WEIGHTED DIFFERENTIATION SUPERPOSITION OPERATOR FROM H^∞ TO nth WEIGHTED-TYPE SPACE

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Abstract. Let $H(\mathbb{D})$ be the set of all analytic functions on the open unit disk $\mathbb{D}$ of $\mathbb{C}$, $u \in H(\mathbb{D})$ and ϕ an entire function on $\mathbb{C}$. In this paper, we characterize the boundedness and compactness of the weighted differentiation superposition operator $D_u^m S_\phi$ from H^∞ to the nth weighted-type space.

1. Introduction

Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$.

First, we present some of the most interesting linear operators studied on some subspaces of $H(\mathbb{D})$. Let $z \in \mathbb{D}$, then the multiplication operator with symbol $u \in H(\mathbb{D})$ is defined by $M_u(f)(z) = u(z)f(z)$, and composition operator with symbol $\varphi \in S(\mathbb{D})$ is defined by $C_\varphi(f)(z) = f(\varphi(z))$.

Let $m \in \mathbb{N}_0$ and $f \in H(\mathbb{D})$, then the mth differentiation operator is defined by

$$D^m f(z) = f^{(m)}(z), \quad z \in \mathbb{D}, \tag{1}$$

where $f^{(0)} = f$. If $m = 1$, then it is the standard differentiation operator D. In recent years, there has been a lot of interest in the study of products of differential operator and others. For example, products DC_φ and $C_\varphi D$, which are the most basic product-type operators involving the differentiation operator, have been studied, for example, in [1–9]. Many other results have evolved from them, for example, the following six operators were studied in [10]

$$DM_uC_\varphi,\ DC_\varphi M_u,\ C_\varphi DM_u,\ C_\varphi M_u D,\ M_u C_\varphi D, M_u DC_\varphi. \tag{2}$$

An operator, namely including all the operators in (2), was introduced and investigated in [11, 12]. In some studies, for example, Wang et al. in [13] generalized operators in (2) and studied the following operators

$$D^n M_u C_\varphi,\ D^n C_\varphi M_u,\ C_\varphi D^n M_u,\ C_\varphi M_u D^n,\ M_u C_\varphi D^n,\ M_u D^n C_\varphi. \tag{3}$$

Some other product-type operators on subspaces of $H(\mathbb{D})$ can be found (see, e.g., [14–17] and the related references therein).

Next, we introduce the superposition operator (see, for example, [18] or [19]). Let ϕ be a complex-valued function on $\mathbb{C}$. Then the superposition operator S_ϕ on $H(\mathbb{D})$ is defined as

$$S_\phi f = \phi(f(z)), \quad z \in \mathbb{D}.$$

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Assume that X and Y are two metric spaces of analytic functions on $\mathbb{D}$ and S_ϕ maps X into Y. Note that if X contain the linear functions, then ϕ must be an entire function. Recently, the boundedness and compactness of S_ϕ have been characterized on or between some analytic function spaces (see, for example, [19–26]).

The following weighted differentiation superposition operator, which is introduced in [27], is a class of nonlinear operators. Let $m \in \mathbb{N}_0$, $u \in H(\mathbb{D})$ and ϕ be an entire function on $\mathbb{C}$. The weighted differentiation superposition operator denoted as $D_u^m S_\phi$ on some subspaces of $H(\mathbb{D})$ is defined by

$$(D_u^m S_\phi f)(z) = u(z)\phi^{(m)}(f(z)), \quad z \in \mathbb{D}.$$

Our goal of this paper is to improve results of Kamal and Eissa in [27]. Here, we rethink the boundedness and compactness of this operator from H^∞ space to nth weighted-type space, which can be regarded as a continuation of our work (see, for example, [19]).

Now, we introduce the important Bell polynomial (see, for example, [13, 15]). Let $n, k \in \mathbb{N}_0$. Then the Bell polynomial is defined as

$$B_{n,k} := B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{\prod_{i=1}^{n-k-1} j_i!} \prod_{i=1}^{n-k-1} \left(\frac{x_i}{i!}\right)^{j_i}, \tag{4}$$

where the sum is taken over all non-negative integer sequences $j_1, j_2, \dots, j_{n-k+1}$ satisfying $\sum_{i=1}^{n-k+1} j_i = k$ and $\sum_{i=1}^{n-k+1} i j_i = n$. In particular, if $k = 0$, we have $B_{0,0} = 1$ and $B_{n,0} = 0$ for $n \in \mathbb{N}$.

