Modular Chromatic Number of Snow Graphs of Some Cycle Related Graph and Its Extended Snow Graph

P. Sumathi[1], S.Tamilselvi[2]*

[1]Department of Mathematics, C. Kandaswami Naidu College for Men,University of Madras Anna Nagar, Chennai 600 102,Tamilnadu, India, Email: sumathipaul@yahoo.com
[2]Department of Mathematics, School of Arts and Science, Vinayaka mission's Chennai Campus, Vinayaka mission's research foundation(Deemed to be University), Chennai 603104, Tamilnadu, India, Email: rat31323@gmail.com
*Corresponding Author

Received: 19.04.2024 Revised : 25.05.2024 Accepted: 10.05.2024

ABSTRACT
For a graph $G = (V(G), E(G))$ a coloring $C: V(G) \to \mathbb{Z}_k$, $k \geq 2$ (not a proper coloring) called the modular coloring if for each pair of neighboring vertices $\mathcal{S}(u) \neq \mathcal{S}(v)$. The minimum k for which G has an modular k - coloring is called modular chromatic number of G. The modular k- coloring of a graph is denoted by $m_c(G)$.In this study we examine the modular chromatic number for some snow graph and extended snow graph of cycle related graph.

Keywords: Snow graph, extended snow graph, wheel graph, gear graph, friendship graph, generalized fan graph, flower graph.

INTRODUCTION

There are many real-world applications for graph theory. That's when a graph coloring comes into play. One of the research's emerging topics is modular coloring. We consider the finite undirected connected graph. F. Okamoto, E. Salehi, and P.Zhang in 2010. In [1], a modular k- coloring, $k \geq 2$ of a graph without isolated vertices is a coloring of the vertices of G with the element in $\mathbb{Z}_k$ (where adjacent vertices may be colored the same) having the property that for every two adjacent vertices of G,The sum of the colors of the neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k- coloring is called modular chromatic number $m_c(G)$.

Let G be any graph and H is a double claw graph. Let $\{v_1, v_2, \dots, v_\mu\}$ are the external vertex (except the center vertex) of a graph G, take μ copies of H and the root vertex of the double claw graph joined with the each external node G is called snow graph and it is denoted by $ST_{dc}(G_\mu)$ where μ is the number of external vertices of graph G and T_{dc} denote the double claw tree graph.

Let G be the cyclic graph and H is any $\gamma -$ ary tree graph. Let $\{v_1, v_2, \dots, v_\mu\}$ are the external vertex (except the center vertex) of a cyclic graph, take μ copies of H and the root vertex of the $\gamma -$ ary tree joined with the external node of G by an edge is called extended snow graph denoted by $ST_\gamma(G_\mu)$ where μ, γ is the number of external vertices of cyclic graph G and height of n-ary tree graph T respectively.

In [2] P. Sumathi, S. Tamilselvi, determined the modular chromatic number of certain cyclic graphs. In [3], we obtain the modular chromatic number of the inflated graphs of the wheel, gear, fan, friendship, and flower graphs. In [4], also we examined the modular coloring of the corona product of a generalized Jahangir graph. In [5] N. Paramaguru, R. Sampathkumar, investigatedmodular colorings of join of two special graphs. In [6], T. Nicholas, G. R, Sanma, discussed the modular colorings of cycle related graphs. In [7], R.Rajarajachozhan, R.Sampathkumar, found the modular coloring of the cartesianproducts $K_m \square K_n$, $K_m \square C_n$, And $K_m \square P_n$. In [8], T. Nicholas, G. R, Sanma, found the modular colorings of circular Halin graphs of level two.In [9] Sanma. G. R, P. Maya obtained the Modular coloring and switching in some planar graphs.

Main results

Let G be a non-trivial undirected connected graph. In this paperwe introduced two new graph structures, snow graph $ST_{dc}(G_\mu)$ and extended snow graph $ST_\gamma(G_\mu)$ of cycle related graphs. The following theorem results that the snow graph and its extended snow graph of cycle related graph admits modular coloring.

Theorem 1. For any integer $\mu \geq 3$,the modular chromatic number of a snow graph of wheel graph $mc(ST_{dc}(W_\mu)) = \begin{cases} 3 & \text{when } \mu \text{ is even} \\ 4 & \text{when } \mu \text{ is odd} \end{cases}$.

Proof: The construction of the snow graph of W_μ is described as, letW_μ be a wheel graph on μ vertices and H is any double claw tree T_{dc}. Each external vertex of W_μ except the center vertex is attached with the root vertex of the T_{dc} is joined by an edge is called snow graph of wheel graph. Let$V\left(ST_{dc}(W_\mu)\right) = c_0 \cup v_\alpha \cup v_{\alpha\beta}$where $1 \leq \alpha \leq \mu$ and $1 \leq \beta \leq 6$ andlet $E\left(ST_{dc}(W_\mu)\right) = c_0 v_\alpha \cup v'_\alpha: 1 \leq \alpha \leq \mu \cup v'_{\alpha\beta}: 1 \leq \alpha \leq \mu$ and $1 \leq \beta \leq 6$. The modular coloring of $ST_{dc}(W_\mu)$ is defined by the following two cases

Case 1: When μ is odd,

Coloring of $ST_{dc}(W_\mu)$ is

$C(c_0) = 0, C(v_\alpha) = 0$ for all α.

When α ranges from 1 to$\mu - 1$ and β ranges from 1 to 6

For all odd α; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$,

For all even α; $C(v_{\alpha\beta}) = \begin{cases} 2: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$

When;$C(v_{\alpha\beta}) = \begin{cases} 3: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$.

Modular coloring of $ST_{dc}(W_\mu)$ is

$\mathcal{S}(c_0) = 0,\ \mathcal{S}(v_\alpha) = \begin{cases} 1: & 1 \leq \alpha \leq \mu - 1, \text{where } \alpha \text{ is odd} \\ 2: & 1 \leq \alpha \leq \mu - 1, \text{where } \alpha \text{ is even} \\ 3: & \alpha = \mu \end{cases}$

When α ranges from 1 to$\mu - 1$ and β ranges from 1 to 6

For all odd α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 1: & \text{otherwise} \end{cases}$, for all even α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: \beta = 1,2,3 \\ 2: \text{otherwise} \end{cases}$.

When $\alpha = \mu$;$\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: \beta = 1,2,3 \\ 3: \text{otherwise} \end{cases}$.

Case 2: When μ is even,

Coloring of $ST_{dc}(W_\mu)$ is

$C(c_0) = 0, C(v_\alpha) = 0$ for all α.

When α ranges from 1 toμ and β ranges from 1 to 6

For all odd α; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$, for all even α; $C(v_{\alpha\beta}) = \begin{cases} 2: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$.

Modular coloring of $ST_{dc}(W_\mu)$ is

$\mathcal{S}(c_0) = 0,\ \mathcal{S}(v_\alpha) = \begin{cases} 1: \text{for all odd } \alpha \\ 2: \text{for all odd } \alpha \end{cases}$

When α ranges from 1 to$\mu - 1$ and β ranges from 1 to 6

For all odd α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 1: & \text{otherwise} \end{cases}$, for all even α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: \beta = 1,2,3 \\ 2: \text{otherwise} \end{cases}$.

From the above cases the graph $ST_{dc}(W_\mu)$ is 4 modular colorable, therefore the modular coloring of$m_c\left(ST_{dc}(W_\mu)\right) = 4$.

