Since B is L-Lipchitz continuous, we have

$$\|Bw_n - By_n\| \leq L\|w_n - y_n\|. \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - L^2r_n^2)\|w_n - y_n\|^2 - 2\langle y_n - u, w_n - y_n + r_n(By_n - Bw_n)\rangle. \tag{3.5}$$

Since $y_n = J^A_{r_n}(w_n - r_nBw_n - r_nCw_n)$, we have $(I - r_nB - r_nC)w_n \in (I + r_nA)y_n$. So, we obtain

$$\frac{1}{r_n}(w_n - r_nBw_n - r_nCw_n - y_n) \in Ay_n. \tag{3.6}$$

Since $0 \in (A+B+C)u$, we have

$$-Bu - Cu \in Au. \tag{3.7}$$

Since the operator A is maximal monotone, one gets

$$\frac{1}{r_n}\langle w_n - r_nBw_n - r_nCw_n - y_n + r_nBu + r_nCu, y_n - u\rangle \geq 0.$$

This implies that

$$\langle w_n - r_nBw_n - r_nCw_n - y_n + r_nBu + r_nCu, y_n - u\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle w_n - y_n + r_n(By_n - Bw_n), y_n - u\rangle &\geq \langle r_nBy_n - r_nBu - r_nCu + r_nCw_n, y_n - u\rangle \\ &= \langle r_nBy_n - r_nBu, y_n - u\rangle + \langle r_nCw_n - r_nCu, y_n - u\rangle \\ &\geq \langle r_nCw_n - r_nCu, y_n - u\rangle \qquad (3.8)\end{aligned}$$

and since C is c-cococercive, we have

$$\begin{aligned}2r_n\langle Cw_n - Cu, y_n - u\rangle &= 2r_n\langle Cw_n - Cu, y_n - w_n\rangle + 2r_n\langle Cw_n - Cu, w_n - u\rangle \\ &\geq -2r_n\|Cw_n - Cu\|\|y_n - w_n\| + 2cr_n\|Cw_n - Cu\|^2 \\ &\geq -2cr_n\|Cw_n - Cu\|^2 - \frac{r_n}{2c}\|y_n - w_n\|^2 + 2cr_n\|Cw_n - Cu\|^2 \\ &= -\frac{r_n}{2c}\|y_n - w_n\|^2. \qquad (3.9)\end{aligned}$$

Combining the equation (3.8) and (3.9), we obtain

$$-2\langle w_n - y_n + r_n(By_n - Bw_n), y_n - u\rangle \leq \frac{r_n}{2c}\|y_n - w_n\|^2. \qquad (3.10)$$

Combining the equation (3.5) and (3.10), we obtain

$$\|z_n - u\|^2 \leq \|w_n - u\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2, \quad \text{for all } u \in \Omega.$$

This completed the proof.

Lemma 3.2. *Let the operators A, B and C satisfies conditions $(A1)-(A4)$. The three sequences $\{x_n\}, \{w_n\}$ and $\{y_n\}$ generated by IHP Algorithm. Assume that $\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|w_n - y_n\| = 0$. If a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to some $x^* \in H$, then $x^* \in \Omega$ where $\Omega := (A+B+C)^{-1}(0)$.*

Proof. Suppose that $(u, v) \in Graph(A+B+C)$. Thus $v - Bu - Cu \in Au$. Since $y_{n_k} = J^A_{r_{n_k}}(w_{n_k} - r_{n_k}Bw_{n_k} - r_{n_k}Cw_{n_k})$, we have $(I - r_n(B+C)) \in (I + r_{n_k}A)y_{n_k}$. This implies that

