# MULTIFARIOUS FUNCTIONAL EQUATIONS IN CONNECTION WITH THREE GEOMETRICAL MEANS

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Abstract. In this article, we introduce a new generalized multifarious radical reciprocal functional equation by generalizing the equation employed by Narasimman *et al.* in [5] and combining three classical Pythagorean means arithmetic, geometric and harmonic. Also, we illustrate the geometrical interpretation. Mainly, we find its general solution and stabilities related to Ulam problem in modular spaces by using fixed point approach.

## 1. Introduction and preliminaries

In the development of broad field functional equations, we come acrossing various types like additive, quadratic, cubic and so on. In recent research many researchers modeled functional

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equations from physical phenomena. In particular, by geometrical construction authors introduced remarkable reciprocal type functional equations.

In 2010, Ravi and Senthil Kumar [6] introduced functional equation of reciprocal type

$$s(z+w) = \frac{s(z)s(w)}{s(z)+s(w)}. \tag{1.1}$$

with solution $s(z) = \frac{c}{z}$.

In 2014, Bodaghi and Kim [1] introduced the quadratic reciprocal functional equation, which was generalized by Song andSong citeAM.

In 2015, Narasimman, Ravi and Pinelas [5] introduced the radical reciprocal quadratic functional equation

$$s\left(\sqrt[2]{z^2+w^2}\right) = \frac{s(z)s(w)}{s(z)+s(w)}, \ z, w \in (0, \infty), \tag{1.2}$$

which is satisfied by $s(z) = \frac{c}{z^2}$. Also, they provided the solution and stability of (1.2) with geometrical interpretation and application.

For the necessary introduction on stability related to Ulam problem and the notion of modular spaces one can refer to [7, 8, 9, 10, 12].

## 2. Main results

**Definition 2.1.** A reciprocal functional equation is a functional equation with solution of the form $\frac{1}{s(z)}$. When $s(z) = z, z^2, z^3 \ldots$ we have various type of reciprocal functional equations like reciprocal additive, reciprocal quadratic, reciprocal cubic and so on.

**Definition 2.2.** Pythagorean means [3] The three classical Pythagorean means are the arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM), which are defined by

$$AM(a_1, a_2, ..., a_n) = \frac{1}{n}(a_1 + ... + a_n),$$
$$GM(a_1, a_2, ..., a_n) = \sqrt[n]{a_1 + ... + a_n},$$
$$HM(a_1, a_2, ..., a_n) = \frac{n}{\frac{1}{a_1} + ... + \frac{1}{a_n}}.$$

**Definition 2.3.** A functional equations which are arisen from the relations between three Pythagorean means (arithmetic, geometric and harmonic) are known as Pythagorean mean functional equations.

**Definition 2.4.** A reciprocal Pythagorean mean functional equation which shall possess the nature of any type of functional equation like additive, quadratic, cubic and so on is said to be a multifarious reciprocal Pythagorean mean functional equation.

In this paper, using Pythagorean means, we introduce the new generalized $2-$dimensional and $3-$dimensional multifarious radical reciprocal functional equations.

The following $2-$dimensional and $3-$dimensional multifarious radical reciprocal functional equations are obtained by generalizing (1.1) and (1.2)

$$s\left(\sqrt[m]{z^m+w^m}\right)=\frac{s(z)s(w)}{s(z)+s(w)}, \tag{2.3}$$

$$s\left(\sqrt[m]{z_1^m+z_2^m+z_3^m}\right)=\frac{s(z_1)s(z_2)s(z_3)}{s(z_1)s(z_2)+s(z_2)s(z_3)+s(z_1)s(z_3)}, \tag{2.4}$$

which are satisfied by $s(z)=\frac{c}{z^m}$, for all $z,w,z_1,z_2,z_3\in(0,\infty),m\in\mathbb{N}$. Observe that if $m=1$ and $m=2$ in (2.3), we have (1.1) and (1.2), respectively. Further, if $m=3,4,\cdots$ in (2.3), then we have various type of functional equations. Hence the functional equation (2.3) is known as two dimensional multifarious radical reciprocal functional equation. By similar argument, (2.4) is known as three dimensional multifarious radical reciprocal functional equation.

2.1. **Geometrical construction and geometrical interpretation of multifarious radical reciprocal functional equations.** Geometric construction of three Pythagorean means of two variables can be constructed geometrically as showed in Figure 1. Geometric construction of geometric mean of three variables are not possible but the other Pythagorean means can be constructed for any number of variables, one can refer [4, 11].

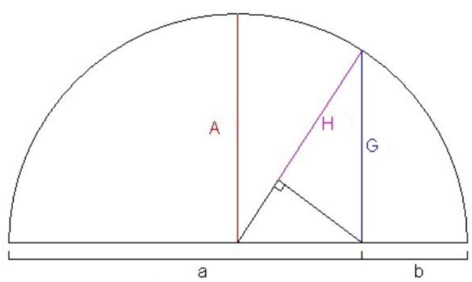


FIGURE 1. Pythagorean means of $a$ and $b$. $A$ is the arithmetic mean, $H$ is the harmonic mean and $G$ is the geometric mean.

The relations between three Pythagorean means of $p-$objects $z_1,z_2,\cdots,z_p$ are represented by the following equation

$$H\left(z_1,z_2,\cdots,z_p\right)=\frac{G\left(z_1,z_2,\cdots,z_p\right)^p}{A\left(\frac{1}{z_1}\prod_{i=1}^p z_i,\frac{1}{z_2}\prod_{i=1}^p z_i,\cdots,\frac{1}{z_p}\prod_{i=1}^p z_i\right)}. \tag{2.5}$$

Consider two spheres $S_1$ and $S_2$ of radii $r_1$ and $r_2$ with $r_1>r_2$, which are located along the $x-$axis centered at $C_1(0,0,0)$ and $C_2(d,0,0)$, respectively.

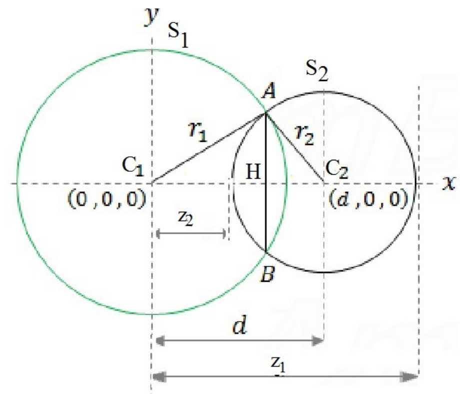


FIGURE 2. Intersecting two spheres $S_1$ and $S_2$.

We can show that the length of $C_2C_1$ is $\frac{z_1+z_2}{2}$ which is the arithmetic mean of $z_1$ and $z_2$. We can find the length $AC_1$, using Pythagoras' theorem, is the geometric mean $\sqrt{z_1 z_2}$ of $z_1$ and $z_2$. Also, we can obtain the length $HC_1$ is $\frac{2z_1z_2}{z_1+z_2}$, which is the harmonic mean of $z_1$ and $z_2$, since $C_2AC_1$ and $AHC_1$ are similar.