Theorem 2. For any integer $\mu \geq 3$,the modular chromatic number of a snow graph of gear graph $mc(ST_{dc}(G_\mu)) = 2$.

Proof: The structure of the snow graph of G_μ, let the graph G_μ be a gear graph with μ vertices then take μ copies of dc, each external vertex of G_μ is affixed by one dc, let $V\left(ST_{dc}(G_\mu)\right) = c_0 \cup v_\alpha \cup v_{\alpha\beta}$where $1 \leq \alpha \leq \mu$ and $1 \leq \beta \leq 6$ and let $E\left(ST_{dc}(G_\mu)\right) = v'_\alpha: 1 \leq \alpha \leq 2\mu \cup v'_{\alpha\beta}: 1 \leq \alpha \leq \mu$ and $1 \leq \beta \leq 6$.

Coloring of $ST_{dc}(G_\mu)$is $C(c_0) = 0$, when α ranges from 1 toμ; $C(v_\alpha) = \begin{cases} 0: \text{for all odd } \alpha \\ 1: \text{for all even } \alpha \end{cases}$.

When α ranges from 1 to μ and β ranges from 1 to 6

For all odd α; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$, for all even α; $C(v_{\alpha\beta}) = 0$.

Modular coloring of $ST_{dc}(G_\mu)$ is

$\mathcal{S}(c_0) = 0,\ \mathcal{S}(v_\alpha) = \begin{cases} 1: \text{for all odd } \alpha \\ 0: \text{for all odd } \alpha \end{cases}$

When α ranges from 1 to$\mu - 1$ and β ranges from 1 to 6

For all odd α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 1: & \text{otherwise} \end{cases}$, for all even α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 1,2,3 \\ 0: & \text{otherwise} \end{cases}$.

From the above cases the graph $ST_{dc}(G_\mu)$ is modular 2- colorable, therefore the modular coloring of $m_c\left(ST_{dc}(G_\mu)\right) = 2$.

Theorem 3. For any integer $\mu \geq 3$,the modular chromatic number of a snow graph of friendship graphmc$(ST_{dc}(F_\mu)) = 3$.

Proof: The construction of the snow graph of F_μ is described as, letF_μ be a friendship graph onμ vertices then take μ copies of dc, each external vertex of F_μ is affixed by one dc, let $V\left(ST_{dc}(F_\mu)\right) = c_0 \cup v_\alpha \cup v_{\alpha\beta}$where$1 \leq \alpha \leq 2\mu$ and $1 \leq \beta \leq 6$ and let$E\left(ST_{dc}(F_\mu)\right) = c_0v_\alpha^{'}: 1 \leq \alpha \leq 2\mu \cup v_\alpha^{'}: 1 \leq \alpha \leq \mu \cup v_{\alpha\beta}^{'}: 1 \leq \alpha \leq \mu$ and $1 \leq \beta \leq 6$.

Coloring of $ST_{dc}(F_\mu)$ is, $C(c_0) = 0$, when α ranges from 1 to 2μ; $C(v_\alpha) = 0$.

When α ranges from 1 to 2μ and β ranges from 1 to 6,

For all odd α; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$, for all even α; $C(v_{\alpha\beta}) = \begin{cases} 2: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$.

Modular coloring of $ST_{dc}(F_\mu)$ is

$\mathcal{S}(c_0) = 0,\ \mathcal{S}(v_\alpha) = \begin{cases} 1: \text{for all odd } \alpha \\ 2: \text{for all even } \alpha \end{cases}$

When α ranges from 1 to 2μ and β ranges from 1 to 6

For all odd α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 1: & \text{otherwise} \end{cases}$, for all even α; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 2: & \text{otherwise} \end{cases}$.

The graph $ST_{dc}(F_\mu)$ is modular 3- colorable, therefore the modular coloring of $m_c\left(ST_{dc}(F_\mu)\right) = 3$.

Theorem 4. For any integer $\mu \geq 3$,the modular chromatic number of a snow graph of generalized fan graphmc$(ST_{dc}(GF_\mu)) = 3$.

Proof:A snow graph of a generalised fan graph $ST_{dc}(GF_\mu)$ is constructed by taking a generalised fan graph $GF_\mu = \overline{K_m} \oplus P_n$ and μ copies double claw graph (dc), since all the vertex of the generalized fan graph GF_μ is external vertices, each vertex of the GF_μ attached by the double claw graph. The vertex set of $ST_{dc}(GF_\mu)$ is defined as $v_\alpha \cup v_{\alpha\beta}$ such that $1 \leq \alpha \leq \mu = m + n$ and $1 \leq \beta \leq 6$. consider bi-vertex sets $V_1(GF_\mu)$ is a vertex set of K_m complement, and $V_2(GF_\mu)$ is a vertex set of a path graph P_n. The graph $ST_{dc}(GF_\mu)$ contains K_3 as an induced sub graph so, it is minimum 3 colors are necessary to color the graph, that implies $m_c(ST_{dc}(GF_\mu)) \leq 3$. The graph'scoloring is accomplished based on the vertex sets, like$C(V_1(GF_\mu)) = 1$and $C(V_2(GF_\mu)) = 0$. The dc graph affixed with the vertex set $V_1(GF_\mu)$ is colored in a way that; $C(v_{\alpha\beta}) = 0$ for all β. For the dc graph that is attached with $V_2(GF_\mu)$, the coloring is given by, when α is odd ; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 1 \text{ to } 3 \\ 0: & \text{otherwise} \end{cases}$, when α is even; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$. It results the modular coloring is $\mathcal{S}(V_1(GF_\mu)) = 0$ and when α ranges from 1 to n$\mathcal{S}(V_2(GF_\mu)) = \begin{cases} 1: & \text{when } \alpha \text{ is odd} \\ 2: & \text{when } \alpha \text{ is even} \end{cases}$. The modular chromatic number of $v_{\alpha\beta}$ that is connected to $V_1(GF_\mu)$is $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 1 \text{ to } 3 \\ 0: & \text{otherwise} \end{cases}$, similarly which is connected to $V_2(GF_\mu)$ is $\mathcal{S}(GF_\mu) = \begin{cases} 0: & \beta = 1 \text{ to } 3 \\ 1: & \text{otherwise} \end{cases}$. This is vividly produce the result that the modular coloring is $m_c\left(ST_{dc}(GF_\mu)\right) = 3$.

Note: The previously mentioned theorem applies to the modular chromatic number of the snow graph of the fan graph.

Theorem 5. For any integer $\mu \geq 3$,the modular chromatic number of a snow graph of flower graphmc$\left(ST_{dc}(Fl_\mu)\right) = 3$ or 4

Proof: The construction of the snow graph of Fl_μ is described as, letFl_μ be a flower graph onμ vertices then take μ copies of dc, each external vertex of Fl_μ is affixed by one dc, let $V\left(ST_{dc}(Fl_\mu)\right) = v_\alpha \cup v_{\alpha\beta}$where $1 \leq \alpha \leq 2\mu$ and $1 \leq \beta \leq 6$ and let $E\left(ST_{dc}(Fl_\mu)\right) = c_0v_\alpha^{'}: 1 \leq \alpha \leq 2\mu \cup v_\alpha^{'}: 1 \leq \alpha \leq \mu \cup v_{\alpha\beta}^{'}: 1 \leq \alpha \leq \mu$ and $1 \leq \beta \leq 6$.

Coloring of $ST_{dc}(Fl_\mu)$ is,

$C(c_0) = 0$, when α ranges from 1 to 2μ; $C(v_\alpha) = 0$.