Next, we collect some needed spaces as follows (see [7]). The symbol H^∞ denotes the space of all bounded analytic functions f on $\mathbb{D}$ such that

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < +\infty.$$

Let μ be a weight function (i.e. a positive continuous function on $\mathbb{D}$) and $n \in \mathbb{N}_0$. Then the nth weighted-type space $\mathcal{W}_\mu^{(n)}(\mathbb{D}) := \mathcal{W}_\mu^{(n)}$ consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}}(f) := \sup_{z \in \mathbb{D}} \mu(z)|f^{(n)}(z)| < +\infty.$$

If $n = 0$, it is the weighted-type space H_μ^∞ (see, for example, [28–30]). If $n = 1$, the Bloch-type space $\mathcal{B}_\mu$, and if $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu$. If $\mu(z) = 1 - |z|^2$, we correspondingly get the classical weighted-type space, Bloch space and Zygmund space. Some information on these classical function spaces and some operators on them can be found, for example, in [31–37].

Let $n \in \mathbb{N}$, then the quantity $b_{\mathcal{W}_\mu^{(n)}}(f)$ is a seminorm on $\mathcal{W}_\mu^{(n)}$ and a norm on $\mathcal{W}_\mu^{(n)}/\mathbb{P}_{n-1}$, where $\mathbb{P}_{n-1}$ is the class of all polynomials whose degrees are less than or equal to $n-1$. A natural norm on $\mathcal{W}_\mu^{(n)}$ can be introduced as follows

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathcal{W}_\mu^{(n)}}(f).$$

The set $\mathcal{W}_\mu^{(n)}$ with this norm becomes a Banach space. The little nth weighted-type space $\mathcal{W}_{\mu,0}^{(n)}$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z| \to 1} \mu(z)|f^{(n)}(z)| = 0.$$

It is easy to see that $\mathcal{W}_{\mu,0}^{(n)}$ is a closed subspace of $\mathcal{W}_{\mu}^{(n)}$ and the set of all polynomials is dense in $\mathcal{W}_{\mu,0}^{(n)}$. If $n = 1$ and $\mu(z) = 1 - |z|^2$, then it is the little Bloch space $\mathcal{B}_0$.

Finally, we will introduce the boundedness and compactness of a operator T. Let X and Y be two Banach spaces, and $T : X \to Y$ be a operator. If there is a positive constant K such that

$$\|Tf\|_Y \leq K\|f\|_X$$

for all $f \in X$, we say that T is bounded. The operator $T : X \to Y$ is compact if it maps bounded sets into relatively compact sets.

As usual, some positive constants are denoted by C, and they may differ from one occurrence to another. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive constant C such that $a \leq Cb$ (resp. $a \geq Cb$). When $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

2. Preliminary results

In this section, we need several auxiliary results for proving the main results. First, we have the following useful result which can be found in [38].

Lemma 2.1. *Let $f \in H^\infty$. Then for every $n \in \mathbb{N}$, there exists a constant $C > 0$ independent of f such that*

$$\sup_{z\in\mathbb{D}}(1 - |z|)^n|f^{(n)}(z)| \leq C\|f\|_\infty.$$

The following lemma is introduced in [31].

Lemma 2.2. *Let $f \in \mathcal{B}$. Then for every $n \in \mathbb{N}$*

$$\|f\|_{\mathcal{B}} \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^n|f^{(n)}(z)|.$$

The following lemma shows that any bounded analytic function on $\mathbb{D}$ is in Bloch space (see Proposition 5.1.2 in [39]).

Lemma 2.3. *$H^\infty \subset \mathcal{B}$. Moreover, $\|f\|_{\mathcal{B}} \leq \|f\|_\infty$ for all $f \in H^\infty$.*

The following gives an important test function (see [40]).

Lemma 2.4. *For fixed $t \geq 0$ and $w \in \mathbb{C}$, the following function is in H^∞*

$$g_{w,t}(z) = \left(\frac{1 - |w|^2}{(1 - \langle z, w\rangle)}\right)^{t+1}.$$

Moreover,

$$\sup_{w\in\mathbb{C}} \|g_{w,t}\|_\infty \lesssim 1.$$

We construct some suitable linear combinations of the functions in Lemma 2.4, which will be used in the proofs of the main results.

