# TERNARY HOM-DERIVATION-HOMOMORPHISM

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Abstract. In this paper, we introduce and solve the following additive-additive $(s,t)$-functional inequality

$$\begin{aligned} &\|g(x+y+z)-g(x)-g(y)-g(z)\| \\ &+\|h(x+y+z)+h(x-2y+z)+h(x+y-2z)-3h(x)\| \\ &\le \left\|s\left(3g\left(\frac{x+y+z}{3}\right)-g(x)-g(y)-g(z)\right)\right\| \\ &+\left\|t\left(3h\left(\frac{x+y+z}{3}\right)+h(x-2y+z)+h(x+y-2z)-3h(x)\right)\right\|, \end{aligned} \tag{0.1}$$

where $s$ and $t$ are fixed nonzero complex numbers with $|s|<1$ and $|t|<1$. Using the direct method and the fixed point method, we prove the Hyers-Ulam stability of ternary hom-derivations and ternary homomorphisms in $C^*$-ternary algebras, associated to the additive-additive $(s,t)$-functional inequality (0.1) and the following functional inequality

$$\begin{aligned} &\|g([x,y,z])-[g(x),h(y),h(z)]-[h(x),g(y),h(z)]-[h(x),h(y),g(z)]\| \\ &+\|h([x,y,z])-[h(x),h(y),h(z)]\| \le \varphi(x,y,z). \end{aligned} \tag{0.2}$$

## 1. Introduction and preliminaries

A $C^*$-ternary algebra is a complex Banach space $A$, equipped with a ternary product $(x,y,z)\mapsto[x,y,z]$ of $A^3$ into $A$, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\le\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see [33]).

Let $A$ be a $C^*$-ternary algebra. A $\mathbb{C}$-linear mapping $g:A\to A$ is a ternary derivation if $g:A\to A$ satisfies

$$g([x,y,z])=[g(x),y,z]+[x,g(y),z]+[x,y,g(z)]$$

for all $x,y,z\in A$, and a $\mathbb{C}$-linear mapping $h:A\to A$ is a ternary homomorphism if $h:A\to A$ satisfies

$$h([x,y,z])=[h(x),h(y),h(z)]$$

for all $x,y,z\in A$ (see [1, 18]). For a ternary derivation $g:A\to A$ and a ternary homomorphism $h:A\to A$,

$$g\circ h([x,y,z])=[g\circ h(x),h(y),h(z)]+[h(x),g\circ h(y),h(z)]+[h(x),h(y),g\circ h(z)]$$

for all $x,y,z\in A$. The $\mathbb{C}$-linear mapping $g\circ h:A\to A$ is called a ternary hom-derivation, which is defined as follows:

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**Definition 1.1.** Let $A$ be a $C^*$-ternary algebra and $H : A \to A$ be ternary homomorphism. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a *ternary hom-derivation* in $A$ if $D : A \to A$ satisfies

$$D([x,y,z]) = [D(x), H(y), H(z)] + [H(x), D(y), H(z)] + [H(x), H(y), D(z)]$$

for all $x, y, z \in A$.

The stability problem of functional equations originated from a question of Ulam [31] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [26] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. Park [21, 22, 24] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [8, 9, 10, 11, 12, 14, 19, 27, 28, 29, 30, 32]).

We recall a fundamental result in fixed point theory.

**Theorem 1.2.** [3, 6] *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 7, 23, 25]).

In this paper, we solve the additive-additive $(s,t)$-functional inequality (0.1). Furthermore, we investigate ternary hom-derivations and ternary homomorphisms in $C^*$-ternary algebras associated to the additive-additive $(s,t)$-functional inequality (0.1) and the functional inequality (0.2) by using the direct method and by the fixed point method.

Throughout this paper, assume that $A$ is a $C^*$-ternary algebra and that $s$ and $t$ are fixed nonzero complex numbers with $|s| < 1$ and $|t| < 1$.