When α ranges from 1 to 2μ except $\alpha = 2\mu - 1$ and β ranges from 1 to 6,

For all odd $\alpha$where $\alpha \equiv 2,3 \pmod 4$; $C(v_{\alpha\beta}) = \begin{cases} 2: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$.
For all odd $\alpha$where $\equiv 0,1 \pmod 4$; $C(v_{\alpha\beta}) = \begin{cases} 1: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$.
When $\alpha = 2\mu - 1$; $C(v_{\alpha\beta}) = \begin{cases} 3: & \beta = 2 \\ 0: & \text{otherwise} \end{cases}$.
The modular coloring of $ST_{dc}(Fl_\mu)$ is,
is $\mathcal{S}(c_0) = 0$,
whenα ranges from 1 to 2μ except $2\mu - 1$,
if$\alpha \equiv 0,1 (\text{mod } 4)$; $\mathcal{S}(v_\alpha) = 1$,
if$\alpha \equiv 2,3 (\text{mod } 4)$; $\mathcal{S}(v_\alpha) = 2$,
if$\alpha = \mu - 1$; $\mathcal{S}(v_\alpha) = 3$.
When α ranges from 1 to 2μ except $2\mu - 1$ and β ranges from 1 to 6,
For all α where $\alpha \equiv 0,1 \pmod 4$; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 1: & \text{otherwise} \end{cases}$,
For all α where $\alpha \equiv 2,3 \pmod 4$; $\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 2: & \text{otherwise} \end{cases}$,
When $\alpha = \mu - 1$;$\mathcal{S}(v_{\alpha\beta}) = \begin{cases} 0: & \beta = 1,2,3 \\ 3: & \text{otherwise} \end{cases}$.
From the above cases the graph $ST_{dc}(Fl_\mu)$ is modular 3- colorable, therefore the modular coloring of $m_c\left(ST_{dc}(Fl_\mu)\right) = 3$.

Extended snow graph

Theorem 6. If $G = W_\mu$ and H is any$\gamma -$ ary graph, then $ST_\gamma(W_\mu)$ is modular 3 or 4 - colorable.
Proof: Let H be a γ-ary graph with γ- children and G be a graph of W_μ with μ vertices. Then, since every vertex in the wheel graph $W_\mu$is an external vertex, the extended snow graph of wheelW_μ is created by taking μ copies of the H = γ-ary graph and attaching them to the μ vertices of the wheel graph by an edge. The $V[ST_\gamma(W_\mu)] = \{v_o \cup v_\alpha \cup v_\beta^{h\alpha} \ni \alpha \to 1 \text{ to}\mu; h \to 1 \text{ to} n \text{and} \beta \to 1 \text{ to} \sum_{i=1}^{h-1} \gamma^i \;\; \forall\, \alpha, \gamma, n, i \in \mathbb{N}\}$ and the $E[ST_\gamma(W_\mu)] = \{e_\alpha \cup v'_\alpha \cup v''_\alpha \cup e_\beta^{h\alpha} \ni \alpha \to 1 \text{ to}\mu; h \to 1 \text{ to} n \text{and} \beta \to 1 \text{ to} \sum_{i=1}^{h} \gamma^i \;\; \forall\, \mu, \gamma \in \mathbb{N}\}$.The following two cases deals the theorem. The modular coloring of $ST_\gamma(W_\mu)$ is defined by an injective mapping $C(C_\mu)$: $v_\alpha \cup v_{\alpha\beta} \to \mathbb{Z}_k$; $k \geq 2$.
Case 1: When μ is odd,
The coloring of $ST_\gamma(W_\mu)$ is,
$C(v_0) = 0$, $C(v_\alpha) = 0$ for all α.
The modular coloring of the above vertices is
$\mathcal{S}(c_0) = 0$, $\mathcal{S}(v_\alpha) = \begin{cases} 1 & \text{if } 1 \leq \alpha \leq \mu - 1, \text{where } \alpha \text{ is odd} \\ 2 & \text{if } 1 \leq \alpha \leq \mu, \text{where } \alpha \text{ is even} \\ 3 & \text{if } \alpha = \mu, \text{where } \mu \text{ is odd} \end{cases}$
Coloring of the γ- aray graph of$ST_\gamma(W_\mu)$ is partitioned into three subcases as follows
Subcase 1: When $\gamma \equiv 0 (\text{mod } 4)$
When α ranges from 1 to $\mu - 1$ and for all β
For all odd α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \leq h \leq n, \text{where h is even} \\ 1 & \text{if } 1 \leq h \leq n, \text{where h is odd} \end{cases}$'
For all even α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \leq h \leq n, \text{where h is even} \\ 1 & \text{if } 2 \leq h \leq n, \text{where h is odd} \\ 2 & \text{if } h = 1 \end{cases}$,
When $\alpha = \mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \leq h \leq n, \text{where h is even} \\ 1 & \text{if } 2 \leq h \leq n, \text{where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$
Modular coloring of $ST_\gamma(W_\mu)$ is
When α ranges from 1 to $\mu - 1$ and for all β
For all odd α; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \leq h \leq n, \text{where h is odd} \\ 1 & \text{if } 1 \leq h \leq n, \text{where h is even} \end{cases}$'
For all even α; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \leq h \leq n, \text{where h is odd} \\ 1 & \text{if } 4 \leq h \leq n, \text{where h is even.} \\ 2 & \text{if } h = 2 \end{cases}$
When $\alpha = \mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \leq h \leq n, \text{where h is odd} \\ 1 & \text{if } \leq h \leq n, \text{where h is even.} \\ 3 & \text{if } h = 2 \end{cases}$

Subcase 2: When $\gamma \equiv 1 (\mathrm{mod}\ 4)$

When α ranges from 1 to $\mu - 1$ and for all β

For all odd α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 1 \le h \le n, \text{where h is odd} \end{cases}$,

For all even α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 2 \le h \le n, \text{where h is odd} \\ 2 & \text{if } h = 1 \end{cases}$,

When $\alpha = \mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 2 \le h \le n, h \equiv 1(\mathrm{mod}\ 4) \text{where h is odd} \\ 2 & \text{if } 2 \le h \le n, h \equiv 3(\mathrm{mod}\ 4) \text{where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to $\mu - 1$ and for all β

For all odd α; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 2 & \text{if } 1 \le h \le n, \text{where h is even} \end{cases}$,

For all even α; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 2 & \text{if } 4 \le h \le n, \text{where h is even} \\ 3 & \text{if } h = 2 \end{cases}$.

When $\alpha = \mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 1 & h = 2, n; n \equiv 2(\mathrm{mod}\ 4) \text{ where h is even} \\ 2 & h = n, n \equiv 0(\mathrm{mod}\ 4), \text{ where h is even} \\ 3 & \text{if } 4 \le h \le n - 1, \text{where h is even} \end{cases}$

Subcase 3: When $\gamma \equiv 2 (\mathrm{mod}\ 4)$

when α ranges from 1 to $\mu - 1$ and for all β

For all odd α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 1 \le h \le n, \text{where h is odd} \end{cases}$,

For all even α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 2 & \text{if } 1 \le h \le n, \text{where h is odd} \end{cases}$,

When $\alpha = \mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 2 \le h \le n, \text{where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$.

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to $\mu - 1$ and for all β

For all odd α; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 3 & \text{if } 1 \le h \le n, \text{where h is even} \end{cases}$,

For all even α; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 2 & \text{if } 1 \le h \le n, \text{where h is even} \end{cases}$.