Lemma 2.5. *Let $w \in \mathbb{C}$. Then there are constants $c_0, c_1, \ldots, c_n$ such that the function*

$$h_w(z) = \sum_{k=0}^{n} c_k g_{w,k}(z)$$

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satisfies

$$h_w^{(s)}(w)=\frac{\overline{w}^s}{(1-|w|^2)^s},\quad 0\le s\le n\quad and\quad h_w^{(l)}(w)=0, \tag{5}$$

where $l\in\{0,1,\dots,n\}\backslash\{s\}$. *Moreover,*

$$\sup_{w\in\mathbb{C}}\|h_w\|_\infty<+\infty.$$

Proof. For the simplicity sake, we write $d_k=k+1$. By a direct calculation, it is easy to see that the system (5) is equivalent to the following system

$$\begin{pmatrix} 1 & 1 & \cdots & 1\\ d_0 & d_1 & \cdots & d_n\\ \vdots & \vdots & \ddots & \vdots\\ \prod\limits_{k=0}^{s-1} d_k & \prod\limits_{k=0}^{s-1} d_{k+1} & \cdots & \prod\limits_{k=0}^{s-1} d_{k+n}\\ \vdots & \vdots & \ddots & \vdots\\ \prod\limits_{k=0}^{n-1} d_k & \prod\limits_{k=0}^{n-1} d_{k+1} & \cdots & \prod\limits_{k=0}^{n-1} d_{k+n} \end{pmatrix}\begin{pmatrix} c_0\\ c_1\\ \vdots\\ c_s\\ \vdots\\ c_n\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}. \tag{6}$$

Since $d_k>0$, $k=\overline{0,n}$, by Lemma 5 in [41], the determinant of system (6) is $D_{n+1}(d_0)=\prod_{j=1}^n j!$, which is different from zero. Therefore, there exist constants $c_0,c_1,\dots,c_n$ such that the system (5) holds. Furthermore, we obtain $\sup_{w\in\mathbb{C}}\|h_w\|_\infty<+\infty$. □

Remark 2.1. In Lemma 2.5, it is clear that, if $s=0$, then there are constants $c_0,c_1,\dots,c_n$ such that the function $h_w(z)$ satisfies $h_w^{(0)}(w)=h_w(w)=1$ and $h_w^{(l)}(w)=0$ for $l=\overline{1,n}$.

We also have the following characterization of compactness which can be proved similar to that in [42] (Proposition 3.11), and so we omit the proof.

Lemma 2.6. *Let* $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $u\in H(\mathbb{D})$ *and* ϕ *be an entire function. Then the bounded operator* $D_u^mS_\phi: H^\infty\to\mathcal{W}_\mu^{(n)}$ *is compact if and only if for each bounded sequence* $\{f_k\}(k\in\mathbb{N})\subset H^\infty$ *such that* $f_k\to 0$ *uniformly on any compact subsets of* $\mathbb{D}$ *as* $k\to\infty$*, it follows that*

$$\lim_{k\to\infty}\|D_u^mS_\phi f_k\|_{\mathcal{W}_\mu^{(n)}}=0.$$

Finally, we need the following result proved in [34]. So, the details are omitted.

Lemma 2.7. *A closed set* K *in* $\mathcal{W}_{\mu,0}^{(n)}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f^{(n)}(z)|=0.$$

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3. Main results and proofs

Now, we begin to characterize the boundedness and compactness of the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ (or $\mathcal{W}_{\mu,0}^{(n)}$).

Theorem 3.1. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $u \in H(\mathbb{D})$ and ϕ an entire function with $\phi^{(m)}(1) \neq 0$ and $\phi^{(m+1)}(0) \neq 0$. Then the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is bounded if and only if*

$$M_i := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u^{(n-i)}(z)|}{(1-|z|^2)^i} < +\infty \tag{7}$$

for $i = \overline{0,n}$.

Moreover, if the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is bounded, then the following asymptotic relationship holds

$$\|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \asymp \sum_{i=0}^n M_i. \tag{8}$$

Proof. Assume that condition (7) holds. Then for each $z \in \mathbb{D}$ and $f \in H^\infty$, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \mu(z)\big|(D_u^m S_\phi f)^{(n)}(z)\big| &= \sup_{z\in\mathbb{D}} \mu(z)\Big|\sum_{j=0}^n \Big(\sum_{i=j}^n C_n^i u^{(n-i)}(z) B_{i,j}(f(z))\Big)\phi^{(m+j)}(f(z))\Big| \\ &\le \sup_{z\in\mathbb{D}} \mu(z)\sum_{j=0}^n \Big(\sum_{i=j}^n C_n^i \big|u^{(n-i)}(z)\big|\big|B_{i,j}(f(z))\big|\Big)\big|\phi^{(m+j)}(f(z))\big|,\end{aligned}$$

where

$$B_{i,j}(f(z)) := B_{i,j}\left(f'(z), f''(z), \dots, f^{(i-j+1)}(z)\right), \quad 0 \le j \le i \le n.$$

Applying formula (4) and Lemma 2.1, we obtain

$$\begin{aligned}\big|B_{i,j}(f(z))\big| &= \left|B_{i,j}\left(f'(z), f''(z), \dots, f^{(i-j+1)}(z)\right)\right| \\ &\le B_{i,j}\left(\frac{\|f\|_\infty}{1-|z|^2}, \frac{\|f\|_\infty}{(1-|z|^2)^2}, \dots, \frac{\|f\|_\infty}{(1-|z|^2)^{i-j+1}}\right).\end{aligned} \tag{9}$$