From Figure 2, we have the equality $HC_1 = \frac{AC_1^2}{C_2C_1}$, that is

$$H(z_1, z_2) = \frac{G(z_1, z_2)^2}{A\left(\frac{1}{z_1}\prod_{i=1}^{2} z_i, \frac{1}{z_2}\prod_{i=1}^{2} z_i\right)}, \tag{2.6}$$

which is the particular case of (2.5) by assuming $p = 2$ and which implies

$$\frac{1}{\frac{1}{z_1}+\frac{1}{z_2}} = \frac{z_1z_2}{z_1+z_2}. \tag{2.7}$$

Assuming $z_1 = \frac{1}{z}$ and $z_2 = \frac{1}{w}$ in (2.7), we obtain

$$\frac{1}{z+w} = \frac{\frac{1}{z}\frac{1}{w}}{\frac{1}{z}+\frac{1}{w}}. \tag{2.8}$$

In that case, (1.1) is valid by (2.8), which is satisfied by $s(z) = \frac{c}{z}$. Assuming $z_1 = \frac{1}{z^2}$ and $z_2 = \frac{1}{w^2}$ in (2.7) leads

$$\frac{1}{z^2+w^2} = \frac{\frac{1}{z^2}\frac{1}{w^2}}{\frac{1}{z^2}+\frac{1}{w^2}}. \tag{2.9}$$

In that case (1.2) is valid by (2.9), which is satisfied by $s(z) = \frac{c}{z^2}$. In general, assuming $z_1 = \frac{1}{z^m}$ and $z_2 = \frac{1}{w^m}$ in (2.7), we have

$$\frac{1}{z^m+w^m} = \frac{\frac{1}{z^m}\frac{1}{w^m}}{\frac{1}{z^m}+\frac{1}{w^m}}. \tag{2.10}$$

In that case, (2.3) is valid by (2.10), which is satisfied by $s(z) = \frac{c}{z^m}$.

In Figure 2, AB is the diameter of common circle. The common circle is the solution of the system

$$z_1^2 + z_2^2 + z_3^2 = r_1^2, \qquad (2.11)$$
$$(z_1 - d)^2 + z_2^2 + z_3^2 = r_2^2,$$

which implies

$$\frac{1}{z_1^2 + z_2^2 + z_3^2} = \frac{1}{r_1^2}, \qquad (2.12)$$
$$\frac{1}{(z_1 - d)^2 + z_2^2 + z_3^2} = \frac{1}{r_2^2}.$$

The system (2.12) can be expressed by radical reciprocal quadratic functional equations of the form

$$s\left(r_1^2\right) = \frac{s(z_1)s(z_2)s(z_3)}{s(z_1)s(z_2) + s(z_2)s(z_3) + s(z_1)s(z_3)}, \qquad (2.13)$$
$$s\left(r_2^2\right) = \frac{s(z_1 - d)s(z_2)s(z_3)}{s(z_1 - d)s(z_2) + s(z_2)s(z_3) + s(z_1 - d)s(z_3)},$$

for $z_1, z_2, z_3, r_1, r_2 \in (0, \infty)$, which is satisfied by $s(z_1) = \frac{c}{z_1^2}$ and the denominators are not equal to zero. Also, observe that the equation (2.13) is the particular case of (2.4) for $m = 2$. By assuming $p = 3$ in (2.5), we obtain

$$H\left(z_1, z_2, z_3\right) = \frac{G\left(z_1, z_2, z_3\right)^3}{A\left(\frac{1}{z_1}\prod_{i=1}^3 z_i, \frac{1}{z_2}\prod_{i=1}^3 z_i, \frac{1}{z_3}\prod_{i=1}^3 z_i\right)}, \qquad (2.14)$$

which gives

$$\frac{1}{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}} = \frac{z_1 z_2 z_3}{z_2 z_3 + z_1 z_3 + z_1 z_2}. \qquad (2.15)$$

Assuming $z_1 = \frac{1}{z_1^m}$, $z_2 = \frac{1}{z_2^m}$ and $z_3 = \frac{1}{z_3^m}$ in (2.15), we have

$$\frac{1}{z_1^m + z_2^m + z_3^m} = \frac{\frac{1}{z_1^m}\frac{1}{z_2^m}\frac{1}{z_3^m}}{\frac{1}{z_1^m} + \frac{1}{z_2^m} + \frac{1}{z_3^m}}. \qquad (2.16)$$

In that case (2.4) is valid by (2.16), which is satisfied by $s(z_1) = \frac{c}{z_1^m}$.

## 3. General solution of the multifarious radical reciprocal functional equations

The following theorems give the solution of (2.3) and (2.4) through motivated by the work of Ger [?].

**Theorem 3.1.** *A general solution of (2.3) is* $s(z) = \frac{c}{z^m}$; $z \in (0, \infty)$ *with* $\frac{s(z)}{\frac{1}{z^m}}$ *a quotient at zero.*

*Proof.* Assuming $z, w = z$ in (2.3), we have

$$s(\sqrt[m]{2}z) = \frac{1}{2}s(z) \tag{3.17}$$

for all $z \in (0, \infty)$. Assuming

$$g(z) = \frac{s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.18}$$

for all $z \in (0, \infty)$, we have

$$\underset{z \to 0^+}{lim} \frac{g(z)}{\frac{1}{z^{\frac{m}{2}}}} =: c \in \mathbb{R}$$

for all $z \in (0, \infty)$. Dividing (3.17) by $\frac{1}{z^{\frac{m}{2}}}$, we obtain

$$\frac{s(\sqrt[m]{2}z)}{\frac{\sqrt{2}}{\sqrt{2}z^{\frac{m}{2}}}} = \frac{\frac{1}{2}s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.19}$$

for all $z \in (0, \infty)$. Using (3.18) in (3.19), we have

$$g(\sqrt[m]{2}z) = \frac{1}{\sqrt{2}}g(z), \tag{3.20}$$

for all $z \in (0, \infty)$. Replacing $z$ by $\frac{z}{\sqrt[m]{2}}$ in (3.20), we get

$$\sqrt{2}g(z) = g\left(\frac{z}{\sqrt[m]{2}}\right). \tag{3.21}$$

Again, replacing $z$ by $\frac{z}{\sqrt[m]{2}}$ in (3.21), we have

$$(\sqrt{2})^2 g(z) = g\left(\frac{z}{(\sqrt[m]{2})^2}\right), \tag{3.22}$$

for all $z \in (0, \infty)$. Continuing the same process $k$ times, we obtain

$$(\sqrt{2})^k g(z) = g\left(\frac{z}{(\sqrt[m]{2})^k}\right), \tag{3.23}$$

for all $z \in (0, \infty)$.

Now,

$$\frac{g(z)}{\frac{1}{z^{\frac{m}{2}}}} = \frac{(\sqrt{2})^k g(z)}{(\sqrt{2})^k \frac{1}{z^{\frac{m}{2}}}} = \frac{g\left(\frac{1}{(\sqrt[m]{2})^k}z\right)}{\frac{(\sqrt{2})^k}{z^{\frac{m}{2}}}} \to c \quad as \quad k \to \infty,$$

for all $z \in (0, \infty)$. Eq. (3.18) implies that

$$s(z) = \frac{1}{z^{\frac{m}{2}}}g(z) = \frac{1}{z^{\frac{m}{2}}}\frac{1}{z^{\frac{m}{2}}}c = \frac{c}{z^m}$$

for all $z \in (0, \infty)$. This completes the proof. □

**Theorem 3.2.** *A general solution of (2.4) is* $s(z) = \frac{c}{z^m}$; $z \in (0, \infty)$ *with* $\frac{s(z)}{\frac{1}{z^m}}$ *a quotient at zero.*