When $\alpha = \mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n - 1, \text{where h is odd} \\ 1 & \text{if} h = 2 \text{ and } h = n, \text{h is even} \\ 3 & \text{if } 4 \le h \le n, \text{where h is even} \end{cases}$.

Subcase 4: When $\gamma \equiv 3 (\mathrm{mod}\ 4)$

When α ranges from 1 to $\mu - 1$ and for all β

For all odd α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n - 1, \text{where h is even} \\ 1 & \text{if } h \equiv 1(\mathrm{mod}\ 4), \text{ where h is odd} \\ 2 & \text{if } h \equiv 3(\mathrm{mod}\ 4), \text{ where h is odd} \end{cases}$,

For all even α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n - 1, \text{where h is even} \\ 1 & \text{if } h \equiv 3(\mathrm{mod}\ 4), \text{ where h is odd} \\ 2 & \text{if } h \equiv 1(\mathrm{mod}\ 4), \text{ where h is odd} \end{cases}$,

When $\alpha = \mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 1 \le h \le n; h \equiv 3(\mathrm{mod}\ 4), \text{ where h is odd} \\ 2 & \text{if } 5 \le h \le n; h \equiv 1(\mathrm{mod}\ 4), \text{ where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$.

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to $\mu - 1$ and for all β

For all odd α;

$$\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 1 & \text{if } 1 \le h \le n - 1; h \equiv 0(\mathrm{mod}\ 4), \text{ and } h = n, h \equiv 2(\mathrm{mod}\ 4) \\ 2 & \text{if } h = n, h \equiv 0(\mathrm{mod}\ 4) \\ 3 & \text{if } 1 \le h \le n - 1; h \equiv 2(\mathrm{mod}\ 4), \text{ where h is even} \end{cases},$$

For all even α;

$$\mathcal{S}\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{and } h=n; h\equiv 0(\text{mod } 4)\\ 2 & \text{if } h=n; h\equiv 2(\text{mod } 4)\\ 3 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{wheren h is even}\end{cases}.$$

When $\alpha=\mu$;

$$\mathcal{S}\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{and } h=n; h\equiv 0(\text{mod } 4)\\ 2 & \text{if } h=2 \text{ and } h=n, h\equiv 2(\text{mod } 4)\\ 3 & \text{if } 1\le h\le n-1; h\equiv 0(\text{mod } 4), \text{ where h is even}\end{cases}.$$

Thus from the above results $ST_\gamma(W_\mu)$ is 4 modular colorable for μ- is odd, therefore the modular coloring of$m_c\left(ST_\gamma(W_\mu)\right)=4$.

Case 2: When μ is even,

$C(v_0)=0, C(v_\alpha)=0$ for all α.

The modular coloring of the above vertices are

$\mathcal{S}(c_0)=0,\ \mathcal{S}(v_\alpha)=\begin{cases}1 & \text{if } 1\le\alpha\le\mu, \text{where } \alpha \text{ is odd}\\ 2 & \text{if } 1\le\alpha\le\mu, \text{where } \alpha \text{ is even}\end{cases}$

Coloring of $ST_\gamma(W_\mu)$ is splitted as three subcases as follows

Subcase 1: When $\gamma\equiv 0(\text{mod } 3)$

When α ranges from 1 to μ and for all β

For all odd α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } 1\le h\le n, \text{where h is odd}\end{cases}'$

For all even α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } 2\le h\le n, \text{where h is odd}\\ 2 & \text{if } h=1\end{cases},$

Modular coloring of $ST_\gamma(W_\mu)$ is

$\mathcal{S}(c_0)=0,\ \mathcal{S}(v_\alpha)=\begin{cases}1 & \text{if } 1\le h\le n, \text{where h is odd}\\ 2 & \text{if } 1\le h\le n, \text{where h is even}\end{cases}'$

When α ranges from 1 to μ and for all β

For all odd α; $\mathcal{S}\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n, \text{where h is even}\end{cases}'$

For all even α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 4\le h\le n, \text{where h is even}\\ 2 & \text{if } h=2\end{cases}.$

Subcase 2: When $\gamma\equiv 1(\text{mod } 3)$

When α ranges from 1 to μ and for all β

For all odd α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } 1\le h\le n, \text{where h is odd}\end{cases}'$

For all even α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 2 & \text{if } 1\le h\le n, \text{where h is odd}\end{cases}'$

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to μ and for all β

For all odd α; $\mathcal{S}\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } h=n, \text{where n is even}\\ 2 & \text{if } 1\le h\le n-1, \text{where h is even}\end{cases},$

For all even α; $\mathcal{S}\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n-1, \text{where h is even}\\ 2 & \text{if } h=n, \text{where n is even}\end{cases}.$

Subcase 3: When $\gamma\equiv 2(\text{mod } 3)$

When α ranges from 1 to μ and for all β

For all odd α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } h\equiv 1(\text{mod4}), \text{where n is odd},\\ 2 & \text{if } h\equiv 3(\text{mod4}), \text{where h is odd}\end{cases}$

For all even α; $C\left(v_\beta^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } h\equiv 3(\text{mod4}), \text{where n is odd},\\ 2 & \text{if } h\equiv 1(\text{mod4}), \text{where h is odd}\end{cases}$

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to μ and for all β

For all odd α; $\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \begin{matrix}\text{if } 1\le h\le n;\ h\equiv 0(\text{mod}4), \text{where n is even}\\ \text{and } h=n, \text{where } n\equiv 2(\text{mod}4)\end{matrix}\\ 2 & \begin{matrix}\text{if } 1\le h\le n\ h\equiv 2(\text{mod}4), \text{where h is even}\\ \text{and } h=n, \text{where } n\equiv 0(\text{mod}4)\end{matrix}\end{cases}$,

For all even α; $C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \begin{matrix}\text{if } 1\le h\le n;\ h\equiv 2(\text{mod}4), \text{where n is even}\\ \text{and } h=n, \text{where } n\equiv 0(\text{mod}4)\end{matrix}\\ 2 & \begin{matrix}\text{if } 1\le h\le n\ h\equiv 0(\text{mod}4), \text{where h is even}\\ \text{and } h=n, \text{where } n\equiv 2(\text{mod}4)\end{matrix}\end{cases}$.

From the above cases the graph $ST_{\gamma}(W_{\mu})$ is 3 modular colorable for μ- is even, therefore the modular coloring of $m_c\left(ST_{\gamma}(W_{\mu})\right)=3$.