For the convenience, we write

$$\widehat{B}_{i,j}(f,z) = B_{i,j}\left(\frac{\|f\|_\infty}{1-|z|^2}, \frac{\|f\|_\infty}{(1-|z|^2)^2}, \dots, \frac{\|f\|_\infty}{(1-|z|^2)^{i-j+1}}\right). \tag{10}$$

From (9) and (10), we get

$$\sup_{z\in\mathbb{D}} \mu(z)\big|(D_u^m S_\phi f)^{(n)}(z)\big| \le \sup_{z\in\mathbb{D}} \mu(z)\sum_{j=0}^n \left(\sum_{i=j}^n C_n^i \big|u^{(n-i)}(z)\big|\widehat{B}_{i,j}(f,z)\right)\big|\phi^{(m+j)}(f(z))\big|. \tag{11}$$

For $i > j$, we have $\widehat{B}_{i,j}(f,z) = 0$. Let $f \in H^\infty$ and $\|f\|_\infty \le M$. Then, we obtain

$$\widehat{B}_{i,j}(f,z) \lesssim \frac{1}{(1-|z|^2)^i}, \quad 0 \le j \le i, \tag{12}$$

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where $i=\overline{0,n}$. From (11) and (12), we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}\mu(z)\big|(D_u^mS_\phi f)^{(n)}(z)\big| \le&\sup_{z\in\mathbb{D}}\mu(z)\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}C_n^i\big|u^{(n-i)}(z)\big|\widehat{B}_{i,j}(f,z)\Big)\big|\phi^{(m+j)}(f(z))\big|\\ \le&C\sup_{z\in\mathbb{D}}\mu(z)\bigg(|u^{(n)}(z)||\phi^{(m)}(f(z))|\\ &+\sum_{i=1}^{n}\frac{|u^{(n-i)}(z)|}{(1-|z|^2)^i}\Big(\sum_{j=1}^{i}\big|\phi^{(m+j)}(f(z))\big|\Big)\bigg). \qquad (13)\end{aligned}$$

Since $f\in H^\infty$ and $\|f\|_\infty\le M$ and ϕ is an entire function, we obtain

$$\big|\phi^{(m+j)}(f(z))\big|\le\max_{|w|=M}\big|\phi^{(m+j)}(w)\big|=L_j<+\infty \qquad (14)$$

for each $z\in\mathbb{D}$ and $j=\overline{0,n}$. From (13) and (14), we have

$$\sup_{z\in\mathbb{D}}\mu(z)\big|(D_u^mS_\phi f)^{(n)}(z)\big|\le C\sup_{z\in\mathbb{D}}\bigg(\mu(z)|u^{(n)}(z)|+\sum_{i=1}^{n}\frac{\mu(z)|u^{(n-i)}(z)|}{(1-|z|^2)^i}\bigg). \qquad (15)$$

On the other hand, we also have that for every $l=\overline{0,n-1}$

$$\big|(D_u^mS_\phi f)^{(l)}(0)\big|\le\Big|\sum_{j=0}^{l}\Big(\sum_{i=j}^{l}C_l^iu^{(l-i)}(0)B_{i,j}(f(0))\Big)\phi^{(m+j)}(f(0))\Big|<+\infty. \qquad (16)$$

From Lemma 2.2, (7), (15) and (16), we see that the operator $D_u^mS_\phi:\mathcal{B}\to\mathcal{W}_\mu^{(n)}$ is bounded. By Lemma 2.3 (or (7) and (15)), it is obvious that the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded. Moreover, it follows that

$$\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\le C\sum_{i=0}^{n}M_i. \qquad (17)$$

Now assume that the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded, then there is a positive constant C independent of f such that

$$\|D_u^mS_\phi f\|_{\mathcal{W}_\mu^{(n)}}\le C\|f\|_\infty \qquad (18)$$

for each $f\in H^\infty$. By Remark 2.1, there is a function $h_w\in H^\infty$ such that

$$h_w(w)=1 \quad\text{and}\quad h_w^{(l)}(w)=0 \qquad (19)$$

for $l=\overline{1,n}$. Let $L_0=\|h_w\|_\infty$. Then, from (18) and (19), we obtain

$$\begin{aligned}L_0\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\ge&\|D_u^mS_\phi h_w\|_{\mathcal{W}_\mu^{(n)}}\\ =&\sup_{z\in\mathbb{D}}\mu(z)\Big|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}C_n^iu^{(n-i)}(z)B_{i,j}(h_w(z))\Big)\phi^{(m+j)}(h_w(z))\Big|\\ \ge&\mu(w)\big|u^{(n)}(w)\big|\big|B_{0,0}(h_w(w))\big|\big|\phi^{(m)}(1)\big|\\ =&\mu(w)\big|u^{(n)}(w)\big|\big|\phi^{(m)}(1)\big|. \qquad (20)\end{aligned}$$