$$\frac{1}{r_{n_k}}(w_{n_k} - y_{n_k} - r_{n_k}(B+C)w_{n_k}) \in Ay_{n_k}.$$

By using the maximal monotonicity of A, we get

$$\langle u - y_{n_k}, v - Bu - Cu - \frac{1}{r_{n_k}}(w_{n_k} - y_{n_k} - r_{n_k}(B+C)w_{n_k})\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle u - y_{n_k}, v\rangle &\geq \langle u - y_{n_k}, (B+C)u + \frac{1}{r_{n_k}}(w_{n_k} - y_{n_k} - r_{n_k}(B+C)w_{n_k})\rangle \\ &= \langle u - y_{n_k}, (B+C)u - (B+C)w_{n_k}\rangle + \frac{1}{r_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle \\ &= \langle u - y_{n_k}, (B+C)u - (B+C)y_{n_k}\rangle + \langle u - y_{n_k}, (B+C)y_{n_k} - (B+C)w_{n_k}\rangle \\ &\quad + \frac{1}{r_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle \\ &\geq \langle u - y_{n_k}, (B+C)y_{n_k} - (B+C)w_{n_k}\rangle + \frac{1}{r_{n_k}}\langle u - y_{n_k}, w_{n_k} - y_{n_k}\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|w_n - y_n\| = 0$ and $B + C$ is Lipschitz continuous, we have $\lim_{n\to\infty}\|(B+C)y_{n_k} - (B+C)w_{n_k}\| = 0$. From $0 < r_n < \min\{c, \frac{1}{2L}\}$, one get

$$\lim_{n\to\infty}\langle u - y_{n_k}, v\rangle = \langle u - x^*, v\rangle \geq 0.$$

Since $A+B+C$ is maximal monotone, we have $0 \in (A+B+C)x^*$. We can conclude that $x^* \in \Omega$. This completed the proof.

Theorem 3.3. *Let the operators A, B and C satisfy conditions $(A1)-(A4)$. Then, the sequence $\{x_n\}$ generated by IHP Algorithm converges strongly to $x^* = P_\Omega(x_0)$.*

Proof. It is obvious that C_n and Q_n are closed convex for every $n \in \mathbb{N}$. First, we will prove that $\Omega \subset C_n$, for all $n \in \mathbb{N}$. By using Lemma 3.1, we obtain $\Omega \subset C_n$, for all $n \in \mathbb{N}$. Next, we prove that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$ by the mathematical induction. By the definition of Q_n in IHP Algorithm, we have $Q_1 = H$. For $n = 1$, we note that $\Omega \subset H = Q_1$. Suppose that $\Omega \subset Q_k$ for some $k \in \mathbb{N}$. Since $C_k \cap Q_k$ is closed and convex, we can define

$$x_{k+1} = P_{C_k\cap Q_k}(x_0).$$

This implies that

$$\langle x_{k+1} - z, x_0 - x_{k+1}\rangle \geq 0 \quad \text{for all } z \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we have $\Omega \subset Q_{k+1}$. It follows that $\Omega \subset Q_n$, for all $n \in \mathbb{N}$. So, $\{x_n\}$ is well defined. Next, we show that $\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty}\|w_n - y_n\|^2 = 0$. Since $\Omega \subset C_n \cap Q_n$, for all $n \in \mathbb{N}$, and $x_{n+1} = P_{C_n\cap Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \leq \|x^* - x_0\|.$$

This mean that $\{x_n\}$ is bounde, so $\{w_n\}$ is also bounded. From the definition of Q_n, we obtain $x_n = P_{Q_n}(x_0)$. Since $x_{n+1} \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \ \text{for all}\, n \in \mathbb{N}.$$

This implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Therefore,

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}$$

It follows that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in C_n \cap Q_n \subset C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2 - (1 - \frac{r_n}{2c} - L^2 r_n^2)\|w_n - y_n\|^2.$$

Since $0 \leq r_n < \min\{c, \frac{1}{2L}\}$, we have $\|z_n - x_{n+1}\| \leq \|w_n - x_{n+1}\|$. Moreover, by the definition of $\{w_n\}$, we get

$$\|w_n - x_n\| = \|x_n + \alpha_n(x_n - x_{n+1}) - x_n\| = |\alpha_n|\|x_n - x_{n+1}\|$$

This implies that $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Therefore,

$$(1 - \frac{r_n}{2c} - L^2 r_n^2)\|w_n - y_n\|^2 \leq \|w_n - x_{n+1}\|^2 - \|z_n - x_{n+1}\|^2.$$

Since $\lim_{n\to\infty} r_n = 0$, we have $\lim_{n\to\infty}(1 - \frac{r_n}{2c} - L^2 r_n^2) = 1$. It follows that $\lim_{n\to\infty}\|w_n - y_n\| = 0$. Finally, we show that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x_0)$. Let $x^* = P_\Omega(x_0)$. Therefore,

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \leq \|x_0 - x^*\|.$$

By Lemma 3.2, we have every sequential weakcluster point of the sequence $\{x_n\}$ belong to Ω. That is $\omega_w(x_n) \subset \Omega$. Hence by Lemma 2.6, we can conclude that the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(x_0)$. This completes the proof. $\square$

3 The Inertial Shrinking projection methods

In this section, we introduce a intertial shrinking projection method and prove a strong convergence theorem.