Theorem 7. If $G=G_{\mu}$ and H is any$\gamma-$ary graph, then $ST_{\gamma}(G_{\mu})$ is modular 2 - colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of G_{μ} with μ vertices. Then, since every vertex in the gear graph $W_{\mu}$is an external vertex, the extended snow graph of gearG_{μ} is created by taking μ copies of the H = γ-ary graph and attaching them to the μ vertices of the gear graph by an edge. The $V[ST_{\gamma}(G_{\mu})]=\{v_o\cup v_{\alpha}\cup v_{\beta}^{h\alpha}\ni \alpha\to 1 \text{ to } 2\mu; h\to 1 \text{ to} n \text{and} \beta\to 1 \text{ to} \sum_{i=1}^{h-1}\gamma^i\ \forall\, \alpha,\gamma,n,i\in\mathbb{N}\}$ and the $E[ST_{\gamma}(G_{\mu})]=\{e_{\alpha}\cup v'_{\alpha}\cup v''_{\frac{\alpha}{2}}\cup e_{\beta}^{h\alpha}\ \ni \alpha\to 1 \text{ to } 2\mu; h\to 1 \text{ to} n \text{and} \beta\to 1 \text{ to} \sum_{i=1}^{h}\gamma^i\ \forall\, \mu,\gamma\in\mathbb{N}\}$.The following two cases deals the theorem. The modular coloring of $ST_{\gamma}(G_{\mu})$ is defined by an injective mapping$C(C_{\mu})$: $V\to\mathbb{Z}_k$; $k\ge 2$.The coloring of $ST_{\gamma}(G_{\mu})$ is,

Case 1: When $\mu\ge 3$, γ is odd

Subcase 1:$h\equiv 0 \pmod 4$

The coloring of $ST_{\gamma}(G_{\mu})$ is given by

$C(v_0)=1, C(v_{\alpha})=0\ \forall\alpha,$

When α ranges from 1 to 2μ, for all β, and $1\le h\le n$

For each odd α; $C\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if h is even and } h\equiv 1 \ (\text{mod } 4); \text{where h is odd}\\ 1 & \text{if } h\equiv 3(\text{mod } 4); \text{where h is odd}\end{cases}$,

For each even α; $C\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if h is odd and } h\equiv 0 \ (\text{mod } 4); \text{where h is even}\\ 1 & \text{if } h\equiv 2 \ (\text{mod } 4); \text{where h is even}\end{cases}$.

The modular coloring is

$\mathcal{S}(v_0)=0, \mathcal{S}(v_{\alpha})=\begin{cases}0 & \text{if } \alpha \text{ is even}\\ 1 & \text{if } \alpha \text{ is odd}\end{cases}\ \forall\alpha,$

When α ranges from 1 to 2μ, for all β, and $1\le h\le n$

For each odd α; $\mathcal{S}\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if } \alpha \text{ is odd}\\ 1 & \text{if } \alpha \text{ is even}\end{cases}$,

For each even α; $\mathcal{S}\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if } \alpha \text{ is even}\\ 1 & \text{if } \alpha \text{ is odd}\end{cases}$.

Subcase 2: Except$h\equiv 0 \pmod 4$

The coloring of $ST_{\gamma}(G_{\mu})$ is given by

$C(v_0)=0, C(v_{\alpha})=\begin{cases}0 & \text{if } \alpha \text{ is even}\\ 1 & \text{if } \alpha \text{ is odd}\end{cases}$,

When α ranges from 1 to 2μ, for all β, and $1\le h\le n$

For each odd α; $C\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if } h\equiv 1 \ (\text{mod } 4); \text{where h is odd}\\ 1 & \text{if h is even and } h\equiv 3(\text{mod } 4); \text{where h is odd}\end{cases}$,

For each even α; $C\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if h is odd and } h\equiv 2 \ (\text{mod } 4); \text{where h is even}\\ 1 & \text{if } h\equiv 0(\text{mod } 4); \text{where h is even}\end{cases}$.

The above applications reflect themodular coloring is

$\mathcal{S}(v_0)=0, \mathcal{S}(v_{\alpha})=\begin{cases}0 & \text{if } \alpha \text{ is odd}\\ 1 & \text{if } \alpha \text{ is even}\end{cases}$,

When α ranges from 1 to 2μ, for all β, and $1\le h\le n$

For each odd α; $\mathcal{S}\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if h is odd}\\ 1 & \text{if h is even}\end{cases}$,

For each even α; $\mathcal{S}\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if h is even}\\ 1 & \text{if h is odd}\end{cases}$

Case 2: When $\mu\ge 3$, γ is even

The coloring of $ST_{\gamma}(G_{\mu})$ is given by

$C(v_0)=0, C(v_{\alpha})=\begin{cases}0 & \text{if } \alpha \text{ is even}\\ 1 & \text{if } \alpha \text{ is odd}\end{cases}$,

When α ranges from 1 to 2μ, for all β, and $1 \le h \le n$
For each odd α; $C(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if h is odd} \\ 1 & \text{if h is even} \end{cases}$,
For each even α; $C(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if h is even} \\ 1 & \text{if h is odd} \end{cases}$.
The above applications reflect the modular coloring is
$\mathcal{S}(v_0) = 0, \mathcal{S}(v_\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is even} \\ 1 & \text{if } \alpha \text{ is odd} \end{cases}$,
When α ranges from 1 to 2μ, for all β, and $1 \le h \le n$
For each odd α; $\mathcal{S}(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if h is odd} \\ 1 & \text{if h is even} \end{cases}$,
For each even α; $\mathcal{S}(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if h is even} \\ 1 & \text{if h is odd} \end{cases}$.
Thus the above case is leads to $ST_\gamma(G_\mu)$ is 3 modular colorable, therefore the modular coloring of $m_c\left(ST_\gamma(G_\mu)\right) = 3$.

Theorem 8. If $G = F_\mu$ and H is any$\gamma -$ ary graph, then $ST_\gamma(F_\mu)$ is modular 3 - colorable.
Proof: Let H be a γ-ary graph with γ- children and G be a graph of F_μ with μ vertices. Then, since every vertex in the friendship graph $F_\mu$is an external vertex except the center vertex, the extended snow graph of friendship graphF_μ is created by taking μ copies of the H = γ-ary graph and attaching them to the μ vertices of the friendship graph by an edge. The $V[ST_\gamma(F_\mu)] = \{v_o \cup v_\alpha \cup v_\beta^{h\alpha} \ni \alpha \to 1 \text{ to } 2\mu; h \to 1 \text{ to } n \text{ and } \beta \to 1 \text{ to } \sum_{i=1}^{h-1} \gamma^i \ \forall \alpha, \gamma, n, i \in \mathbb{N}\}$and the $E[ST_\gamma(F_\mu)] = \{e_{2\alpha} \cup v_\alpha v_{\alpha+1} \forall \text{ odd } \alpha \cup e_\beta^{h\alpha} \ni \alpha \to 1 \text{ to } \mu; h \to 1 \text{ to } n \text{ and } \beta \to 1 \text{ to } \sum_{i=1}^{h} \gamma^i \ \forall \mu, \gamma \in \mathbb{N}\}$.The following three cases deals the theorem. The modular coloring of $ST_\gamma(F_\mu)$ is defined by an injective mapping$C: V \to \mathbb{Z}_k; k \ge 2$.The coloring of $ST_\gamma(F_\mu)$ is,
$C(v_\alpha) = 0$; whereα ranges from 0 to 2μ. The coloring patterns and the modular coloring of the graph$ST_\gamma(F_\mu)$ are followed from the case 2 of theorem 6.
The graph $ST_\gamma(F_\mu)$ is 3 modular colorable, therefore the modular coloring of $m_c\left(ST_\gamma(F_\mu)\right) = 3$.