## 2. Stability of additive-additive $(s,t)$-functional inequality (0.1): a direct method

In this section, we solve and investigate the additive-additive $(s,t)$-functional inequality (0.1) in $C^*$-ternary algebras.

**Lemma 2.1.** *If mappings $g, h : A \to A$ satisfy $g(0) = h(0) = 0$ and*

$$\begin{aligned} &\|g(x+y+z) - g(x) - g(y) - g(z)\| \\ &+\|h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x)\| \\ &\le \left\|s\left(3g\left(\frac{x+y+z}{3}\right) - g(x) - g(y) - g(z)\right)\right\| \\ &+ \left\|t\left(3h\left(\frac{x+y+z}{3}\right) + h(x-2y+z) + h(x+y-2z) - 3h(x)\right)\right\| \end{aligned} \tag{2.1}$$

*for all $x, y, z \in A$, then the mappings $g, h : A \to A$ are additive.*

*Proof.* Letting $x = y = z$ in (2.1), we get

$$\|g(3x) - 3g(x)\| + \|h(3x) - 3h(x)\| \le 0$$

for all $x \in A$. So $g(3x) = 3g(x)$ and $h(3x) = 3h(x)$ for all $x \in A$. It follows from (2.1) that

$$\begin{aligned} &\|g(x+y+z) - g(x) - g(y) - g(z)\| \\ &+\|h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x)\| \\ &\le \|s(g(x+y+z) - g(x) - g(y) - g(z))\| \\ &+ \|t(h(x+y+z) + h(x-2y+z) + h(x+y-2z) - 3h(x))\| \end{aligned}$$

for all $x, y, z \in A$. Thus

$$g(x+y+z) = g(x) + g(y) + g(z),$$

$$h(x+y+z) + h(x-2y+z) + h(x+y-2z) = 3h(x)$$

for all $x, y, z \in A$, since $|s| < 1$ and $|t| < 1$. So the mappings $g, h : A \to A$ are additive. □

**Lemma 2.2.** [20, Theorem 2.1] *Let $f : A \to A$ be an additive mapping such that*

$$f(\lambda a) = \lambda f(a)$$

*for all $\lambda \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $a \in A$. Then the mapping $f : A \to A$ is $\mathbb{C}$-linear.*

Using the direct method, we prove the Hyers-Ulam stability of pairs of ternary hom-derivations and ternary homomorphisms in $C^*$-ternary algebras associated to the additive-additive $(s,t)$-functional inequality (2.1).

**Theorem 2.3.** *Let $\varphi : A^3 \to [0, \infty)$ be a function such that*

$$\sum_{j=1}^{\infty} 27^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) < \infty \tag{2.2}$$

*for all $x, y, z \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and*

$$\begin{aligned} &\|g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \\ &+\|h(\lambda(x+y+z)) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\| \\ &\le \left\| s\left(3g\left(\lambda\frac{x+y+z}{3}\right) - \lambda(g(x)+g(y)+g(z))\right)\right\| \\ &+\left\| t\left(3h\left(\lambda\frac{x+y+z}{3}\right) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\right)\right\| + \varphi(x,y,z) \end{aligned} \tag{2.3}$$

*for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. If the mappings $g, h : A \to A$ satisfy*

$$\begin{aligned} &\|g([x,y,z]) - [g(x),h(y),h(z)] - [h(x),g(y),h(z)] - [h(x),h(y),g(z)]\| \\ &+\|h([x,y,z]) - [h(x),h(y),h(z)]\| \le \varphi(x,y,z) \end{aligned} \tag{2.4}$$

*for all $x, y, z \in A$, then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \sum_{j=1}^{\infty} 3^{j-1}\varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) \tag{2.5}$$