Algorithm 3.2 : Inertial shrinking projection algorithm (ISP Algorithm) Initialization : Choose $x_0, x_1 \in H, \alpha_n \in [0,1)$. Let $C_1 = H$

Iterative step : Compute x_{n+1} via

$$\begin{cases} w_n = x_n + \alpha_n(x_n + x_{n-1}), \\ y_n = J^A_{r_n}(w_n - r_nBw_n - r_nCw_n), \\ z_n = y_n - r_n(By_n - Bw_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 \le \|w_n - z\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \end{cases} \tag{3.11}$$

where

$$0 < r_n < \min\{c, \frac{1}{2L}\} \text{ and } \lim_{n\to\infty} r_n = 0.$$

Theorem 3.4. *Let the operators A, B and C satisfy conditions $(A1) - (A4)$. Then, the sequence $\{x_n\}$ generated by ISP Algorithm converges strongly to $x^* = P_\Omega(x_0)$.*

Proof. By Lemma 3.1, we obtain

$$\|z_n - u\|^2 \le \|w_n - u\|^2 - (1 - \frac{r_n}{2c} - L^2r_n^2)\|w_n - y_n\|^2, \quad \text{for all } u \in \Omega.$$

It follows from $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$ that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|.$$

On the other hand, since $x^* \in \Omega \in C_n$ and $x_n = P_{C_n}(x_0)$, we have $\|x_n - x_0\| \le \|x^* - x_0\|$. Thus $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Similarly proof of Theorem 3.3, we can proof that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|w_n - y_n\| = 0$. By Lemma 2.6 and Lemma 3.2, we can conclude that $\{x_n\}$ converges strongly to $x^* = P_\Omega(x_0)$. This completes the proof. □

4 Numerical results

In this section, we firstly present by following the ideas of He et al. [14] and Dong et al. [15]. For $C = H$, we can write the algorithm 3.1 as in the following

$$\begin{cases} x_0, z_0 \in H, \\ y_n = \alpha_n z_n + (1 - \alpha_n)x_n, \\ z_{n+1} = J^A_{r_n}(y_n - r_n(B + C)y_n), \\ u_n = \alpha_n z_n + (1 - \alpha_n)x_n - z_{n+1}, \\ v_n = (\alpha_n\|z_n\|^2 + (1 - \alpha_n)\|x_n\|^2 - \|z_{n+1}\|^2)/2, \\ C_n = \{z \in C : \langle u_n, z\rangle \le v_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = p_n, \quad \text{if } p_n \in Q_n, \\ x_{n+1} = q_n, \quad \text{if } p_n \notin Q_n, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} p_n &= x_0 - \frac{\langle u_n, x_0\rangle - v_n}{\|u_n\|^2} u_n, \\ q_n &= \left(1 - \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, w_n - p_n\rangle}\right) p_n + \frac{\langle x_0 - x_n, x_n - p_n\rangle}{\langle x_0 - x_n, w_n - p_n\rangle} w_n, \\ w_n &= x_n - \frac{\langle u_n, x_n\rangle - v_n}{\|u_n\|^2}. \end{aligned}$$

Next, we will applies the above to image inpainting. We consider the degradation model that represents an actual image restoration problems or through the least useful mathematical abstractions thereof.

$$y = Hx + w$$

where y, H, x and w are the degraded image, degradation operator, or blurring operator; original image; and noise operator, respectively.

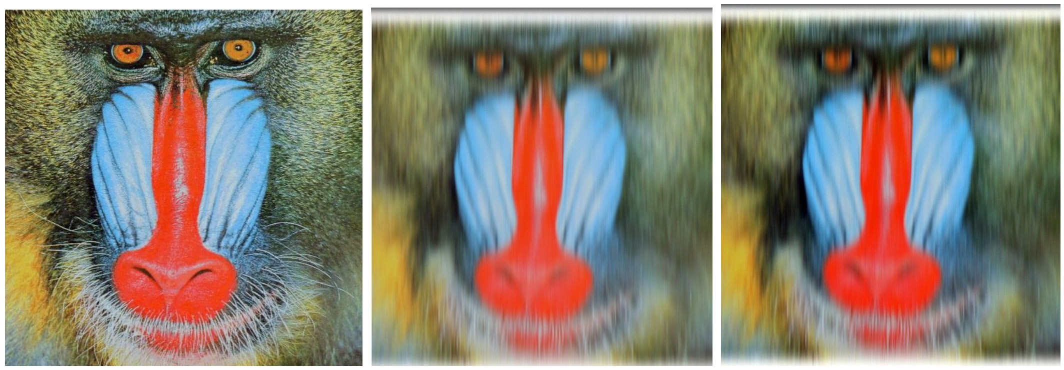
The regularized least-squares problem can be solve to obtain the reconstructed image is the following