*Proof.* Assuming $z_1, z_2, z_3 = z$ in (2.4), we have

$$s(\sqrt[m]{3}z) = \frac{1}{3}s(z), \tag{3.24}$$

and assuming

$$h(z) = \frac{s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.25}$$

we obtain

$$\lim_{z \to 0^+} \frac{h(z)}{\frac{1}{z^{\frac{m}{2}}}} =: c \in \mathbb{R}.$$

Dividing (3.24) by $\frac{1}{z^{\frac{m}{2}}}$, we get

$$\frac{s(\sqrt[m]{3}z)}{\frac{\sqrt{3}}{\sqrt{3}z^{\frac{m}{2}}}} = \frac{\frac{1}{3}s(z)}{\frac{1}{z^{\frac{m}{2}}}}, \tag{3.26}$$

and substituting (3.25) in (3.26), we obtain

$$h(\sqrt[m]{3}z) = \frac{1}{\sqrt{3}}h(z), \tag{3.27}$$

and replacing $z$ by $\frac{z}{\sqrt[m]{3}}$ in (3.27), we have

$$\sqrt{3}h(z) = h\left(\frac{z}{\sqrt[m]{3}}\right). \tag{3.28}$$

Again, replacing $z$ by $\frac{z}{\sqrt[m]{3}}$ in (3.28), we get

$$(\sqrt{3})^2 h(z) = h\left(\frac{z}{(\sqrt[m]{3})^2}\right), \tag{3.29}$$

for all $z \in (0, \infty)$. Continuing the same process $k$ times, we have

$$(\sqrt{3})^k h(z) = h\left(\frac{z}{(\sqrt[m]{3})^k}\right), \tag{3.30}$$

for all $z \in (0, \infty)$. Now,

$$\frac{h(z)}{\frac{1}{z^{\frac{m}{2}}}} = \frac{(\sqrt{3})^k h(z)}{(\sqrt{3})^k \frac{1}{z^{\frac{m}{2}}}} = \frac{h\left(\frac{1}{(\sqrt[m]{3})^k}z\right)}{\frac{(\sqrt{3})^k}{z^{\frac{m}{2}}}} \to c \quad as \quad k \to \infty,$$

for all $z \in (0, \infty)$. Eqs. (3.25) and (3.30) imply that

$$s(z) = \frac{1}{z^{\frac{m}{2}}}h(z) = \frac{1}{z^{\frac{m}{2}}}\frac{1}{z^{\frac{m}{2}}}c = \frac{c}{z^m}$$

for all $z \in (0, \infty)$. This completes the proof. □

In the following theorem, we obtain general solution of (2.3) and (2.4) by derivative method.

**Theorem 3.3.** *Let $s : (0,\infty) \to \mathbb{R}$ be a continuously differentiable function with nowhere vanishing derivatives $s'$. Then $s$ yields a solution to the functional equation (2.3) if and only if there exists a nonzero real constant $c$ such that $s(z) = \frac{c}{z^m}$, $z \in (0,\infty)$.*

*Proof.* Differentiating (2.3) with respect to $z$ on both side, we get

$$s'(\sqrt[m]{z^m+w^m})\frac{(z)^{m-1}}{(\sqrt[m]{z^m+w^m})^{m-1}} = \frac{\Big(s'(z)s(w)\Big)\Big[s(z)+s(w)\Big] - \Big(s(z)s(w)\Big)\Big[s'(z)\Big]}{\Big(s(z)+s(w)\Big)^2}. \tag{3.31}$$

Assuming $z, w = z$ in (3.31), we obtain

$$s'(\sqrt[m]{2}\ z) = \frac{1}{2\sqrt[m]{2}}s'(z), \tag{3.32}$$

and setting $z = \sqrt[m]{2}z$ and $w = z$ in (3.31) and making use of (3.17) and (3.32), we get

$$s'(\sqrt[m]{3}\ z) = \frac{1}{(3)\sqrt[m]{3}}s'(z) \tag{3.33}$$

for all $z \in (0,\infty)$. By making use of (3.32) and (3.33), we have

$$s'\Big((\sqrt[m]{2})^k(\sqrt[m]{3})^l\ z\Big) = \frac{1}{2^k(\sqrt[m]{2})^k}\frac{1}{(3)^l(\sqrt[m]{3})^l}s'(z)$$

for all integers $k, l$. We derive its linearity by assuming $\lambda = (\sqrt[m]{2})^k(\sqrt[m]{3})^l$ and $z = 1$,

$$s'(\lambda) = s'(1)\frac{1}{(\lambda)^{m+1}}$$

for $\lambda \in (0,\infty)$. Therefore, there exist real numbers $c \neq 0, d$ such that $s(z) = \frac{c}{z^m} + d$ for $z \in (0,\infty)$. Note that we have $d = 0$ because of the equality $s(\sqrt[m]{2}z) = \frac{1}{2}s(z)$ valid for all positive $z$. This completes the proof. □

**Theorem 3.4.** *Let $s : (0,\infty) \to \mathbb{R}$ be a continuously differentiable function with nowhere vanishing derivatives $s'$. Then $s$ yields a solution to the functional equation (2.4) if and only if there exists a nonzero real constant $c$ such that $s(z) = \frac{c}{z^m}$, $z \in (0,\infty)$.*

*Proof.* Differentiating (2.4) with respect to $z_1$ on both side, we obtain

$$\begin{aligned} s'(\sqrt[m]{z_1^m+z_2^m})\frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m+z_2^m})^{m-1}} + s'(\sqrt[m]{z_1^m+z_{p+1}^m})\frac{(z_1)^{m-1}}{(\sqrt[m]{z_1^m+z_{p+1}^m})^{m-1}} \\ = \frac{s'(z_1)\left(s(z_2)\right)^2}{\left(s(z_1)+s(z_2)\right)^2} + \frac{s'(z_1)\left(s(z_{p+1})\right)^2}{\left(s(z_1)+s(z_{p+1})\right)^2}, \end{aligned} \tag{3.34}$$

and (3.24) implies

$$s'(\sqrt[m]{2}z) = \frac{1}{2\sqrt[m]{2}}s'(z). \tag{3.35}$$

Assuming $z_1 = z$ and $z_2 = z_{p+1} = \sqrt[m]{2}z$ in (3.34) and making use of (3.24) and (3.35), we get

$$s'(\sqrt[m]{3}z) = \frac{1}{3\sqrt[m]{3}}s'(z), \tag{3.36}$$

and from (3.35) and (3.36), we get

$$s'\left((\sqrt[m]{2})^k(\sqrt[m]{3})^l z\right) = \frac{1}{2^k(\sqrt[m]{2})^k}\frac{1}{3^l(\sqrt[m]{3})^l}s'(z),$$

for all integers $k, l$. We derive its linearity by assuming $\lambda = (\sqrt[m]{2})^k(\sqrt[m]{3})^l$ and $z = 1$,

$$s'(\lambda) = s'(1)\frac{1}{(\lambda)^{m+1}}$$

for $\lambda \in (0, \infty)$. Therefore, there exist real numbers $c \neq 0, d$ such that $s(z) = \frac{c}{z^m} + d$ for $z \in (0, \infty)$. Note that we have to have $d = 0$ because of the equality $s(\sqrt[m]{2}z) = \frac{1}{2}s(z)$ exists. This completes the proof. □

## 4. Generalized Hyers-Ulam stability of two dimensional multifarious functional equation

This section deals the generalized Hyers-Ulam stability of two dimensional multifarious functional equation (2.3) in modular spaces by making use of fixed point approach.

