A New Introduction to Fuzzy Group Algebra

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ABSTRACT
In this paper, the concept of fuzzy group algebra is introduced using a finite group G and a fuzzy group μ on G. we had proved certain basic properties of fuzzy group algebra, including its behaviour as a fuzzy algebra and as a fuzzy G - module. The work is a continuation of the concept of semi simplicity of fuzzy G - modules and its properties including relationships with complete reducibility and injectiviness, which was already defined by the authors. The intersections and α- cuts of fuzzy group algebra are also analyzed.

Key words: Fuzzy G - Module, Fuzzy Group, Group Algebra, Fuzzy Group Algebra, Intersections and α - cuts of Fuzzy Group Algebra

1. INTRODUCTION

The introduction of fuzzy sets by Lofti A Zadeh led way to the fuzzification of algebraic structures. Fuzzy groups and groupoids were defined by[Rosenfield, A. (1971)]. The theory of fuzzy G – modules was studied by [Fernandez, S. (2004)]. The concept of semi simplicity for fuzzy G - modules was introduced by the authors [Abraham, P., & Sebastian, S. (2012)]. Semi simplicity of fuzzy G - modules is related with complete reducibility and injectiviness of fuzzy G - modules. The chain conditions on fuzzy G - modules are also introduced by [Abraham, P., & Sebastian, S. (2017)].
We turn our attention towards fuzzification of Maschke's Theorem on semi simplicity of group algebra. The primary objective in this process was to introduce the fuzzy version of a group algebra. In this paper, we introduce the concept of fuzzy group algebra using a finite group G and a fuzzy group μ on G. we had observed the basic properties of fuzzy group algebra, including its behaviour as a fuzzy algebra and as a fuzzy G – module. The intersections and α - cuts of fuzzy group algebra are also analyzed. This will led to desirable results asserting the main objectives.

2. Preliminaries

Given a finite group G, a vector space M over a field K is said to be a G - module if for every $g \in G$ and $m \in M$ there exists a product 'gm' called action of G on M satisfying

i) $1_G m = m, \forall m \in M$
ii) $(gh)m = g(hm) \forall g, h \in G$ and $m \in M$
iii) $g(k_1 m_1 + k_2 m_2) = k_1(gm_1) + k_2(gm_2)\ \forall\ m_1, m_2 \in M, g, h \in G$ and $k_1, k_2 \in K$

A subspace of M, which itself is a G - module with the same action is called G - sub module. A non zero G - module M is irreducible if the only G - sub modules of M are M and {0}. Otherwise it is reducible. A non-zero G - module M is completely reducible if for every G - sub module N of M there exists a G - sub module N^* of M such that $M = N \oplus N^*$. A G - module M is semi simple if there exists a family of irreducible G - sub modules M_i such that $M = {}_{i=1}^{n}\oplus M_i$

A fuzzy G - module over a G - module M is a fuzzy set μ on Msuch that

i) $\mu(ax + by) \geq \min(\mu(x), \mu(y)) \forall a, b \in K$ and $x, y \in M$
ii) $\mu(gm) \geq \mu(m) \forall\ m \in M$ and $g \in G$

The standard fuzzy intersection of finite number of fuzzy G - modules is again a fuzzy G- module.
Let G be a group and K be a field. The K vector space having G as Hamel basis is called the **Group Algebra** denoted by K(G). It contains elements of the forma $= \sum_{g \in G} a_g g, a_g \in K$ and $a_g = 0$ for all but a finite number of elements of G. The addition and multiplication in K(G)are defined by the following operations.
For $a = \sum_{g \in G} a_g g$ and $b = \sum_{g \in G} b_g g$ in K(G)

$$a + b = \sum_{g\in G}(a_g + b_g)g \;= \sum_{g\in G} a_g b_{g^{-1}t}$$

With these two operations K(G) is a K algebra with identity element

$1_{K\,(G)} = \sum_{g\in G} a_g g$ where $a_g = 1_K$ if $g = 1_G$ and $a_g = 0$ otherwise.

With the action of G on K(G) defined by

$$\left(\sum_{g\in G} a_g x\right) g = \sum_{x\in G} a_x xg = \sum_{x\in G} a_{xg^{-1}}\, x$$

K(G) can be considered as a G - module. It can be noted that If H is a subgroup of G then K(H) is a subgroup of K(G)

3. Fuzzy Group Algebra

3.1 Definition

Let G is a finite group and μ is a fuzzy group on G. The fuzzy set K(μ) on K(G) defined by

$$K(\mu)\left(\sum a_g g\right) = \min_{a_g\neq 0} \mu(g)$$

is called the **fuzzy group algebra** of μ over the group algebra K(G).

μ is called the fuzzy group corresponding to fuzzy group algebra K(μ).