*for all $x \in A$.*

*Proof.* Letting $\lambda = 1$ and $y = z = x$ in (2.3), we get

$$\|g(3x) - 3g(x)\| + \|h(3x) - 3h(x)\| \le \varphi(x,x,x) \tag{2.6}$$

and so

$$\left\|g(x) - 3g\left(\frac{x}{3}\right)\right\| + \left\|h(x) - 3h\left(\frac{x}{3}\right)\right\| \le \varphi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right)$$

for all $x \in A$. Thus

$$\begin{aligned} &\left\|3^l g\left(\frac{x}{3^l}\right) - 3^m g\left(\frac{x}{3^m}\right)\right\| + \left\|3^l h\left(\frac{x}{3^l}\right) - 3^m h\left(\frac{x}{3^m}\right)\right\| \\ &\le \sum_{j=l}^{m-1} \left\|3^j g\left(\frac{x}{3^j}\right) - 3^{j+1} g\left(\frac{x}{3^{j+1}}\right)\right\| + \sum_{j=l}^{m-1} \left\|3^j h\left(\frac{x}{3^j}\right) - 3^{j+1} h\left(\frac{x}{3^{j+1}}\right)\right\| \\ &\le \sum_{j=l+1}^{m} 3^{j-1}\varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right) \end{aligned} \tag{2.7}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (2.7) that the sequences $\{3^k g(\frac{x}{3^k})\}$ and $\{3^k h(\frac{x}{3^k})\}$ are Cauchy for all $x \in A$. Since $Y$ is a Banach space, the sequences $\{3^k g(\frac{x}{3^k})\}$ and $\{3^k h(\frac{x}{3^k})\}$ converge. So one can define the mappings $D, H : A \to A$ by

$$D(x) := \lim_{k\to\infty} 3^k g\left(\frac{x}{3^k}\right), \qquad \& \qquad H(x) := \lim_{k\to\infty} 3^k h\left(\frac{x}{3^k}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.7), we get (2.5).

It folllows from (2.3) that

$$
\begin{aligned}
&\|D(\lambda(x+y+z))-\lambda(D(x)+D(y)+D(z))\| \\
&+\|H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\| \\
&=\lim_{n\to\infty}3^n\left\|g\left(\lambda\frac{x+y+z}{3^n}\right)-\lambda\left(g\left(\frac{x}{3^n}\right)+g\left(\frac{y}{3^n}\right)+g\left(\frac{z}{3^n}\right)\right)\right\| \\
&+\lim_{n\to\infty}3^n\left\|h\left(\lambda\frac{x+y+z}{3^n}\right)+h\left(\lambda\frac{x-2y+z}{3^n}\right)+h\left(\lambda\frac{x+y-2z}{3^n}\right)-3\lambda h\left(\frac{x}{3^n}\right)\right\| \\
&\le\lim_{n\to\infty}3^n\left\|s\left(3g\left(\lambda\frac{x+y+z}{3^{n+1}}\right)-\lambda\left(g\left(\frac{x}{3^n}\right)+g\left(\frac{y}{3^n}\right)+g\left(\frac{z}{3^n}\right)\right)\right)\right\| \\
&+\lim_{n\to\infty}3^n\left\|t\left(3h\left(\lambda\frac{x+y+z}{3^{n+1}}\right)+h\left(\lambda\frac{x-2y+z}{3^n}\right)+h\left(\lambda\frac{x+y-2z}{3^n}\right)-3\lambda h\left(\frac{x}{3^n}\right)\right)\right\| \\
&\quad+\lim_{n\to\infty}3^n\varphi\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right) \\
&=\left\|s\left(3D\left(\lambda\frac{x+y+z}{3}\right)-\lambda(D(x)+D(y)+D(z))\right)\right\| \\
&+\left\|t\left(3H\left(\lambda\frac{x+y+z}{3}\right)+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\right)\right\|
\end{aligned}
$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. So

$$
\begin{aligned}
&\|D(\lambda(x+y+z))-\lambda(D(x)+D(y)+D(z))\| \\
&+\|H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\| \\
&\le\left\|s\left(3D\left(\lambda\frac{x+y+z}{3}\right)-\lambda(D(x)+D(y)+D(z))\right)\right\| \\
&+\left\|t\left(3H\left(\lambda\frac{x+y+z}{3}\right)+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\right)\right\|
\end{aligned}
\tag{2.8}
$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$.