**Theorem 4.1.** *Consider a mapping $\eta : M^2 \to [0, +\infty)$ with*

$$\lim_{k\to\infty} \frac{1}{\left(\frac{1}{2}\right)^k}\eta\left((2)^{\frac{k}{m}}z, (2)^{\frac{k}{m}}w\right) = 0, \tag{4.37}$$

*and*

$$\eta\left((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}w\right) \le \frac{1}{2}\psi\eta\{z, w\}, \forall z, w \in M, \tag{4.38}$$

*for $\psi < 1$. Assume that $s : M \to Z_\xi$ fulfills*

$$\xi\left(M_1 s(z, w)\right) \le \eta(z, w), \tag{4.39}$$

*for all $z, w \in M$. In that case, there is a unique reciprocal mapping $R : M \to Z_\xi$ such that*

$$\xi(R(z) - s(z)) \le \frac{1}{\frac{1}{2}(1 - \psi)}\eta(z, z),\ \forall z \in M. \tag{4.40}$$

*Proof.* Assume $N = \xi'$ and define $\xi'$ on $N$ as,

$$\xi'(q) =: \inf\{(2)^{\frac{1}{m}} > 0 : \xi(h(j)) \leq (2)^{\frac{1}{m}}\eta(z,w), \forall z \in M\}.$$

One can easily prove that $\xi'$ is a convex modular with Fatou property on $N$ and $N_{\xi'}$ is $\xi-$complete, see [2]. Consider the function $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma q(z) = \frac{1}{2}q(2^{\frac{1}{m}}z), \tag{4.41}$$

for all $z \in M$ and $q \in N_{\xi'}$. Let $q, r \in N_{\xi'}$ and $(2)^{\frac{1}{m}} \in [0,1]$ with $\xi'(q-r) < (2)^{\frac{1}{m}}$. By definition of $\xi'$, we get

$$\xi(q(z) - r(z)) \leq (2)^{\frac{1}{m}}\eta(z,w), \forall z, w \in M. \tag{4.42}$$

By making use of (4.38) and (4.42), we get

$$\begin{aligned}\xi\left(\frac{q((2)^{\frac{1}{m}}z)}{\frac{1}{2}} - \frac{r((2)^{\frac{1}{m}}z)}{\frac{1}{2}}\right) &\leq \frac{1}{\frac{1}{2}}\xi\left(q((2)^{\frac{1}{m}}z) - r((2)^{\frac{1}{m}}z)\right)\\ &\leq \frac{1}{\frac{1}{2}}(2)^{\frac{1}{m}}\eta\left((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}w\right) \leq (2)^{\frac{1}{m}}\psi\eta\left(z,w\right),\end{aligned}$$

for all $z, w \in M$. In that case, $\sigma$ is a $\xi'-$contraction and (4.39) implies

$$\xi\left(\frac{s((2)^{\frac{1}{m}}z)}{\frac{1}{2}} - s(z)\right) \leq \frac{1}{\frac{1}{2}}\eta(z,z), \forall z \in M, \tag{4.43}$$

and replacing $z$ by $(2)^{\frac{1}{m}}z$ in (4.43), we get

$$\xi\left(\frac{s((2)^{\frac{2}{m}}z)}{\frac{1}{2}} - s((2)^{\frac{1}{m}}z)\right) \leq \frac{\eta((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}z)}{\frac{1}{2}},\ \forall z \in M. \tag{4.44}$$

By making use of (4.43) and (4.44), we get

$$\xi\left(\frac{s((2)^{\frac{2}{m}}z)}{\frac{1}{2^2}} - s(z)\right) \leq \frac{1}{\frac{1}{2^2}}\eta((2)^{\frac{1}{m}}z, (2)^{\frac{1}{m}}z) + \frac{1}{\frac{1}{p}}\eta(z,z), \tag{4.45}$$

for all $z \in M$ and by generalization, we get

$$\begin{aligned}\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}} - s(z)\right) &\leq \sum_{i=1}^{k}\frac{1}{\frac{1}{2^i}}\eta(((2)^{\frac{1}{m}})^{i-1}z, ((2)^{\frac{1}{m}})^{i-1}z)\\ &\leq \frac{1}{\psi\frac{1}{2}}\eta(z,z)\sum_{i=1}^{k}\psi^i\\ &\leq \frac{1}{\frac{1}{2}(1-\psi)}\eta(z,z),\ \forall z \in M. \end{aligned} \tag{4.46}$$

We obtain from (4.46),

$$\begin{aligned}
&\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}\right) \\
&\quad\leq \frac{1}{2}\xi\left(2\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-2s(z)\right)+\frac{1}{2}\xi\left(2\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}-2s(z)\right) \\
&\quad\leq \frac{\kappa}{2}\xi\left(\frac{s((2)^{\frac{k}{m}}z)}{\frac{1}{2^k}}-s(z)\right)+\frac{\kappa}{2}\xi\left(\frac{s((2)^{\frac{u}{m}}z)}{\frac{1}{2^u}}-s(z)\right) \\
&\quad\leq \frac{\kappa}{\frac{1}{2}(1-\psi)}\eta(z,z),\ \forall z\in M
\end{aligned}\tag{4.47}$$

where $k,u\in\mathfrak{N}$. Thus

$$\xi'(\sigma^k s-\sigma^u s)\leq\frac{\kappa}{\frac{1}{2}(1-\psi)},$$

and hence the boundedness of an orbit of $\sigma$ at $s$ is given. $\{\tau^k s\}$ is $\xi'-$converges to $R\in N_{\xi'}$ by Theorem 1.5 in [2]. By $\xi'-$contractivity of $\sigma$, we get

$$\xi'(\sigma^k s-\sigma R)\leq\psi\xi'(\sigma^{k-1}s-R).$$

Letting $k\to\infty$ and by Fatou property of $\xi'$, we get

$$\begin{aligned}
\xi'(\sigma R-R)&\leq\lim_{2\to\infty}\inf\xi'(\sigma R-\sigma^k s)\\
&\leq\psi\lim_{k\to\infty}\inf\xi'(R-\sigma^{k-1}s)=0.
\end{aligned}$$

Hence $R$ is a fixed point of $\sigma$. In (4.39), replacing $(z,W)$ by $\left((2)^{\frac{k}{m}}z,(2)^{\frac{k}{m}}w\right)$, we get

$$\xi\left(\frac{1}{\frac{1}{2^k}}M_1 s((2)^{\frac{k}{m}}z,(2)^{\frac{k}{m}}w)\right)\leq\frac{1}{\frac{1}{2^k}}\eta((2)^{\frac{k}{m}}z,(2)^{\frac{k}{m}}w).\tag{4.48}$$

By Theorems 3.1, 3.3 and letting $k\to\infty$, we obtain that $R$ is a reciprocal mapping and using (4.46), we obtain (4.40). For the uniqueness of $R$, consider another multifarious type reciprocal mapping $T:M\to Z_\xi$ satisfying (4.40). Then $T$ is a fixed point of $\sigma$ such that

$$\xi'(R-T)=\xi'(\sigma R-\sigma T)\leq\psi\xi'(R-T).\tag{4.49}$$

From (4.49), we get $R=T$. This completes the proof. □

The proofs of the following corollaries 4.2 and 4.4 follow from the fact that, each normed space implies a modular space with modular $\xi(z)=\|z\|$.

**Corollary 4.2.** *Assume $\eta$ is a function from $M^2$ to $[0, +\infty)$ for*

$$\lim_{k\to\infty} \frac{1}{\frac{1}{2^k}} \eta\{(2^{\frac{k}{m}})z, (2^{\frac{k}{m}})w\} = 0, \tag{4.50}$$

*and*

$$\eta\{(2^{\frac{1}{m}})z, (2^{\frac{1}{m}})w\} \leq \frac{1}{2}\psi\eta\{z, w\},\ \psi < 1. \tag{4.51}$$

*Assume that $s : M \to Z$ satisfies the condition, for a Banach space $Z$,*

$$\|M_1 s(z, w)\| \leq \eta(z, w), \tag{4.52}$$

*for all $z, w \in M$. Then there is a unique reciprocal mapping $R : M \to Z$ such that*

$$\|R(z) - s(z)\| \leq \frac{\eta(z, z)}{\frac{1}{2}(1 - \psi)}, \tag{4.53}$$

*for all $z \in M$.*

**Theorem 4.3.** *Assume $\eta$ is a function from $M^2$ to $[0, +\infty)$ with*

$$\lim_{k\to\infty} \frac{1}{\kappa^k} \eta\left(\frac{z}{(2)^{\frac{k}{m}}}, \frac{w}{(2)^{\frac{k}{m}}}\right) = 0, \tag{4.54}$$