Since $|\phi^{(m)}(1)|\ne0$, we have

$$L_0\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\ge\|D_u^mS_\phi h_w\|_{\mathcal{W}_\mu^{(n)}}\ge C\mu(z)|u^{(n)}(z)|, \qquad (21)$$

for each $z\in\mathbb{D}$, which implies that $M_0<+\infty$.

By Lemma 2.4, there is a function $\tilde{h}_w \in H^\infty$ such that

$$\tilde{h}_w^{(n)}(w) = \frac{\overline{w}^n}{(1-|w|^2)^n} \quad \text{and} \quad \tilde{h}_w^{(l)}(w) = 0 \tag{22}$$

for $l = \overline{0, n-1}$. Let $L_n = \|\tilde{h}_w\|_\infty$. Then, from (18) and (22), we have

$$\begin{aligned} L_n \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \geq & \|D_u^m S_\phi \tilde{h}_w\|_{\mathcal{W}_\mu^{(n)}} \\ = & \sup_{z\in\mathbb{D}} \mu(z) \Big| \sum_{j=0}^{n} \Big(\sum_{i=j}^{n} C_n^i u^{(n-i)}(z) B_{i,j}(\tilde{h}_w(z)) \Big) \phi^{(m+j)}(\tilde{h}_w(z)) \Big| \\ \geq & \mu(w) \big| u(w) B_{n,1}(\tilde{h}_w(w)) \phi^{(m+1)}(0) + u^{(n)}(w) B_{0,0}(\tilde{h}_w(w)) \phi^{(m)}(0) \big| \\ = & \mu(w) \Big| \frac{u(w)\overline{w}^n}{(1-|w|^2)^n} \phi^{(m+1)}(0) + u^{(n)}(w)\phi^{(m)}(0) \Big| \\ \geq & \mu(w) \Big| \frac{u(w)\overline{w}^n}{(1-|w|^2)^n} \phi^{(m+1)}(0) \Big| - \mu(w) \big| u^{(n)}(w)\phi^{(m)}(0) \big|, \end{aligned} \tag{23}$$

where

$$B_{i,j}(\tilde{h}_w(z)) := B_{i,j}\left(\tilde{h}'_w(z), \tilde{h}''_w(z), \ldots, \tilde{h}_w^{(i-j+1)}(z)\right).$$

From (21) and (23), we have

$$\begin{aligned} \mu(w) \Big| \frac{u(w)\overline{w}^n}{(1-|w|^2)^n} \phi^{(m+1)}(0) \Big| &\leq L_n \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} + \mu(w) \big| u^{(n)}(w)\phi^{(m)}(0) \big| \\ &\leq (L_n + CL_0) \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}}. \end{aligned}$$

Since $|\phi^{(m+1)}(0)| \neq 0$, we have

$$(L_n + CL_0) \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \geq \|D_u^m S_\phi \tilde{h}_w\|_{\mathcal{W}_\mu^{(n)}} \geq C \frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n}. \tag{24}$$

From (24), we have

$$(L_n + CL_0) \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \geq C \sup_{|z|>1/2} \frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n} \geq \frac{C}{2^n} \sup_{|z|>1/2} \frac{\mu(z)|u(z)|}{(1-|z|^2)^n}. \tag{25}$$

One the other hand, we have

$$\sup_{|z|\leq 1/2} \frac{\mu(z)|u(z)|}{(1-|z|^2)^n} \leq \left(\frac{4}{3}\right)^n \sup_{|z|\leq 1/2} \mu(z)|u(z)|. \tag{26}$$

From (25) and (26), we get that $M_n < +\infty$.