Let $\lambda = 1$ in (2.8). By Lemma 2.1, the mappings $D, H : A \to A$ are additive.

It follows from (2.8) and the additivity of $D$ and $H$ that

$$
\begin{aligned}
&\|D(\lambda(x+y+z))-\lambda(D(x)+D(y)+D(z))\| \\
&+\|H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x)\| \\
&\le\|s(D(\lambda(x+y+z))-\lambda(D(x)+D(y)+D(z)))\| \\
&+\|t(H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x))\|
\end{aligned}
$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. Since $|s| < 1$ and $|t| < 1$,

$$
\begin{aligned}
D(\lambda(x+y+z))-\lambda(D(x)+D(y)+D(z)) &= 0, \\
H(\lambda(x+y+z))+H(\lambda(x-2y+z))+H(\lambda(x+y-2z))-3\lambda H(x) &= 0
\end{aligned}
$$

and so $D(\lambda x) = \lambda D(x)$ and $H(\lambda x) = \lambda H(x)$ for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. Thus by Lemma 2.2, the additive mappings $D, H : A \to A$ are $\mathbb{C}$-linear.

It follows from (2.4) and the additivity of $D, H$ that

$$\begin{aligned}
&\|D([x,y,z]) - [D(x),H(y),H(z)] - [H(x),D(y),H(z)] - [H(x),H(y),D(z)]\| \\
&+\|H([x,y,z]) - [H(x),H(y),H(z)]\| \\
&= 27^n \left\| g\left(\frac{[x,y,z]}{27^n}\right) - \left[g\left(\frac{x}{3^n}\right), h\left(\frac{y}{3^n}\right), h\left(\frac{z}{3^n}\right)\right] \right. \\
&\qquad \left. - \left[h\left(\frac{x}{3^n}\right), g\left(\frac{y}{3^n}\right), h\left(\frac{z}{3^n}\right)\right] - \left[h\left(\frac{x}{3^n}\right), h\left(\frac{y}{3^n}\right), g\left(\frac{z}{3^n}\right)\right] \right\| \\
&+27^n \left\| h\left(\frac{[x,y,z]}{27^n}\right) - \left[h\left(\frac{x}{3^n}\right), h\left(\frac{y}{3^n}\right), h\left(\frac{z}{3^n}\right)\right] \right\| \le 27^n \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right),
\end{aligned}$$

which tends to zero as $n \to \infty$, by (2.2). So

$$\begin{aligned}
D([x,y,z]) - [D(x),H(y),H(z)] - [H(x),D(y),H(z)] - [H(x),H(y),D(z)] &= 0, \\
H([x,y,z]) - [H(x),H(y),H(z)] &= 0
\end{aligned}$$

for all $x, y, z \in A$. Hence the mapping $D : A \to A$ is a ternary hom-derivation and the mapping $H : A \to A$ is a ternary homomorphism. $\square$

**Corollary 2.4.** *Let $r > 3$ and $\theta$ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$ and*

$$\begin{aligned}
&\|g(\lambda(x+y+z)) - \lambda(g(x)+g(y)+g(z))\| \\
&+\|h(\lambda(x+y+z)) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\| \\
&\le \left\| s\left(3g\left(\lambda\frac{x+y+z}{3}\right) - \lambda(g(x)+g(y)+g(z))\right)\right\| \\
&+\left\| t\left(3h\left(\lambda\frac{x+y+z}{3}\right) + h(\lambda(x-2y+z)) + h(\lambda(x+y-2z)) - 3\lambda h(x)\right)\right\| \\
&\qquad +\theta(\|x\|^r + \|y\|^r)
\end{aligned} \tag{2.9}$$