$$\min\{\frac{1}{2}\|H(x) - y\|_2^2 + \mu\varphi(y)\} \tag{4.2}$$

where $\mu > 0$ is the regularization parameter and $\varphi(.)$ is the regularization functional. A well-known regularization function used to remove noise in the restoration problem is the l_1 norm, which is called Tikhonov regularization [**?**]. The problem (4.2) can be written in the form of the following problem as:

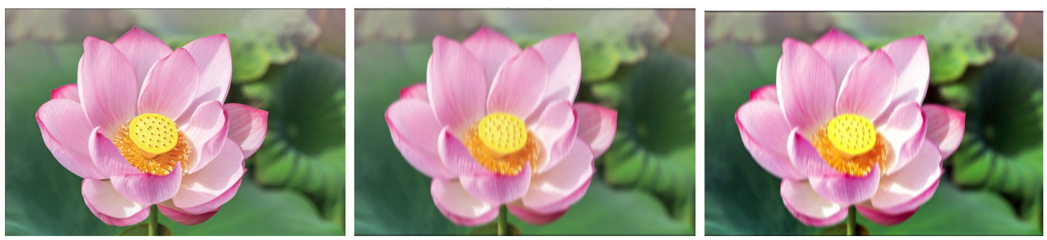
$$\min_{x\in\mathbb{R}^k}\{\frac{1}{2}\|H(x) - y\|_2^2 + \mu\|x\|_1\} \tag{4.3}$$

Note that problem (4.3) is a spacial case of the problem (1.1) by setting $A = \partial f(.), B = 0$, and $C = \nabla L(.)$ where $f(x) = \|x\|_1$ and $L(x) = \frac{1}{2}\|Hx - y\|_2^2$ This setting we have that $C(x) = \nabla L(x) = H'(Hx-y)$, where H' is a transpose of H. We begin the problem by choosing images and degrade them by random noise and different types of blurring. The random noise in this study is provided by Gaussian white noise of zero mean and 0.0001 variance. We solve the problem in (4.3) by using the above algorithm. We set $c = 70n^2, L = 0.001$ and $r_n = \frac{1}{100n+1}$. All the experiments were implemented in Matlab R2015 running on a Desktop with Intel(R) Core(TM) i5-7200u CPU 2.50 GHz, and 4 GB RAM. We obtain the following results.



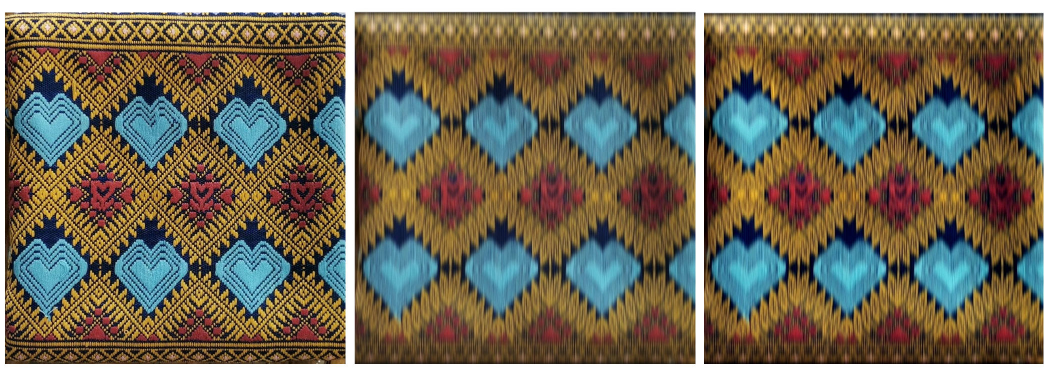
(a) Mandril **(b)** Gaussian blur **(c)** Our algorithm

Figure 1: Pictures of animals



(a) Lotus **(b)** Gaussian blur **(c)** Our algorithm

Figure 2: Pictures of lotus



(a) Fabric **(b)** Gaussian blur **(c)** Our algorithm

Figure 3: Pictures of Thai fabric

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