By Lemma 2.4, there is a function $\hat{h}_w \in H^\infty$ such that

$$\hat{h}_w^{(n-1)}(w) = \frac{\overline{w}^{n-1}}{(1-|w|^2)^{n-1}} \quad \text{and} \quad \hat{h}_w^{(l)}(w) = 0, \tag{27}$$

where $l \in \{0, 1, \ldots, n\} \setminus \{n-1\}$. Let $L_{n-1} = \|\hat{h}_w\|_\infty$. From (18) and (27), we have

$$\begin{aligned}L_{n-1}\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} \geq&\|D_u^m S_\phi \hat{h}_w\|_{\mathcal{W}_\mu^{(n)}}\\ =&\sup_{z\in\mathbb{D}}\mu(z)\Big|\sum_{j=0}^{n}\Big(\sum_{i=j}^{n}C_n^i u^{(n-i)}(z)B_{i,j}(\hat{h}_w(z))\Big)\phi^{(m+j)}(\hat{h}_w(z))\Big|\\ \geq&\mu(w)\Big|C_n^{n-1}u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0)\\ &+\sum_{j=1}^{n}u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)+u^{(n)}(w)\phi^{(m)}(0)\Big|\\ \geq&\mu(w)\Big|C_n^{n-1}u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0)\\ &+\sum_{j=1}^{n}u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big|-\mu(w)\big|u^{(n)}(w)\phi^{(m)}(0)\big|,\end{aligned}\tag{28}$$

where $B_{i,j}(\hat{h}_w(z)) := B_{i,j}\Big(\hat{h}'_w(z),\hat{h}''_w(z),\dots,\hat{h}_w^{(i-j+1)}(z)\Big)$. From (21) and (28), by using the triangle inequality, we obtain

$$\begin{aligned}&(L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\\ \geq&\mu(w)\Big|C_n^{n-1}u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0)+\sum_{j=1}^{n}u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big|\\ \geq&\mu(w)\Big|u'(w)B_{n-1,1}(\hat{h}_w(w))\phi^{(m+1)}(0)\Big|-\mu(w)\Big|\sum_{j=1}^{n}u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big|.\end{aligned}\tag{29}$$

From (29), we have

$$\begin{aligned}&\mu(w)\big|u'(w)B_{n-1,1}(\hat{h}_w)\phi^{(m+1)}(0)\big|\\ \leq&(L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+\mu(w)\Big|\sum_{j=1}^{n}u(z)B_{n,j}(\hat{h}_w(w))\phi^{(m+j)}(0)\Big|\\ \leq&(L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+\frac{\mu(w)|u(w)||w|^n}{(1-|w|^2)^n}\Big(\sum_{j=1}^{n}\big|\phi^{(m+j)}(0)\big|\Big).\end{aligned}\tag{30}$$

Since $|\phi^{(m+1)}(0)|\neq 0$, by using (24) and (30), we obtain

$$\begin{aligned}C\frac{\mu(z)|u'(z)||z|^{n-1}}{(1-|z|^2)^{n-1}}\Big| &\leq (L_{n-1}+CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+C\frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n}\\ &\leq (L_n+L_{n-1}+2CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}.\end{aligned}\tag{31}$$

From (31), we have

$$\begin{aligned}(L_n+L_{n-1}+2CL_0)\|D_u^m S_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}} &\geq C\sup_{|z|>1/2}\frac{\mu(z)|u'(z)||z|^{n-1}}{(1-|z|^2)^{n-1}}\\ &\geq \frac{C}{2^{n-1}}\sup_{|z|>1/2}\frac{\mu(z)|u'(z)|}{(1-|z|^2)^{n-1}}.\end{aligned}\tag{32}$$

One the other hand, we have

$$\sup_{|z|\le 1/2}\frac{\mu(z)|u'(z)|}{(1-|z|^2)^{n-1}}\le\left(\frac{4}{3}\right)^{n-1}\sup_{|z|\le 1/2}\mu(z)|u'(z)|. \tag{33}$$

From (32) and (33), we get that $M_{n-1}<+\infty$.

Now, assume that (7) holds for $k\le i\le n$, where $1\le k\le n-2$. Let $L_{k-1}=\|h_w\|_\infty$. By using the function in Lemma 2.4, we have

$$\begin{aligned}L_{k-1}\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\ge&\|D_u^mS_\phi h_w\|_{\mathcal{W}_\mu^{(n)}}\\
=&\sup_{z\in\mathbb{D}}\mu(z)\Big|\sum_{j=0}^n\Big(\sum_{i=j}^nC_n^iu^{(n-i)}(z)B_{i,j}(h_w(z))\Big)\phi^{(m+j)}(h_w(z))\Big|\\
\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\\
&+\sum_{i=k}^n\sum_{j=1}^iC_n^iu^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)+u^{(n)}(w)\phi^{(m)}(0)\Big|\\
\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\\
&+\sum_{i=k}^n\sum_{j=1}^iC_n^iu^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|-\mu(w)\big|u^{(n)}(w)\phi^{(m)}(0)\big|\end{aligned} \tag{34}$$

for each $w\in\mathbb{D}$. From (21) and (34), we have

$$\begin{aligned}(L_{k-1}+CL_0)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\\
&+\sum_{i=k}^n\sum_{j=1}^iC_n^iu^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|\\
\ge&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\Big|\\
&-\mu(w)\Big|\sum_{i=k}^n\sum_{j=1}^iC_n^iu^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|.\end{aligned}$$