*for all $\lambda \in \mathbb{T}^1$ and all $x, y, z \in A$. If the mappings $g, h : A \to A$ satisfy*

$$\begin{aligned}
&\|g([x,y,z]) - [g(x),h(y),h(z)] - [h(x),g(y),h(z)] - [h(x),h(y),g(z)]\| \\
&+\|h([x,y,z]) - [h(x),h(y),h(z)]\| \le \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned} \tag{2.10}$$

*for all $x, y, z \in A$, then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{3\theta}{3^r - 3}\|x\|^r$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 2.3 by $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. $\square$

**Theorem 2.5.** *Let $\varphi : A^3 \to [0, \infty)$ be a function and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3), (2.4) and*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{3^j}\varphi(3^j x, 3^j y, 3^j z) < \infty \tag{2.11}$$

*for all $x, y, z \in A$. Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{1}{3}\Phi(x, x, x) \tag{2.12}$$

*for all $x \in A$.*

*Proof.* It follows from (2.6) that

$$\left\|g(x) - \frac{1}{3}g(3x)\right\| + \left\|h(x) - \frac{1}{3}h(3x)\right\| \leq \frac{1}{3}\varphi(x, x, x) \tag{2.13}$$

for all $x \in A$. Thus

$$\begin{aligned} &\left\|\frac{1}{3^l}g\left(\frac{x}{3^l}\right) - \frac{1}{3^m}g\left(3^m x\right)\right\| + \left\|\frac{1}{3^l}h\left(3^l x\right) - \frac{1}{3^m}h\left(3^m x\right)\right\| \\ &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}g\left(3^j x\right) - \frac{1}{3^{j+1}}g\left(3^{j+1}x\right)\right\| + \sum_{j=l}^{m-1}\left\|\frac{1}{3^j}h\left(3^j x\right) - \frac{1}{3^{j+1}}h\left(3^{j+1}x\right)\right\| \\ &\leq \frac{1}{3}\sum_{j=l}^{m-1}\frac{1}{3^j}\varphi\left(3^j x, 3^j x, 3^j x\right) \end{aligned} \tag{2.14}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (2.14) that the sequences $\{\frac{1}{3^k}g(3^k x)\}$ and $\{\frac{1}{3^k}h(3^k x)\}$ are Cauchy for all $x \in A$. Since $Y$ is a Banach space, the sequences $\{\frac{1}{3^k}g(3^k x)\}$ and $\{\frac{1}{3^k}h(3^k x)\}$ converge. So one can define the mappings $D, H : A \to A$ by

$$D(x) := \lim_{k\to\infty}\frac{1}{3^k}g\left(3^k x\right),$$
$$H(x) := \lim_{k\to\infty}\frac{1}{3^k}h\left(3^k x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing to the limit $m \to \infty$ in (2.14), we get (2.12).

By the same reasoning as in the proof of Theorem 2.3, one can show that the mappings $D, H : A \to A$ are $\mathbb{C}$-linear.

It follows from (2.4) and the additivity of $D$ and $H$ that

$$\begin{aligned} &\|D([x, y, z]) - [D(x), H(y), H(z)] - [H(x), D(y), H(z)] - [H(x), H(y), D(z)]\| \\ &+\|H([x, y, z]) - [H(x), H(y), H(z)]\| \\ &= \frac{1}{27^n}\|g\left(27^n[x, y, z]\right) - [g\left(3^n x\right), h(3^n y), h(3^n z)] \\ &\quad -[h\left(3^n x\right), g(3^n y), h(3^n z)] - [h\left(3^n x\right), h(3^n y), g(3^n z)]\| \\ &+\frac{1}{27^n}\|h\left(27^n[x, y, z]\right) - [h\left(3^n x\right), h\left(3^n y\right), h(3^n z)]\| \\ &\leq \frac{1}{27^n}\varphi\left(3^n x, 3^n y, 3^n z\right) \leq \frac{1}{3^n}\varphi\left(3^n x, 3^n y, 3^n z\right), \end{aligned}$$