Theorem 9. If $G = \bar{k}_m \oplus P_n$ and H is any$\gamma -$ ary graph, then $ST_\gamma(\bar{k}_m \oplus P_n)$ is modular 3 - colorable.
Proof: Let H be a γ-ary graph with γ- children and G be a graph of $\bar{k}_m \oplus P_l$ with μ vertices. Then, since every vertex in the generalized fan graph $\bar{k}_m \oplus P_l$is an external vertex, the extended snow graph of generalized fan graph$\bar{k}_m \oplus P_l$ is created by taking $\mu = m + l$ copies of the H = γ-ary graph and attaching them to the μ vertices of the generalized fan graph by an edge.The $V[ST_\gamma(\bar{k}_m \oplus P_l)] = \{u_\alpha \ni \alpha \to 1 \text{ to } m \cup v_\alpha \ni \alpha \to 1 \text{ to } l \cup v_\beta^{h\alpha} \ni h \to 1 \text{ to } n \text{ and } \beta \to 1 \text{ to } \sum_{i=1}^{h-1} \gamma^i \ \forall \alpha, \gamma, n, i \in \mathbb{N}\}$and the $E[ST_\gamma(\bar{k}_m \oplus P_n)] = \{u'_\alpha \ni \alpha \to 1 \text{ to } m \cup v'_\alpha \ni \alpha \to 1 \text{ to } l \cup e_\beta^{h\alpha} \ni h \to 1 \text{ to } n \text{ and } \beta \to 1 \text{ to } \sum_{i=1}^{h} \gamma^i \ \forall \mu, \gamma \in \mathbb{N}\}$.The following three cases deals the theorem. The modular coloring of $ST_\gamma(\bar{k}_m \oplus P_l)$ is defined by an injective mapping$C: V \to \mathbb{Z}_k; k \ge 2$ .the coloring of$ST_\gamma(\bar{k}_m \oplus P_l)$ is given by

$$C(u_\alpha) = 0 \ \forall \ \alpha \text{ rangeas from 1 to m}$$
$$C(v_\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is odd} \\ 2 & \text{if } \alpha \text{ is even} \end{cases}$$

Case 1: when $\gamma \equiv 0 \pmod 3$
When α ranges from 1 to m, $1 \le h \le n$, and $1 \le \beta \le \sum_{i=1}^{h} \gamma^i$

$$C(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if } 1 \le h \le n; \text{h is odd} \\ 1 & \text{if } 4 \le h \le n; \text{h is even} \\ 2 & \text{if } h = 2 \text{ and } \beta = 1 \end{cases}$$

the modular coloring is

$$\mathcal{S}(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if } 1 \le h \le n; \text{h is even} \\ 1 & \text{if } 1 \le h \le n; \text{h is odd, except } 1 \le \beta \le \gamma \text{ in } h = 3 \\ 2 & \text{if } 1 \le \beta \le \gamma \text{ in } h = 3 \end{cases}$$

When α ranges from 1 to l, $1 \le h \le n$, and $1 \le \beta \le \sum_{i=1}^{h} \gamma^i$
The coloring of the vertices of $ST_\gamma(\bar{k}_m \oplus P_l)$ is followed from subcase 1 of case 2 in the theorem 6.
Case 2: when $\gamma \equiv 1 \pmod 3$
When α ranges from1 to m, $1 \le h \le n$, and $1 \le \beta \le \sum_{i=1}^{h} \gamma^i$

$$C(v_\beta^{h\alpha}) = \begin{cases} 0 & \text{if h is odd} \\ 1 & \text{if h is even} \end{cases}$$

the modular coloring is

$$\mathcal{S}\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if h is even}\\ 1 & \text{if h is odd}\end{cases}$$

When α ranges from 1 to l, $1 \le h \le n$, and $1 \le \beta \le \sum_{i=1}^{h}\gamma^{i}$

Coloring of the vertices $v_{\beta}^{h\alpha}$ is described in the same way of coloring of subcase 1 of case 2 in the theorem 6.

Case 3: when $\gamma \equiv 2(\mathrm{mod}\ 3)$

When α ranges from 1 to m, $1 \le h \le n$, and $1 \le \beta \le \sum_{i=1}^{h}\gamma^{i}$

For $1 \le h \le n$; where h is odd, $C\left(v_{\beta}^{h\alpha}\right)=0$,

For $h=2$; $C\left(v_{\beta}^{h\alpha}\right)=1$,

For $h \ge 4$; where h is even, the subsequent of the vertex(h) of the vertex $(h-2)$ which is colored by 1 then the coloring is given by $C\left(v_{\beta}^{h\alpha}\right)=\begin{cases}2 & h \equiv 1(\mathrm{mod}\ \gamma)\\ 1 & \text{otherwise}\end{cases}$, that is colored by 2 then the coloring is given by $C\left(v_{\beta}^{h\alpha}\right)=1$.

The modular coloring is

$$C\left(v_{\beta}^{h\alpha}\right)=\begin{cases}0 & \text{if } 1 \le h \le n; \text{where h is even}\\ 1 & \text{if } 3 \le h \le n, ; \text{where h is odd}\\ 2 & \text{if} h=1\end{cases}$$

When α ranges from 1 to l, $1 \le h \le n$, and $1 \le \beta \le \sum_{i=1}^{h}\gamma^{i}$.

Coloring of $C\left(v_{\beta}^{h\alpha}\right)$ described in the same way of coloring of subcase 2 of case 2 in the theorem 6.

The above results that, the graph $ST_{\gamma}(\bar{k}_m \oplus P_l)$ is modular -3 colorable, thus the $m_c(ST_{\gamma}(\bar{k}_m \oplus P_l))$ =3.

Theorem 10. If $G = Fl_{\mu}$ and H is any$\gamma-$ ary graph, then $ST_{\gamma}\left(Fl_{\mu}\right)$ is modular 3 - colorable.

Proof: Let H be a γ-ary graph with γ- children and G be a graph of Fl_{μ} with μ vertices. Then, since every vertex in the flower graph $Fl_{\mu}$is an external vertex except the center vertex, the extended snow graph of flower graphFl_{μ} is created by taking μ copies of the H = γ-ary graph and attaching them to the μ vertices of the flower graph by an edge. The $V\left[ST_{\gamma}\left(Fl_{\mu}\right)\right]=\{v_o \cup v_{\alpha} \cup v_{\beta}^{h\alpha} \ni \alpha \to 1 \text{ to } 2\mu; h \to 1 \text{ to} n \text{and} \beta \to 1 \text{ to} \sum_{i=1}^{h-1}\gamma^{i}\ \ \forall\, \alpha,\gamma,n,i \in \mathbb{N}\}$and the $E\left[ST_{\gamma}\left(Fl_{\mu}\right)\right]=\{e_{2\alpha} \cup v_{\alpha}v_{\alpha+1} \forall \text{ odd } \alpha \cup\ e_{\beta}^{h\alpha}\ \ni \alpha \to 1 \text{ to}\mu; h \to 1 \text{ to} n \text{and} \beta \to 1 \text{ to} \sum_{i=1}^{h}\gamma^{i}\ \ \forall\, \mu,\gamma \in \mathbb{N}\}$.The following three cases deals the theorem. The modular coloring of $ST_{\gamma}(Fl_{\mu})$ is defined by an injective mapping$C\colon\ V \to \mathbb{Z}_k;\ k \ge 2$.

The coloring of $ST_{\gamma}(W_{\mu})$ is,

Case 1: When μ is odd,

$C(v_0)=0$, $C(v_{\alpha})=0$ for all α ranges from 1 to 2μ .