Then, we have

$$\begin{aligned}&\mu(w)\Big|C_n^{k-1}u^{(n-(k-1))}(w)B_{k-1,1}(h_w(w))\phi^{(m+1)}(0)\Big|\\
\le&(L_{k-1}+CL_0)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\\
&+\mu(w)\Big|\sum_{i=k}^n\sum_{j=1}^iC_n^iu^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|\\
\le&(L_{k-1}+CL_0)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\\
&+C\sum_{i=k}^n\sum_{j=1}^i\mu(w)\Big|u^{(n-i)}(z)B_{i,j}(h_w(w))\phi^{(m+j)}(0)\Big|\\
\le&(L_{k-1}+CL_0)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\end{aligned}$$

$$+C\sum_{i=k}^{n}\frac{\mu(w)|u^{(n-i)}(w)||w|^i}{(1-|w|^2)^i}\Big(\sum_{j=1}^{i}\big|\phi^{(m+j)}(0)\big|\Big) \tag{35}$$

Since $|\phi^{(m+1)}(0)|\neq 0$, from (35) and the assumption (7), we have

$$\begin{aligned}&C\frac{\mu(z)|u^{(n-(k-1))}(z)||z|^{k-1}}{(1-|z|^2)^{k-1}}\\ \leq&(L_{k-1}+CL_0)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}+C\sum_{i=k}^{n}\frac{\mu(w)|u^{(n-i)}(w)||w|^i}{(1-|w|^2)^i}\\ \leq&\Big(\sum_{t=k-1}^{n}L_t+(n-k+2)CL_0\Big)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}.\end{aligned} \tag{36}$$

From (36), we have

$$\begin{aligned}\Big(\sum_{t=k-1}^{n}L_t+(n-k+2)CL_0\Big)\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}\geq& C\frac{\mu(z)|u^{(n-(k-1))}(z)||z|^{k-1}}{(1-|z|^2)^{k-1}}\\ \geq&\frac{C}{2^{k-1}}\frac{\mu(z)|u^{(n-(k-1))}(z)|}{(1-|z|^2)^{k-1}}\end{aligned} \tag{37}$$

One the other hand, we have

$$\sup_{|z|\leq 1/2}\frac{\mu(z)|u^{(n-(k-1))}(z)|}{(1-|z|^2)^{k-1}}\leq\left(\frac{4}{3}\right)^{n-(k-1)}\sup_{|z|\leq 1/2}\mu(z)|u^{(n-(k-1))}(z)|. \tag{38}$$

From (37) and (38), we get that $M_{k-1}<+\infty$. Hence, from the mathematical induction it follows that (7) holds for every $i=\overline{0,n}$. Moreover, we also obtain

$$\sum_{i=0}^{n}M_i\leq C\|D_u^mS_\phi\|_{H^\infty\to\mathcal{W}_\mu^{(n)}}. \tag{39}$$

From (17) and (39), then the asymptotic relation (8) follows, as desired. □

Theorem 3.2. *Let $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, $u\in H(\mathbb{D})$ and ϕ an entire function with $\phi^{(m)}(1)\neq 0$ and $\phi^{(m+1)}(0)\neq 0$. Then the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_{\mu,0}^{(n)}$ is bounded if and only if the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded and for each $i\in\{0,1,\dots,n\}$*

$$\lim_{|z|\to 1}\mu(z)|u^{(n-i)}(z)|=0. \tag{40}$$

Proof. Assume that $D_u^mS_\phi:H^\infty\to\mathcal{W}_{\mu,0}^{(n)}$ is bounded. Then for each $f\in H^\infty$, we have

$$\lim_{|z|\to 1}\mu(z)|(D_u^mS_\phi f)^{(n)}(z)|=0. \tag{41}$$

Clearly, the operator $D_u^mS_\phi:H^\infty\to\mathcal{W}_\mu^{(n)}$ is bounded. Hence, from (24), we obtain

$$\frac{\mu(z)|u(z)||z|^n}{(1-|z|^2)^n}\leq C\mu(z)|(D_u^mS_\phi\tilde{h}_w)^{(n)}(z)|. \tag{42}$$

From (42), we obtain

$$\mu(z)|u(z)||z|^n\leq C\mu(z)|(D_u^mS_\phi\tilde{h}_w)^{(n)}(z)| \tag{43}$$

By taking $|z|\to 1$ in (43) and using (41), it follows that (40) holds for $i=n$. Hence, by the proof of Theorem 3.1, we get that (40) holds for each $i=\overline{0,n}$.