*and*

$$\eta\left(\frac{z}{(2)^{\frac{1}{m}}}, \frac{w}{(2)^{\frac{1}{m}}}\right) \leq \frac{\psi}{\frac{1}{2}}\rho\{z, w\}, \tag{4.55}$$

*for all $z, w \in M, \psi < 1$. Assume that $s : M \to Z_\xi$ fulfills*

$$\xi\left(M_1 s(z, w)\right) \leq \eta(z, w). \tag{4.56}$$

*Then there is a unique reciprocal mapping $R : M \to Z_\xi$ such that*

$$\xi(R(z) - s(z)) \leq \frac{p\psi}{1 - \psi}\eta(z, z),\ \forall z \in M. \tag{4.57}$$

*Proof.* Replacing $z$ by $\frac{z}{(2)^{\frac{1}{m}}}$ in (4.41) of Theorem 4.1 and using a similar method to that of Theorem 4.1, we complete the proof. □

**Corollary 4.4.** *Assume $\eta$ is a function from $M^2$ to $[0, +\infty)$ with*

$$\lim_{k\to\infty} \frac{1}{\sigma^k} \eta\left(\frac{z}{(2)^{\frac{k}{m}}}, \frac{w}{(2)^{\frac{k}{m}}}\right) = 0, \tag{4.58}$$

*and*

$$\eta\left(\frac{z}{(2)^{\frac{1}{m}}}, \frac{w}{(2)^{\frac{1}{m}}}\right) \leq \frac{\psi}{\frac{1}{2}}\eta\{z, w\}, \psi < 1. \tag{4.59}$$

*Assume that $s : M \to Z$ fulfills*

$$\|M_1 s(z, w)\| \leq \eta(z, w), \tag{4.60}$$

*for all $z, w \in M$. Then there is a unique reciprocal mapping $R : M \to Z$ such that*

$$\|R(z) - s(z)\| \leq \frac{p\psi}{1 - \psi} \eta(z, z), \tag{4.61}$$

*for all $z \in M$.*

Using Corollaries 4.2 and 4.4, we obtain the following corollaries.

**Corollary 4.5.** *Assume $\eta$ is a function from $M^2$ to $[0, +\infty)$, $Z$ is a Banach space and $\epsilon \geq 0$ is a real number such that*

$$\lim_{k\to\infty} \frac{1}{\frac{1}{2^k}} \eta\{(2)^{\frac{k}{m}} z, (2)^{\frac{k}{m}} w\} = 0, \tag{4.62}$$

*and*

$$\eta\{(2)^{\frac{1}{m}} z, (2)^{\frac{1}{m}} w\} \leq \frac{1}{2} \psi \eta\{z, w\}, \psi < 1. \tag{4.63}$$

*Assume that $s : M \to Z$ fulfills*

$$\|M_1 s(z, w)\| \leq \epsilon, \tag{4.64}$$

*for all $z, w \in M$. Then there is a unique reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((2)^{\frac{k}{m}} z\right)}{\frac{1}{2^k}}$, such that*

$$\|R(z) - s(z)\| \leq 2\epsilon, \tag{4.65}$$

*for all $z \in M$.*

*Proof.* Assume that $\eta(z, w) = \epsilon$ for all $z, w \in Z$. The Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq 2\epsilon,$$

for all $z \in Z$ and making use of Corollary 4.4, we get

$$\|R(z) - s(z)\| \leq 2\epsilon,$$

for all $z \in Z$. □

**Corollary 4.6.** *Assume that $s : M \to X$ fulfills the following, for a linear space $M$ and a Banach space $Z$ , respectively,*

$$\|M_1 s(z, w)\| \leq \epsilon \left(\|z\|^u + \|w\|^u\right), \tag{4.66}$$

*for all $z, w \in M$ with $0 \leq u < -m$ or $u > -m$ for some $\epsilon \geq 0$. Then there is a reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((2)^{\frac{k}{m}} z\right)}{\frac{1}{2^k}}$, such that*

$$\|R(z) - s(z)\| \leq \frac{4\epsilon}{\left|1 - 2^{\frac{m+u}{m}}\right|} \|z\|^u, \ \ \forall z \in M. \tag{4.67}$$

*Proof.* If we choose $\eta(z, w) = \epsilon\big(\|z\|^u + \|w\|^u\big)$, then Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq \frac{4\epsilon}{1 - 2^{\frac{m+u}{m}}} \|z\|^u,$$

for all $z \in Z$ and $u < -m$. Using Corollary-4.4, we obtain

$$\|R(z) - s(z)\| \leq \frac{4\epsilon}{2^{\frac{m+u}{m}} - 1} \|z\|^u,$$

for all $z \in Z$ and $u > -m$. □

The following is an example to elucidate (2.3), which is not stable for $u = -m$ in Corollary 4.6.

*Example* 4.7. Define $\phi : \mathbb{R} \to \mathbb{R}$ with $a > 0$ as

$$\phi(z) = \begin{cases} \frac{a}{z^m}, & \text{if } z \in (1, \infty) \\ a\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}z)}{2^{mk}}$. Then $s$ fulfills

$$\|M_1 s(z, w)\| \leq \frac{a 2^{2m}(3)}{2(2^m - 1)} \times \left(\left|\frac{1}{z^m}\right| + \left|\frac{1}{w^m}\right|\right) \tag{4.68}$$

for all $z_1, w \in \mathbb{R}$. In that case there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ as

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|, \ \beta > 0, \forall z \in \mathbb{R}. \tag{4.69}$$

**Corollary 4.8.** *Let $s : Z_1 \to Z_2$ be a mapping. Assume that there exists $\epsilon \geq 0$ such that*

$$\|M_1 s(z, w)\| \leq \epsilon \left(\|z\|^{\frac{u}{2}} \|w\|^{\frac{u}{2}}\right)$$

*for all $z, w \in Z_1$. Then there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ satisfying (2.3) and*

$$\|r(z) - s(z)\| \leq \begin{cases} \frac{2\epsilon}{1-2^{\frac{m+u}{m}}}\|z\|^u & \text{for } u < -m \\ \frac{2\epsilon}{2^{\frac{m+u}{m}}-1}\|z\|^u & \text{for } u > -m \end{cases}$$

*for all $z \in Z_1$.*

*Proof.* Replace $\eta(z, w)$ by $\epsilon\left(\|z\|^{\frac{u}{2}}\|w\|^{\frac{u}{2}}\right)$. Then Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{1 - 2^{\frac{m+u}{m}}}\|z\|^2,$$

for $u < -m$ and for all $z \in Z_1$ and making use of Corollary-4.4, we get

$$\|R(z) - s(z)\| \leq \frac{2\epsilon}{2^{\frac{m+u}{m}} - 1}\|z\|^2, \tag{4.70}$$

for $u > -m$ and for all $z \in Z_1$. □

**Corollary 4.9.** *Let $\epsilon > 0$ and $\alpha < -\frac{m}{2}$ or $\alpha > -\frac{m}{2}$ be real numbers, and $s : Z_1 \to Z_2$ be a mapping satisfying the functional inequality*

$$\|M_1 s(z, w)\| \leq \epsilon \left\{\|z\|^{2\alpha} + \|w\|^{2\alpha} + (\|z\|^\alpha\|w\|^\alpha)\right\}.$$

*Then e there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ fulfilling (2.3) and*

$$\|R(z) - s(z)\| \leq \begin{cases} \frac{6\epsilon}{1-2^{\frac{2\alpha+m}{m}}}\|z\|^{2\alpha} & \text{for } \alpha < -\frac{m}{2} \\ \frac{6\epsilon}{2^{\frac{2\alpha+m}{m}}-1}\|z\|^{2\alpha} & \text{for } \alpha > -\frac{m}{2} \end{cases}$$

*for all $z \in Z_1$.*

*Proof.* Set $\epsilon\Big\{\|z\|^{2\alpha} + \|w\|^{2\alpha} + (\|z\|^\alpha\|w\|^\alpha)\Big\}$ instead of $\eta(z, w)$. Then Corollary 4.4 implies

$$\|R(z) - s(z)\| \leq \frac{6\epsilon}{1 - 2^{\frac{2\alpha+m}{m}}}\|z\|^{2\alpha},$$

for $\alpha < -\frac{m}{2}$ and for all $z \in Z_1$ and making use of Corollary-4.4, we get

$$\|R(z) - s(z)\| \leq \frac{6\epsilon}{2^{\frac{2\alpha+m}{m}} - 1}\|z\|^{2\alpha},$$

for $\alpha > -\frac{m}{2}$ and for all $z \in Z$. □

The following is an example to elucidate (2.3), which is not stable for $\alpha = -\frac{m}{2}$ in Corollary 4.9.