which tends to zero as $n \to \infty$, by (2.11). So

$$\begin{aligned} D([x, y, z]) - [D(x), H(y), H(z)] - [H(x), D(y), H(z)] - [H(x), H(y), D(z)] &= 0, \\ H([x, y, z]) - [H(x), H(y), H(z)] &= 0 \end{aligned}$$

for all $x, y, z \in A$. Hence the mapping $D : A \to A$ is a ternary hom-derivation and the mapping $H : A \to A$ is a ternary homomorphism. □

**Corollary 2.6.** *Let $r < 1$ and $\theta$ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.9) and (2.10). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{3\theta}{3 - 3^r}\|x\|^r$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 2.5 by $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

## 3. Stability of additive-additive $(s, t)$-functional inequality (0.1): a fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of pairs of hom-derivations and homomorphisms in $C^*$-ternary algebras associated to the additive-additive $(s, t)$-functional inequality (0.1).

**Theorem 3.1.** *Let $\varphi : A^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) \leq \frac{L}{27}\varphi(x, y, z) \leq \frac{L}{3}\varphi(x, y, z) \tag{3.1}$$

*for all $x, y, z \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3) and (2.4). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \leq \frac{L}{3(1 - L)}\varphi(x, x, x) \tag{3.2}$$

*for all $x \in A$.*

*Proof.* It follows from (3.1) that

$$\sum_{j=1}^{\infty} 27^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) \leq \sum_{j=1}^{\infty} 27^j \frac{L^j}{27^j}\varphi(x, y) = \frac{L}{1 - L}\varphi(x, y, z) < \infty$$

for all $x, y, z \in A$. By Theorem 2.3, there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ satisfying (2.5).

Letting $\lambda = 1$ and $y = z = x$ in (2.3), we get

$$\|g(3x) - 3g(x)\| + \|h(3x) - 3h(x)\| \leq \varphi(x, x, x) \tag{3.3}$$

for all $x \in A$.

Consider the set

$$S := \{(g, h) : (A, A) \to (A, A), \quad g(0) = h(0) = 0\}$$

and introduce the generalized metric on $S$:

$$d((g, h), (g_1, h_1)) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - g_1(x)\| + \|h(x) - h_1(x)\| \leq \mu\varphi(x, x, x),\ \forall x \in A\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see [17]).

Now we consider the linear mapping $J : S \to S$ such that

$$J(g,h)(x) := \left(3g\left(\frac{x}{3}\right), 3h\left(\frac{x}{3}\right)\right)$$

for all $x \in A$.

Let $(g,h), (g_1,h_1) \in S$ be given such that $d((g,h),(g_1,h_1)) = \varepsilon$. Then

$$\|g(x) - g_1(x)\| + \|h(x) - h_1(x)\| \le \varepsilon\varphi(x,x,x)$$

for all $x \in A$. Since

$$\begin{aligned}&\left\|3g\left(\frac{x}{3}\right) - 3g_1\left(\frac{x}{3}\right)\right\| + \left\|3h\left(\frac{x}{3}\right) - 3h_1\left(\frac{x}{3}\right)\right\| \\ &\le 3\varepsilon\varphi\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right) \le 3\varepsilon\frac{L}{3}\varphi(x,x,x) = L\varepsilon\varphi(x,x,x)\end{aligned}$$

for all $x \in A$, t $d(J(g,h), J(g_1,h_1)) \le L\varepsilon$. This means that

$$d(J(g,h), J(g_1,h_1)) \le Ld((g,h),(g_1,h_1))$$

for all $(g,h), (g_1,h_1) \in S$.