The modular coloring of the above vertices is

$$\mathcal{S}(c_0)=0,\ \mathcal{S}(v_{\alpha})=\begin{cases}1 & \text{if } 1 \le \alpha \le \mu-1 \text{ and } \mu+1 \le \alpha \le 2\mu, \text{where } \alpha \text{ is odd;}\\ & \alpha=2\mu, \text{where } \mu \text{ is even}\\ 2 & \text{if } 1 \le \alpha \le \mu-1, \text{and } \mu+1 \le \alpha \le 2\mu; \text{where } \alpha \text{ is even}\\ 3 & \text{if } \alpha=\mu, \text{where } \mu \text{ is odd}\end{cases}$$

Coloring of the γ- aray graph of$ST_{\gamma}\left(W_{\mu}\right)$ is partitioned into three subcases as follows

Subcase 1: When $\gamma \equiv 0(\mathrm{mod}\ 4)$

When α ranges from 1 to 2μ except μ and for all β

For all odd $\alpha$and $\alpha=2\mu$, α is even; $C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1 \le h \le n, \text{where h is even}\\ 1 & \text{if } 1 \le h \le n, \text{where h is odd}\end{cases}$'

For all even αexcept $\alpha=2\mu$;$C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1 \le h \le n, \text{where h is even}\\ 1 & \text{if } 2 \le h \le n, \text{where h is odd}\\ 2 & \text{if } h=1\end{cases}$,

When $\alpha=\mu$; $C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1 \le h \le n, \text{where h is even}\\ 1 & \text{if } 2 \le h \le n, \text{where h is odd}\\ 3 & \text{if } h=1\end{cases}$

Modular coloring of $ST_{\gamma}\left(W_{\mu}\right)$ is

whenα ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha=2\mu$, α is even; $\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1 \le h \le n, \text{where h is odd}\\ 1 & \text{if } 1 \le h \le n, \text{where h is even}\end{cases}$'

For all even αexcept $\alpha=2\mu$; $\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1 \le h \le n, \text{where h is odd}\\ 1 & \text{if } 4 \le h \le n, \text{where h is even.}\\ 2 & \text{if } h=2\end{cases}$

When $\alpha=\mu$; $\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1 \le h \le n, \text{where h is odd}\\ 1 & \text{if } \le h \le n, \text{where h is even.}\\ 3 & \text{if } h=2\end{cases}$

Subcase 2: When $\gamma \equiv 1 (\mathrm{mod}\ 4)$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha = 2\mu$, α is even; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 1 \le h \le n, \text{where h is odd} \end{cases}$,

For all even α except $\alpha = 2\mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 2 \le h \le n, \text{where h is odd} \\ 2 & \text{if } h = 1 \end{cases}$,

When $\alpha = \mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 2 \le h \le n, h \equiv 1(\mathrm{mod}\ 4) \text{where h is odd} \\ 2 & \text{if } 2 \le h \le n, h \equiv 3(\mathrm{mod}\ 4) \text{where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha = 2\mu$, α is even; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 2 & \text{if } 1 \le h \le n, \text{where h is even} \end{cases}$,

For all even α except $= 2\mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 2 & \text{if } 4 \le h \le n, \text{where h is even} \\ 3 & \text{if } h = 2 \end{cases}$.

When $\alpha = \mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 1 & h = 2, n; n \equiv 2(\mathrm{mod}\ 4) \text{ where h is even} \\ 2 & h = n, n \equiv 0(\mathrm{mod}\ 4), \text{ where h is even} \\ 3 & \text{if } 4 \le h \le n-1, \text{where h is even} \end{cases}$

Subcase 3: When $\gamma \equiv 2(\mathrm{mod}\ 4)$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha = 2\mu$, α is even; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 1 \le h \le n, \text{where h is odd} \end{cases}$,

For all even α; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 2 & \text{if } 1 \le h \le n, \text{where h is odd} \end{cases}$,

For all even α except $\alpha = 2\mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 2 \le h \le n, \text{where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$.

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha = 2\mu$, α is even; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 3 & \text{if } 1 \le h \le n, \text{where h is even} \end{cases}$,

For all even α except $\alpha = 2\mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 2 & \text{if } 1 \le h \le n, \text{where h is even} \end{cases}$.

When $\alpha = \mu$; $\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n-1, \text{where h is odd} \\ 1 & \text{if } h = 2 \text{ and } h = n, \text{h is even} \\ 3 & \text{if } 4 \le h \le n, \text{where h is even} \end{cases}$.

Subcase 4: When $\gamma \equiv 3(\mathrm{mod}\ 4)$

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha = 2\mu$, α is even; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n-1, \text{where h is even} \\ 1 & \text{if } h \equiv 1(\mathrm{mod}\ 4), \text{ where h is odd} \\ 2 & \text{if } h \equiv 3(\mathrm{mod}\ 4), \text{ where h is odd} \end{cases}$,

For all even α except $\alpha = 2\mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n-1, \text{where h is even} \\ 1 & \text{if } h \equiv 3(\mathrm{mod}\ 4), \text{ where h is odd} \\ 2 & \text{if } h \equiv 1(\mathrm{mod}\ 4), \text{ where h is odd} \end{cases}$,

When $\alpha = \mu$; $C(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } 1 \le h \le n; h \equiv 3(\mathrm{mod}\ 4), \text{ where h is odd} \\ 2 & \text{if } 5 \le h \le n; h \equiv 1(\mathrm{mod}\ 4), \text{ where h is odd} \\ 3 & \text{if } h = 1 \end{cases}$.

Modular coloring of $ST_\gamma(W_\mu)$ is

When α ranges from 1 to 2μ except μ and for all β

For all odd α and $\alpha = 2\mu$, α is even;

$$\mathcal{S}(v_\beta^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 1 & \text{if } 1 \le h \le n-1; h \equiv 0(\mathrm{mod}\ 4), \text{ and } h = n, h \equiv 2(\mathrm{mod}\ 4) \\ 2 & \text{if } h = n, h \equiv 0(\mathrm{mod}\ 4) \\ 3 & \text{if } 1 \le h \le n-1; h \equiv 2(\mathrm{mod}\ 4), \text{ where h is even} \end{cases},$$

For all even α except $\alpha = 2\mu$;

$$\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{and } h=n; h\equiv 0(\text{mod } 4)\\ 2 & \text{if } h=n; h\equiv 2(\text{mod } 4)\\ 3 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{wheren h is even}\end{cases}.$$

When $\alpha=\mu$;

$$\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n-1; h\equiv 2(\text{mod } 4), \text{and } h=n; h\equiv 0(\text{mod } 4)\\ 2 & \text{if } h=2 \text{ and } h=n, h\equiv 2(\text{mod } 4)\\ 3 & \text{if } 1\le h\le n-1; h\equiv 0(\text{mod } 4), \text{ where h is even}\end{cases}.$$

Thus from the above results $ST_{\gamma}\left(W_{\mu}\right)$ is 4 modular colorable for μ- is odd, therefore the modular coloring of$m_c\left(ST_{\gamma}\left(W_{\mu}\right)\right)=4$.

Case 2: When μ is even,

$C(v_0)=0, C(v_\alpha)=0$ for all α ranges from 1 to 2μ.