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Conversely, assume that $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is bounded and condition (40) holds. Let $\hat{p} \in H^\infty$ and $\|\hat{p}\|_\infty \leq M$. Then, we have

$$\left|\phi^{(m+j)}(\hat{p}(z))\right| < +\infty.$$

For every polynomial $\hat{p}$, we have

$$\begin{aligned}\mu(z)\left|(D_u^m S_\phi \hat{p})^{(n)}(z)\right| &= \sup_{z\in\mathbb{D}} \mu(z) \left| \sum_{j=0}^{n} \Big(\sum_{i=j}^{n} C_n^i u^{(n-i)}(z) B_{i,j}(\hat{p}(z)) \Big) \phi^{(m+j)}(\hat{p}(z)) \right| \\ &\leq \sup_{z\in\mathbb{D}} \mu(z) \sum_{j=0}^{n} \Big(\sum_{i=j}^{n} C_n^i \big|u^{(n-i)}(z)\big| \big|B_{i,j}(\hat{p}(z))\big| \Big) \big|\phi^{(m+j)}(\hat{p}(z))\big| \to 0\end{aligned}$$

as $|z| \to 1$. From this, we have that for every polynomial $\hat{p}$, $D_u^m S_\phi \hat{p} \in \mathcal{W}_{\mu,0}^{(n)}$. Since the set of all polynomials is dense in H^∞, we have that for each $f \in H^\infty$ there exist a sequence of polynomial $\{\hat{p}_k\}$ such that

$$\lim_{k\to\infty} \|f - \hat{p}_k\|_\infty = 0. \tag{44}$$

From (44) and using the boundedness of $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$, we obtain

$$\|D_u^m S_\phi f - D_u^m S_\phi \hat{p}_k\|_{\mathcal{W}_\mu^{(n)}} \leq \|D_u^m S_\phi\|_{H^\infty \to \mathcal{W}_\mu^{(n)}} \|f - \hat{p}_k\|_\infty \to 0 \tag{45}$$

as $k \to \infty$. Hence, $D_u^m S_\phi(H^\infty) \subseteq \mathcal{W}_{\mu,0}^{(n)}$ and the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_{\mu,0}^{(n)}$ is bounded. The proof is finished. □

Theorem 3.3. *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $u \in H(\mathbb{D})$ and ϕ an entire function with $\phi^{(m)}(1) \neq 0$ and $\phi^{(m+1)}(0) \neq 0$. Then the following statements are equivalent:*

(a) *The operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is compact.*

(b) *The operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_{\mu,0}^{(n)}$ is compact.*

(c) *For each $i \in \{0, 1, \ldots, n\}$, it follows that*

$$\lim_{|z|\to 1} \frac{\mu(z)|u^{(n-i)}(z)|}{(1-|z|^2)^i} = 0. \tag{46}$$

Proof. $(c) \Rightarrow (b)$. From (13) and using (46), we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_\infty \leq 1} \mu(z)|(D_u^m S_\phi f)^n(z)| = 0.$$

Obviously, the set is bounded. Hence, by Lemma 2.6 the compactness of the operator $D_u^m S_\phi : H^\infty \to \mathcal{W}_{\mu,0}^{(n)}$ follows.

$(b) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (c)$. Suppose that $D_u^m S_\phi : H^\infty \to \mathcal{W}_\mu^{(n)}$ is compact. Then it is clear that the operator is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|z_k| \to 1$ as $k \to \infty$. If such a sequence does not exist, then condition (46) is vacuously satisfied. Let $\tilde{h}_k = \tilde{h}_{z_k}$, where $\tilde{h}_w$ is defined in the proof of the Theorem 3.1 (or Lemma 2.4). Since $\lim_{k\to\infty} \tilde{h}_{z_k} = 0$, we have $\tilde{h}_k \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $k \to \infty$. Hence, by Lemma 2.5 we have

$$\lim_{k\to\infty} \|D_u^m S_\phi \tilde{h}_k\|_{\mathcal{W}_\mu^{(n)}} = 0. \tag{47}$$

On the other hand, from (25), for sufficiently large k it follows that

$$\|D_u^m S_\phi \tilde{h}_k\|_{\mathcal{W}_\mu^{(n)}} \geq \frac{\mu(z_k)|u(z_k)|}{(1-|z_k|^2)^n}, \tag{48}$$

which along with (47) and letting $k \to \infty$ in inequality (48) and since $\{z_k\}$ is an arbitrary sequence such that $|z_k| \to 1$ as $k \to \infty$, implies that (46) holds for $i = n$. By the proof of the Theorem 3.1, we get that equality (46) holds for each $i \in \{0, 1, ..., n\}$. This completes the proof. □

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