*Example* 4.10. Define $\phi : \mathbb{R} \to \mathbb{R}$ with a constant $l > 0$ as

$$\phi(z) = \begin{cases} \frac{l}{z^m}, & \text{if } z \in (1, \infty) \\ l\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(2^{-k}z)}{2^{mk}}$. Then $s$ fulfills

$$\|M_1 s(z,w)\| \leq \frac{a2^{2m}(3)}{2(2^m-1)} \times \left( \left|\frac{1}{z^m}\right| + \left|\frac{1}{w^m}\right| + \left|\frac{1}{z^m}\right| \left|\frac{1}{w^m}\right| \right) \tag{4.71}$$

for all $z, w \in \mathbb{R}$. In that case, there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ as

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|, \ \beta > 0, \forall z \in \mathbb{R}. \tag{4.72}$$

## 5. Generalized Hyers-Ulam stability of three dimensional multifarious functional equation

This section deals the Hyers-Ulam stability of the three dimensional multifarious functional equation (2.4) in modular spaces by making use of fixed point approach.

**Theorem 5.1.** *Consider a mapping* $\eta : M^2 \to [0, +\infty)$ *with*

$$\lim_{k\to\infty} \frac{1}{\left(\frac{1}{3}\right)^k} \eta\left((3)^{\frac{k}{m}} z_1, (3)^{\frac{k}{m}} z_2, (3)^{\frac{k}{m}} z_3\right) = 0, \tag{5.73}$$

*and*

$$\eta\left((3)^{\frac{1}{m}} z_1, (3)^{\frac{1}{m}} z_2, (3)^{\frac{1}{m}} z_3\right) \leq \frac{1}{3}\psi\eta\{z_1, z_2, z_3\}, \forall z_1, z_2, z_3 \in M, \tag{5.74}$$

*for* $\psi < 1$. *Assume that* $s : M \to Z_\xi$ *fulfills*

$$\xi\left(M_1 s(z_1, z_2, z_3)\right) \leq \eta(z_1, z_2, z_3), \tag{5.75}$$

*for all* $z_1, z_2, z_3 \in M$. *Then there is a unique reciprocal mapping* $R : M \to Z_\xi$ *such that*

$$\xi(R(z) - s(z)) \leq \frac{1}{\frac{1}{3}(1-\psi)} \eta(z, z, z), \ \forall z \in M. \tag{5.76}$$

*Proof.* Assume $N = \xi'$ and define $\xi'$ on $N$ by

$$\xi'(q) =: \inf\{(3)^{\frac{1}{m}} > 0 : \xi(h(j)) \leq (3)^{\frac{1}{m}} \eta(z_1, z_2, z_3), \forall z \in M\}.$$

One can easily prove that $\xi'$ is a convex modular with Fatou property on $N$ and $N_{\xi'}$ is $\xi-$complete, see [2]. Consider the mapping $\sigma : N_{\xi'} \to N_{\xi'}$ defined by

$$\sigma q(z) = \frac{1}{3} q(3^{\frac{1}{m}} z), \tag{5.77}$$

for all $z \in M$ and $q \in N_{\xi'}$. Let $q, r \in N_{\xi'}$ and $(3)^{\frac{1}{m}} \in [0,1]$ with $\xi'(q-r) < (3)^{\frac{1}{m}}$. By definition of $\xi'$, we get

$$\xi(q(z) - r(z)) \leq (3)^{\frac{1}{m}}\eta(z_1, z_2, z_3), \forall z_1, z_2, z_3 \in M. \tag{5.78}$$

By making use of (5.74) and (5.78), we have

$$\begin{aligned}\xi\left(\frac{q((3)^{\frac{1}{m}}z)}{\frac{1}{3}} - \frac{r((3)^{\frac{1}{m}}z)}{\frac{1}{3}}\right) &\leq \frac{1}{\frac{1}{3}}\xi\left(q((3)^{\frac{1}{m}}z) - r((3)^{\frac{1}{m}}z)\right)\\ &\leq \frac{1}{\frac{1}{3}}(3)^{\frac{1}{m}}\eta\left((3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z_2, (3)^{\frac{1}{m}}z_3\right) \leq (3)^{\frac{1}{m}}\psi\eta\left(z_1, z_2, z_3\right),\end{aligned}$$

for all $z_1, z_2, z_3 \in M$. Then $\sigma$ is a $\xi'-$contraction and (5.75) implies

$$\xi\left(\frac{s((3)^{\frac{1}{m}}z)}{\frac{1}{3}} - s(z)\right) \leq \frac{1}{\frac{1}{3}}\eta(z, z, z), \forall z \in M, \tag{5.79}$$

and replacing $z$ by $(3)^{\frac{1}{m}}z$ in (5.79), we get

$$\xi\left(\frac{s((3)^{\frac{2}{m}}z)}{\frac{1}{3}} - s((3)^{\frac{1}{m}}z)\right) \leq \frac{\eta((3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z, \ldots, (3)^{\frac{1}{m}}z)}{\frac{1}{3}},\ \forall z \in M \tag{5.80}$$

and by making use of (5.79) and (5.80), we get

$$\xi\left(\frac{s((3)^{\frac{2}{m}}z)}{\frac{1}{9}} - s(z)\right) \leq \frac{1}{\frac{1}{9}}\eta((3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z, (3)^{\frac{1}{m}}z) + \frac{1}{\frac{1}{3}}\eta(z, z, z),$$

for all $z \in M$ and by generalization, we get

$$\begin{aligned}\xi\left(\frac{s((3)^{\frac{k}{m}}z)}{\frac{1}{3^k}} - s(z)\right) &\leq \sum_{i=1}^{k}\frac{1}{\frac{1}{3^i}}\eta(((3)^{\frac{1}{m}})^{i-1}z, ((3)^{\frac{1}{m}})^{i-1}z, ((3)^{\frac{1}{m}})^{i-1}z)\\ &\leq \frac{1}{\psi\frac{1}{3}}\eta(z, z, z)\sum_{i=1}^{k}\psi^i\\ &\leq \frac{1}{\frac{1}{3}(1-\psi)}\eta(z, z, z),\ \forall z \in M.\end{aligned} \tag{5.81}$$