It follows from (3.3) that

$$\left\|g(x) - 3g\left(\frac{x}{3}\right)\right\| + \left\|h(x) - 3h\left(\frac{x}{3}\right)\right\| \le \varphi\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right) \le \frac{L}{3}\varphi(x,x,x)$$

for all $x \in A$. So $d((g,h),(Jg,Jh)) \le \frac{L}{3}$.

By Theorem 1.2, there exist mappings $D, H : A \to A$ satisfying the following:

(1) $(D,H)$ is a fixed point of $J$, i.e.,

$$D(x) = 3D\left(\frac{x}{3}\right), \qquad H(x) = 3H\left(\frac{x}{3}\right) \tag{3.4}$$

for all $x \in A$. The mapping $(D,H)$ is a unique fixed point of $J$. This implies that $(D,H)$ is a unique mapping satisfying (3.4) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \quad \le \quad \mu\varphi(x,x,x)$$

for all $x \in A$;

(2) $d(J^l(g,h),(D,H)) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 3^l g\left(\frac{x}{3^l}\right) = D(x), \quad \lim_{l\to\infty} 3^l h\left(\frac{x}{3^l}\right) = H(x)$$

for all $x \in A$;

(3) $d((g,h),(D,H)) \le \frac{1}{1-L}d((g,h),J(g,h))$, which implies

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{L}{3(1-L)}\varphi(x,x,x)$$

for all $x \in A$. Thus we get the inequality (3.2).

The rest of the proof is the same as in the proof of Theorem 2.3. □

**Corollary 3.2.** *Let $r > 3$ and $\theta$ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.9) and (2.10). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{3\theta}{3^r - 3}\|x\|^r$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 3.1 by taking $L = 3^{1-r}$ and $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

**Theorem 3.3.** *Let $\varphi : A^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \le 27L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \tag{3.5}$$

*for all $x, y, z \in A$. Let $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.3) and (2.4). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{1}{3(1-L)}\varphi(x, x, x) \tag{3.6}$$

*for all $x \in A$.*

*Proof.* It follows from (3.5) that

$$\sum_{j=1}^{\infty} \frac{1}{27^j}\varphi\left(3^j x, 3^j y, 3^j z\right) \le \sum_{j=1}^{\infty} \frac{1}{27^j}(27L)^j \varphi(x, y, z) = \frac{L}{1-L}\varphi(x, y, z) < \infty$$

for all $x, y, z \in A$. By Theorem 2.5, there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ satisfying (2.12).

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the linear mapping $J : S \to S$ such that

$$J(g, h)(x) := \left(\frac{1}{3}g(3x), \frac{1}{3}h(3x)\right)$$

for all $x \in A$.

It follows from (3.3) that

$$\left\|g(x) - \frac{1}{3}g(3x)\right\| + \left\|h(x) - \frac{1}{3}h(3x)\right\| \le \frac{1}{3}\varphi(x, x, x)$$

for all $x \in A$. Thus we get the inequality (3.6).

The rest of the proof is similar to the proof of Theorem 3.1. □

**Corollary 3.4.** *Let $r < 1$ and $\theta$ be nonnegative real numbers and $g, h : A \to A$ be mappings satisfying $g(0) = h(0) = 0$, (2.9) and (2.10). Then there exist a unique ternary hom-derivation $D : A \to A$ and a unique ternary homomorphism $H : A \to A$ such that*

$$\|g(x) - D(x)\| + \|h(x) - H(x)\| \le \frac{3\theta}{3 - 3^r}\|x\|^r$$

*for all $x \in A$.*

*Proof.* The proof follows from Theorem 3.3 by taking $L = 3^{r-1}$ and $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in A$. □

## 4. Conclusions

We have introduced the additive-additive $(s, t)$-functional inequality (0.1), and using the direct method and the fixed point method, we have proved the Hyers-Ulam stability of ternary hom-derivations and ternary homomorphisms in $C^*$-ternary algebras, associated to the additive-additive $(s, t)$-functional inequality (0.1) and the functional inequality (0.2).

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