Every fuzzy group on G can be used to construct a fuzzy group algebra on K(G), the restriction of which to G yields the original fuzzy group. The mapping f defined by $f(g) = 1.g$ is an isomorphism from G into K(G). This f is a fuzzy homomorphism from any fuzzy group μ of G to fuzzy group algebra K(μ) of K(G). It is evident from the fact that $f(\mu)(1.g) = \mu(g) = K(\mu)(1.g)$

3.2 Proposition

For any fuzzy group μ on a finite group G and field K, the fuzzy group algebra K(μ) on K(G) is a fuzzy algebra. In general, fuzzy group algebras are fuzzy algebras

Proof: Consider a fuzzy group μ on a finite group G,

then for $x, y \in K\,(G)$, $a, b \in K$,

$$K\,(\mu)(ax + by) = K\,(\mu)\left(\sum_{g\in G} a\, a_g g + \sum_{g\in G} b\, b_g g\right)$$
$$= K\,(\mu)\left(\sum_{g\in G}(a\, a_g + b\, b_g)g\right)$$
$$= \min_{a\, a_g + b\, b_g \neq 0} \mu(g)$$
$$\geq \min_{a_g\neq 0,\, b_g\neq 0} \mu(g)$$
$$\geq \text{Min}\left(\min_{a_g\neq 0}\mu(g), \min_{b_g\neq 0}\mu(g)\right)$$
$$= \text{Min}\big(K(\mu)(x), K(\mu)(y)\big)$$
$$K\,(\mu)(xy) = K\,(\mu)\left(\sum_{t\in G} c_t t\right)$$
$$= \min_{c_t\neq 0}\big(\mu(t)\big)$$
$$= \min_{\sum a_g b_{g^{-1}t}\neq 0}\big(\mu(t)\big)$$
$$\geq \min_{a_g\neq 0, b_{g^{-1}t}\neq 0}\big(\mu(g), \mu(g^{-1}t)\big)$$
$$\geq \text{Min}\left(\min_{a_g\neq 0}\mu(g), \min_{b_{g^{-1}t}\neq 0}\mu(g^{-1}t)\right)$$
$$= \text{Min}\big(K(\mu)(x), K(\mu)(y)\big)$$

This concludes the proof that fuzzy group algebras are fuzzy algebras.

3.3 Proposition

The fuzzy group algebra K(μ)is a fuzzy G - module on K(G), if K(G) is considered as a G - module.

Proof: By Proposition 3.2, it is evident that $K(\mu)(ax + by) = \mathrm{Min}(K(\mu)(x), K(\mu)(y))$

$$\text{For } m \in K(G) \text{and } g \in G, \quad K(\mu)(g.m) = K(\mu)\left(g\sum_{x\in G} a_x x\right)$$
$$= K(\mu)[(1.g)\textstyle\sum_{x\in G} (a_x x)]$$
$$= K(\mu)\left[\textstyle\sum_{x\in G} (a_{xg^{-1}} x)\right]$$
$$= \min_{a_{xg^{-1}}\neq 0}(\mu(x))$$
$$\geq \min_{a_x\neq 0}(\mu(x))$$
$$= K(\mu)(m)$$

This shows that all fuzzy group algebras behave as a fuzzy G - module over K(G).

3.4 Proposition

If μ and ϑare two fuzzy groups defined on a group G, then $K(\mu \cap \vartheta) = K(\mu) \cap K(\vartheta)$ on the fuzzy group algebra K(G). The fuzzy group algebra of the intersection of two fuzzy groups is the fuzzy intersection of the respective fuzzy group algebras of them.

Proof: For fuzzy groups μ and ϑ on group G, $\mu \cap \vartheta$ is also a fuzzy group.

$$K(\mu \cap \vartheta)(x) = K(\mu \cap \vartheta)\left(\sum_{g\in G} a_g g\right)$$
$$= \min_{a_g\neq 0}(\mu \cap \vartheta(g))$$
$$= \min_{a_g\neq 0}(\mu(g), \vartheta(g))$$
$$= \min\left[\min_{a_g\neq 0}\mu(g), \min_{a_g\neq 0}\vartheta(g)\right]$$
$$= \min[K(\mu)(x), K(\vartheta)\vartheta(x)]$$
$$= K(\mu) \cap K(\vartheta)(x)$$

3.5 Proposition

If μ and ϑ two fuzzy groups on a group G with $\mu \leq \vartheta$, then on the group algebra K(G) the fuzzy group algebras satisfy, $K(\mu) \leq K(\vartheta)$.

Fuzzy group algebras preserve the ordering of corresponding fuzzy groups.

Proof: By Definition, $\mu \leq \vartheta$gives $\mu(g) \leq \vartheta(g)$ for every g in G.

For any $x \in K(G)$,

$$K(\mu)(x) = K(\mu)\left(\sum a_g g\right) = \min_{a_g\neq 0}\mu(g)$$
$$\leq \min_{a_g\neq 0}\vartheta(g)$$
$$= K(\vartheta)\left(\sum a_g g\right) = K(\vartheta)(x)$$

3.6 Proposition

For $\alpha \in [0,1]$, $(K(\mu))^{\alpha} = K(\mu^{\alpha})$. The α cut of a fuzzy group algebra K(μ) is the group algebra of the α cut of the corresponding fuzzy group μ.

Proof: Let $\alpha \in [0,1]$ and $x \in K(G)$

$$x \in (K(\mu))^{\alpha} \quad \rightarrow K(\mu)(x) \geq \alpha$$
$$\rightarrow \min_{a_g\neq 0}\mu(g) \geq \alpha$$
$$\rightarrow \mu(g) \geq \alpha \text{ for all } a_g \neq 0 \text{ in } x = \sum a_g g$$
$$\rightarrow g \in \mu^{\alpha} \text{ for all } a_g \neq 0 \text{ in } x = \sum a_g g$$
$$\rightarrow g \in \mu^{\alpha} \text{ for all } a_g \neq 0 \text{ in } x = \sum a_g g$$
$$\rightarrow x = \sum a_g g \in K(\mu^{\alpha})$$

CONCLUSION

We had succeeded in the primary objective of introducing the fuzzy version of group algebra. The properties of fuzzy group algebra as fuzzy G - module will help to study its property of semi simplicity. This will turn out to be a great step towards fuzzification of Maschke's Theorem on semi simplicity of group algebras.

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