The modular coloring of the above vertices are

$$\mathcal{S}(c_0)=0,\ \mathcal{S}(v_\alpha)=\begin{cases}1 & \text{if } 1\le\alpha\le\mu, \text{where } \alpha \text{ is odd}\\ 2 & \text{if } 1\le\alpha\le\mu, \text{where } \alpha \text{ is even}\end{cases}$$

Coloring of $ST_{\gamma}\left(W_{\mu}\right)$ is splitted as three subcases as follows

Subcase 1: When $\gamma\equiv 0(\text{mod } 3)$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and ;$\mu+1\le\alpha\le 2\mu$, where α is odd

$$C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } 1\le h\le n, \text{where h is odd}\end{cases}'$$

For all $1\le\alpha\le\mu$, where α is odd and ;$\mu+1\le\alpha\le 2\mu$, where α is even

$$C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } 2\le h\le n, \text{where h is odd}\\ 2 & \text{if } h=1\end{cases},$$

Modular coloring of $ST_{\gamma}\left(W_{\mu}\right)$ is

$$\mathcal{S}(c_0)=0,\ \mathcal{S}(v_\alpha)=\begin{cases}1 & \text{if } 1\le h\le n, \text{where h is odd}\\ 2 & \text{if } 1\le h\le n, \text{where h is even}\end{cases}$$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and ;$\mu+1\le\alpha\le 2\mu$, where α is odd

$$\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n, \text{where h is even}\end{cases}'$$

For all $1\le\alpha\le\mu$, where α is odd and ;$\mu+1\le\alpha\le 2\mu$, where α is even

$$C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 4\le h\le n, \text{where h is even.}\\ 2 & \text{if } h=2\end{cases}$$

Subcase 2: When $\gamma\equiv 1(\text{mod } 3)$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and ;$\mu+1\le\alpha\le 2\mu$, where α is odd

$$C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 1 & \text{if } 1\le h\le n, \text{where h is odd}\end{cases}'$$

For all $1\le\alpha\le\mu$, where α is odd and ;$\mu+1\le\alpha\le 2\mu$, where α is even

$$C\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is even}\\ 2 & \text{if } 1\le h\le n, \text{where h is odd}\end{cases}'$$

Modular coloring of $ST_{\gamma}\left(W_{\mu}\right)$ is

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and ;$\mu+1\le\alpha\le 2\mu$, where α is odd

$$\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } h=n, \text{where n is even}\\ 2 & \text{if } 1\le h\le n-1, \text{where h is even}\end{cases},$$

For all $1\le\alpha\le\mu$, where α is odd and ;$\mu+1\le\alpha\le 2\mu$, where α is even

$$\mathcal{S}\left(v_{\beta}^{\alpha h}\right)=\begin{cases}0 & \text{if } 1\le h\le n, \text{where h is odd}\\ 1 & \text{if } 1\le h\le n-1, \text{where h is even.}\\ 2 & \text{if } h=n, \text{where n is even}\end{cases}$$

Subcase 3: When $\gamma\equiv 2(\text{mod } 3)$

When α ranges from 1 to 2μ and for all β

For all $1\le\alpha\le\mu$, where α is even and ;$\mu+1\le\alpha\le 2\mu$, where α is odd

$$C(v_{\beta}^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } h \equiv 1(\text{mod}4), \text{where n is odd,} \\ 2 & \text{if } h \equiv 3(\text{mod}4), \text{where h is odd} \end{cases}$$

For all $1 \le \alpha \le \mu$, where α is odd and ;$\mu + 1 \le \alpha \le 2\mu$, where α is even

$$C(v_{\beta}^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is even} \\ 1 & \text{if } h \equiv 3(\text{mod}4), \text{where n is odd,} \\ 2 & \text{if } h \equiv 1(\text{mod}4), \text{where h is odd} \end{cases}$$

Modular coloring of $ST_{\gamma}(W_{\mu})$ is

When α ranges from 1 to μ and for all β

For all $1 \le \alpha \le \mu$, where α is even and ;$\mu + 1 \le \alpha \le 2\mu$, where α is odd

$$\mathcal{S}(v_{\beta}^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 1 & \begin{array}{c}\text{if } 1 \le h \le n;\ h \equiv 0(\text{mod}4), \text{where n is even} \\ \text{and } h = n, \text{where } n \equiv 2(\text{mod}4)\end{array} \\ 2 & \begin{array}{c}\text{if } 1 \le h \le n\ h \equiv 2(\text{mod}4), \text{where h is even} \\ \text{and } h = n, \text{where } n \equiv 0(\text{mod}4)\end{array} \end{cases},$$

For all $1 \le \alpha \le \mu$, where α is odd and ;$\mu + 1 \le \alpha \le 2\mu$, where α is even

$$C(v_{\beta}^{\alpha h}) = \begin{cases} 0 & \text{if } 1 \le h \le n, \text{where h is odd} \\ 1 & \begin{array}{c}\text{if } 1 \le h \le n;\ h \equiv 2(\text{mod}4), \text{where n is even} \\ \text{and } h = n, \text{where } n \equiv 0(\text{mod}4)\end{array} \\ 2 & \begin{array}{c}\text{if } 1 \le h \le n\ h \equiv 0(\text{mod}4), \text{where h is even} \\ \text{and } h = n, \text{where } n \equiv 2(\text{mod}4)\end{array} \end{cases}.$$

From the above cases the graph $ST_{\gamma}(W_{\mu})$ is 3 modular colorable for μ- is even, therefore the modular coloring of $m_c\left(ST_{\gamma}(W_{\mu})\right) = 3$.

CONCLUSION

In this paper, we developed two new graphs called snow graph and extended snow graph, and in addition to that, we examined the modular coloring for some famous graphs, such as snow graph and its extended snow graph of wheel, gear, friendship, generalized fan, and flower graph. It results in the graph being modular - K colorable, and the modular chromatic number is obtained. The application of the result is extended to the traffic signal of a busy road with huge passengers, and in the future we can develop many applications related to this work using various graph structures.

REFERENCES

[1] F. Okamoto, E. Salehi, P. Zhang. A Checkerboard problem and modular colorings of graph, Bull.Inst.combin.appl58(2010),29-47

[2] P. Sumathi, S. Tamilselvi, Modular chromatic number of certain cyclic graphs. Journal of algebraic statistics, Volume 13, No. 3, 2022, p. 4459-4466.

[3] P. Sumathi, S. Tamilselvi, Modular coloring on inflated graphs. Industrial engineering journal, ISSN 0970-2555 Volume 52, Issue 4,No. 2,april 2023.

[4] P. Sumathi, S. Tamilselvi, The modular chromatic number of the corona product of a generalized Jahangir graph. Indian journal of science and technology, Year: 2023, Volume: 16, Issue: 46, Pages: 4309-4327. DOI: 10.17485/IJST/v16i46.2028.

[5] N. Paramaguru, R. Sampathkumar, Modular Colorings Of Join Of Two Special Graphs. Electronic Journal of Graph Theory and Applications 2 (2) (2014), 139–149 https://dx.doi.org/10.5614/ejgta.2014.2.2.6

[6] T. Nicholas, G. R, Sanma, Modular Colorings Of Cycle Related Graphs, Global Journal of Pure and Applied Mathematics, 13 (7) (2017), 3779-3788 © Research India Publications http://www.ripublication.com

[7] R.Rajarajachozhan, R.Sampathkumar, Modular Coloring Of The Cartesian Products $K_m \square K_n$, $K_m \square C_n$, And $K_m \square P_n$, Discrete mathematics, Algorithms and Applications / Vol. 09, No. 06, 1750075 (2017) https://doi.org/10.1142/S1793830917500756

[8] T.Nicolas, Sanma, G.R Modular colorings of circular Halin graphs of level two, Asian journal of Mathematics and Computer Research 17(1): 48-55, 2017.

[9] Sanma. G. R, P. Maya Modular Coloring and Switching In Some Planar Graphs, Ratio Mathematica. Volume 45, 2023, DOI: http://dx.doi.org/10.23755/rm.v45i0.984