We obtain from (5.81),

$$\begin{aligned}
&\xi\left(\frac{s((3)^{\frac{k}{m}}3)}{\frac{1}{3^k}}-\frac{s((3)^{\frac{u}{m}}z)}{\frac{1}{3^u}}\right)\\
&\quad\le \frac{1}{2}\xi\left(2\frac{s((3)^{\frac{k}{m}}z)}{\frac{1}{3^k}}-2s(z)\right)+\frac{1}{2}\xi\left(2\frac{s((3)^{\frac{u}{m}}z)}{\frac{1}{3^u}}-2s(z)\right)\\
&\quad\le \frac{\kappa}{2}\xi\left(\frac{s((3)^{\frac{k}{m}}z)}{\frac{1}{3^k}}-s(z)\right)+\frac{\kappa}{2}\xi\left(\frac{s((3)^{\frac{u}{m}}z)}{\frac{1}{3^u}}-s(z)\right)\\
&\quad\le \frac{\kappa}{\frac{1}{3}(1-\psi)}\eta(z,z,z),\ \forall z\in M
\end{aligned}$$

where $k,u\in\mathfrak{N}$. Thus

$$\xi'(\sigma^k s-\sigma^u s)\le\frac{\kappa}{\frac{1}{3}(1-\psi)},$$

and hence the boundedness of an orbit of $\sigma$ at $s$ is given. So $\{\tau^k s\}$ is $\xi'-$convergent to $R\in N_{\xi'}$ by Theorem 1.5 in [2]. By $\xi'-$contractivity of $\sigma$, we get

$$\xi'(\sigma^k s-\sigma R)\le\psi\xi'(\sigma^{k-1}s-R).$$

Taking $k\to\infty$ and by Fatou property of $\xi'$, we get

$$\xi'(\sigma R-R)\le\lim_{k\to\infty}\inf\xi'(\sigma R-\sigma^k s)\le\psi\lim_{k\to\infty}\inf\xi'(R-\sigma^{k-1}s)=0.$$

Hence $R$ is a fixed point of $\sigma$. In (5.75), replacing $(z_1,z_2,z_3)$ by $\left((3)^{\frac{k}{m}}z_1,(3)^{\frac{k}{m}}z_2,(3)^{\frac{k}{m}}z_3\right)$, we get

$$\xi\left(\frac{1}{\frac{1}{3^k}}M_1 s((3)^{\frac{k}{m}}z_1,(3)^{\frac{k}{m}}z_2,(3)^{\frac{k}{m}}z_3)\right)\le\frac{1}{\frac{1}{3^k}}\eta((3)^{\frac{k}{m}}z_1,(3)^{\frac{k}{m}}z_2,(3)^{\frac{k}{m}}z_3).$$

By Theorems 3.1, 3.3 and taking $k\to\infty$, we obtain that $R$ is a reciprocal mapping and using (5.81), we have (5.76). For the uniqueness of $R$, consider another multi-type reciprocal mapping $T:M\to Z_\xi$ satisfying (5.76). Then $T$ is a fixed point of $\sigma$ such that

$$\xi'(R-T)=\xi'(\sigma R-\sigma T)\le\psi\xi'(R-T). \tag{5.82}$$

From (5.82), we get $R=T$. This completes the proof. □

The proofs of Corollaries 5.2 and 5.4 follows from the fact that every normed space is a modular space of modular $\xi(z)=\|z\|$.

**Corollary 5.2.** *Assume $\eta$ is a function from $M^2$ to $[0,+\infty)$ such that*

$$\lim_{k\to\infty}\frac{1}{\frac{1}{3^k}}\eta\{(3^{\frac{k}{m}})z_1,(3^{\frac{k}{m}})z_2,(3^{\frac{k}{m}})z_3\}=0,$$

*and*

$$\eta\{(3^{\frac{1}{m}})z_1,(3^{\frac{1}{m}})z_2,(3^{\frac{1}{m}})z_3\}\leq\frac{1}{3}\psi\eta\{z_1,z_2,z_3\},\ \psi<1.$$

*Assume that* $s:M\to Z$ *satisfies the following, for a Banach space* $Z$*,*

$$\|M_1 s(z_1,z_2,z_3)\|\leq\eta(z_1,z_2,z_3),$$

*for all* $z_1,z_2,z_3\in M$*. Then there is a unique reciprocal mapping* $R:M\to Z$ *such that*

$$\|R(z)-s(z)\|\leq\frac{\eta(z,z,z)}{\frac{1}{3}(1-\psi)},$$

*for all* $z\in M$*.*

**Theorem 5.3.** *Assume* $\eta$ *is a function from* $M^2$ *to* $[0,+\infty)$ *with*

$$\lim_{k\to\infty}\frac{1}{\kappa^k}\eta\left(\frac{z_1}{(3)^{\frac{k}{m}}},\frac{z_2}{(3)^{\frac{k}{m}}},\frac{z_3}{(3)^{\frac{k}{m}}}\right)=0,$$

*and*

$$\eta\left(\frac{z_1}{(3)^{\frac{1}{m}}},\frac{z_2}{(3)^{\frac{1}{m}}},\frac{z_3}{(3)^{\frac{1}{m}}}\right)\leq\frac{\psi}{\frac{1}{3}}\rho\{z_1,z_2,z_3\},$$

*for all* $z_1,z_2,z_3\in M,\psi<1$*. Assume that* $s:M\to Z_\xi$ *fulfills*

$$\xi\left(M_1 s(z_1,z_2,z_3)\right)\leq\eta(z_1,z_2,z_3).$$

*Then there is a unique reciprocal mapping* $R:M\to Z_\xi$ *such that*

$$\xi(R(z)-s(z))\leq\frac{p\psi}{1-\psi}\eta(z,z,z),\ \forall z\in M.$$

*Proof.* Replacing $z$ by $\frac{z}{(3)^{\frac{1}{m}}}$ in (5.77) of Theorem 5.1 and by a similar method to that of Theorem 5.1, we complete the proof. □

**Corollary 5.4.** *Assume* $\eta$ *is a function from* $M^2$ *to* $[0,+\infty)$ *with*

$$\lim_{k\to\infty}\frac{1}{\sigma^k}\eta\left(\frac{z_1}{(3)^{\frac{k}{m}}},\frac{z_2}{(3)^{\frac{k}{m}}},\frac{z_3}{(3)^{\frac{k}{m}}}\right)=0,$$

*and*

$$\eta\left(\frac{z_1}{(3)^{\frac{1}{m}}},\frac{z_2}{(3)^{\frac{1}{m}}},\frac{z_3}{(3)^{\frac{1}{m}}}\right)\leq\frac{\psi}{\frac{1}{3}}\eta\{z_1,z_2,z_3\},\psi<1.$$

*Assume that* $s:M\to Z$ *fulfills*

$$\|M_1 s(z_1,z_2,z_3)\|\leq\eta(z_1,z_2,z_3),$$

*for all $z_1, z_2, z_3 \in M$. Then there is a unique reciprocal mapping $R : M \to Z$ such that*

$$\|R(z) - s(z)\| \leq \frac{p\psi}{1-\psi}\eta(z, z, z),$$

*for all $z \in M$.*

Using Corollaries 5.2 and 5.4, we obtain the following corollaries.

**Corollary 5.5.** *Assume $\eta$ is a function from $M^2$ to $[0, +\infty)$, $Z$ is a Banach space and $\epsilon \geq 0$ is a real number such that*

$$\lim_{k\to\infty} \frac{1}{\frac{1}{3^k}} \eta\{(3)^{\frac{k}{m}} z_1, (3)^{\frac{k}{m}} z_2, (3)^{\frac{k}{m}} z_3\} = 0,$$

*and*

$$\eta\{(3)^{\frac{1}{m}} z_1, (3)^{\frac{1}{m}} z_2, (3)^{\frac{1}{m}} z_3\} \leq \frac{1}{3}\psi\eta\{z_1, z_2, z_3\}, \psi < 1.$$

*Assume that $s : M \to Z$ fulfills*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon,$$

*for all $z_1, z_2, z_3 \in M$. Then there is a unique reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((3)^{\frac{k}{m}} z\right)}{\frac{1}{3^k}}$, such that*

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{2},$$

*for all $z \in M$.*

*Proof.* Assume that $\eta(z_1, z_2, z_3) = \epsilon$ for all $z_1, z_2, z_3 \in Z$. Then Corollary 5.2 implies

$$\|R(z) - s(z)\| \leq \frac{p\epsilon}{2},$$

for all $z \in Z$ and $p \neq 0, \pm 1$ and making use of Corollary 5.4, we get

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{2},$$

for all $z \in Z$. □

**Corollary 5.6.** *If $s : M \to X$ fulfills the following inequality, for a linear space $M$ and a Banach space $Z$, respectively,*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon \left(\|z_1\|^u + \|z_2\|^u + \|x_3\|^u\right),$$

*for all $z_1, z_2, z_3 \in M$ with $0 \leq u < -m$ or $u > -m$ for some $\epsilon \geq 0$. Then there is a reciprocal mapping $R : M \to Z$, defined by $R(z) = \lim_{k\to\infty} \frac{s\left((3)^{\frac{k}{m}} z\right)}{\frac{1}{3^k}}$, such that*

$$\|R(z) - s(z)\| \leq \frac{9\epsilon}{\left|1 - 3^{\frac{m+u}{m}}\right|} \|z\|^u, \ \ \forall z \in M.$$

*Proof.* If we choose $\eta(z_1, z_2, z_3) = \epsilon\big(\|z_1\|^u + \|z_2\|^u + \|z_3\|^u\big)$, then Corollary 4.2 implies

$$\|R(z) - s(z)\| \leq \frac{9\epsilon}{1 - 3^{\frac{m+u}{m}}} \|z\|^u,$$

for all $z \in Z$ and $u < -m$. Using Corollary 4.4, we obtain

$$\|R(z) - s(z)\| \leq \frac{9\epsilon}{3^{\frac{m+u}{m}} - 1} \|z\|^u,$$

for all $z \in Z$ and $u > -m$. □

The following is an example to elucidate (2.4), which is not stable for $u = -m$ in Corollary 5.6.

*Example* 5.7. Define $\phi : \mathbb{R} \to \mathbb{R}$ with $a > 0$ as

$$\phi(z) = \begin{cases} \frac{a}{z^m}, & \text{if } z \in (1, \infty) \\ a\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(3^{-k} z)}{3^{mk}}$. Then $s$ fulfills

$$\|M_1 s(z_1, z_2, z_3)\| \leq \frac{a 3^{2m}(4)}{3(3^m - 1)} \times \left( \left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \left|\frac{1}{z_3^m}\right| \right)$$

for all $z_1, z_2, z_3 \in \mathbb{R}$. In that case, there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ such that

$$|s(z) - R(z)| \leq \beta \left|\frac{1}{z^m}\right|, \beta > 0, \forall z \in \mathbb{R}.$$

**Corollary 5.8.** *Assume $s : Z_1 \to Z_2$ is a mapping. Assume that there exists $\epsilon \geq 0$ such that*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon \left( \|z_1\|^{\frac{u}{3}} \|z_2\|^{\frac{u}{3}} \|z_3\|^{\frac{u}{p}} \right)$$

*for all $z_1, z_2, z_3 \in Z_1$. Then there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ fulfilling (2.4) and*

$$\|r(z) - s(z)\| \leq \begin{cases} \frac{3\epsilon}{1 - 3^{\frac{m+u}{m}}} \|z\|^u & \text{for } u < -m \\ \frac{3\epsilon}{3^{\frac{m+u}{m}} - 1} \|z\|^u & \text{for } u > -m \end{cases}$$

*for all $z \in Z_1$.*

*Proof.* Replace $\eta(z_1, z_2, z_3)$ by $\epsilon\left(\|z_1\|^{\frac{u}{3}}\|z_2\|^{\frac{u}{3}}\|z_3\|^{\frac{u}{3}}\right)$. Then Corollary 5.2 implies

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{1 - 3^{\frac{m+u}{m}}}\|z\|^3,$$

for $u < -m$ and for all $z \in Z_1$ and making use of Corollary 5.4, we get

$$\|R(z) - s(z)\| \leq \frac{3\epsilon}{3^{\frac{m+u}{m}} - 1}\|z\|^3,$$

for $u > -m$ and for all $z \in Z_1$. □

**Corollary 5.9.** *Let $\epsilon > 0$ and $\alpha < -\frac{m}{3}$ or $\alpha > -\frac{m}{3}$ be real numbers, and $s : Z_1 \to Z_2$ be a mapping satisfying the functional inequality*

$$\|M_1 s(z_1, z_2, z_3)\| \leq \epsilon\left\{\|z_1\|^{3\alpha} + \|z_2\|^{3\alpha} + \|z_3\|^{3\alpha} + (\|z_1\|^{\alpha}\|z_2\|^{\alpha}\|z_3\|^{\alpha})\right\}.$$

*Then there exists a unique reciprocal mapping $R : Z_1 \to Z_2$ fulfilling (2.4) and*

$$\|R(z) - s(z)\| \leq \begin{cases} \frac{12\epsilon}{1-3^{\frac{3\alpha+m}{m}}}\|z\|^{3\alpha} & \text{for } \alpha < -\frac{m}{3} \\ \frac{12\epsilon}{3^{\frac{3\alpha+m}{m}}-1}\|z\|^{3\alpha} & \text{for } \alpha > -\frac{m}{3} \end{cases}$$

*for all $z \in Z_1$.*

*Proof.* Replace $\eta(z_1, z_2, z_3)$ by $\epsilon\Big\{\|z_1\|^{3\alpha} + \|z_2\|^{3\alpha} + \|z_3\|^{3\alpha} + (\|z_1\|^{\alpha}\|z_2\|^{\alpha}\|z_3\|^{\alpha})\Big\}$. Then Corollary 5.4 implies

$$\|R(z) - s(z)\| \leq \frac{12\epsilon}{1 - 3^{\frac{3\alpha+m}{m}}}\|z\|^{3\alpha},$$

for $\alpha < -\frac{m}{3}$ and for all $z \in Z_1$ and making use of Corollary 5.4, we get

$$\|R(z) - s(z)\| \leq \frac{12\epsilon}{3^{\frac{3\alpha+m}{m}} - 1}\|z\|^{3\alpha},$$

for $\alpha > -\frac{m}{3}$ and for all $z \in Z$. □

The following is an example to elucidate (2.4), which is not stable for $\alpha = -\frac{m}{3}$ in Corollary 5.9.

*Example* 5.10. Define $\phi : \mathbb{R} \to \mathbb{R}$ with a constant $l > 0$ as

$$\phi(z) = \begin{cases} \frac{l}{z^m}, & \text{if } z \in (1, \infty) \\ l\ , & \text{otherwise} \end{cases}$$

and a function $s : \mathbb{R} \to \mathbb{R}$ by $s(z) = \sum_{k=0}^{\infty} \frac{\phi(3^{-k}z)}{3^{mk}}$. Then $s$ fulfills

$$\|M_1 s(z_1, z_2, z_3)\| \leq \frac{a3^{2m}(4)}{3(3^m - 1)} \times \left(\left|\frac{1}{z_1^m}\right| + \left|\frac{1}{z_2^m}\right| + \left|\frac{1}{z_3^m}\right| + \left|\frac{1}{z_1^m}\right|\left|\frac{1}{z_2^m}\right|\left|\frac{1}{z_3^m}\right|\right)$$

for all $z_1, z_2, z_3 \in \mathbb{R}$. In that case, there does not exist a reciprocal mapping $R : \mathbb{R} \to \mathbb{R}$ such that

$$|s(z) - R(z)| \leq \beta \left| \frac{1}{z^m} \right|, \ \beta > 0, \forall z \in \mathbb{R}.$$

## 6. Conclusion

In this work, we introduced the new generalized multifarious type radical reciprocal functional equations combining three classical Pythagorean means arithmetic, geometric and harmonic. Importantly, we obtained their general solution and stabilities related to Ulam problem with suitable counter examples in modular spaces by using fixed point approach. Furthermore, we illustrated their geometrical interpretation.

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