On a class of $k+1$ th-order difference equations with variable coefficients

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Abstract

A Lie point symmetry analysis of a class of higher order difference equations with variable coefficients is considered and new symmetries are found. These symmetries are utilized to investigate the existence of solutions. The results in this paper generalize some results in the literature.

Key words: Difference equation; symmetry; reduction; group invariant solutions; periodicity

1 Introduction

Recently, rational difference equations have become a centre of interest of many authors, see [1–4]. Many methods have been developed to solve difference equations in closed form, that is, when every solution can be written in terms of the initial values and the indexing variable index n only. Among others, is the Lie symmetry approach used for differential equations. This differential equations method for difference equations was studied by P. Hydon and others (see [5–7, 9–11]). In [6], the author introduced an algorithm for obtaining symmetries and conservation laws of second-order difference equations. Now, it is known that these tools can be used to lower the order, via the invariants of the Lie group of transformations, as it was established for differential equations.

In this work, we aim to use the Lie symmetry approach to solve the following difference equations:

$$x_{n+1} = \frac{x_{n-k}}{\beta_n + \gamma_n \prod_{i=0}^{k} x_{n+i}}, \tag{1}$$

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where β_n and γ_n are real sequences. The definitions and notation in this paper follow the ones used by Hydon in [6]. Therefore, we will have to shift the equation k times and study

$$u_{n+k+1} = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}}, \tag{2}$$

instead.
Our work is a natural generalization of the results by Elabbasy, et. al. [1]. These authors used induction method to give solutions of

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n+i}}, \tag{3}$$

where the parameters α, β and γ are non-negative real numbers and the initial values are positive numbers.

2 Definitions and algorithm

As mentioned earlier, the definitions and notation used in this paper follow those adopted by Hydon in [6].

Definition 2.1 *A parameterized set of point transformations,*

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon), \tag{4}$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. *Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$*

2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0*

3. *Each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore*

$$\hat{x}_i(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, ..., p. \tag{5}$$

Consider the $k+1$th-order difference equation

$$u_{n+k+1} = \Omega(u_n, u_{n+1}, \dots, u_{n+k}), \tag{6}$$

for some function Ω. We shall restrict our attention to Lie point symmetries where $\hat{u}_n$ is a function of n and u_n only. In other words, we assume that the Lie point symmetries are of the form

$$\hat{n} = n; \qquad \hat{u}_n = u_n + \epsilon Q(n, u_n) \tag{7}$$

and that the analogous prolonged infinitesimal symmetry generator takes the form

$$X^{[k]} = \sum_{i=0}^{k} Q(n+i, u_{n+i}) \frac{\partial}{\partial u_{n+i}}, \tag{8}$$

where $Q = Q(n, u_n)$ is referred to as the characteristic. We define the symmetry condition as

$$\hat{u}_{n+k+1} = \Omega(n, \hat{u}_n, \hat{u}_{n+1}, ..., \hat{u}_{n+k}) \tag{9}$$

whenever (6) holds. Substituting the Lie point symmetries (7) into the symmetry condition (9) leads to the linearized symmetry condition

$$Q(n+k+1, u_{n+k+1}) - X^{[k]}\Omega = 0, \tag{10}$$

whenever (6) holds.
One can solve for the characteristic $Q(n, u_n)$ using the method of elimination and thereafter lower the order the difference equation (6) via the canonical coordinate [8]

$$S_n = \int \frac{du_n}{Q(n, u_n)}. \tag{11}$$

3 Main results

3.1 Symmetries

Consider the $k+1$ th-order difference equations of the form (2), i.e.,

$$u_{n+k+1} = \Omega = \frac{u_n}{B_n + A_n \prod_{i=0}^{k} u_{n+i}}.$$

We impose the symmetry condition (10) on (2) to get

$$Q(n+k+1, u_{n+k+1}) - \sum_{i=0}^{k} \Omega_{,u_{n+i}} Q(n+i, u_{n+i}) = 0, \tag{12}$$

where $\Omega_{,y}$ denotes the partial derivative of Ω with respect to y.
The characteristic in (12) takes different arguments and one can eliminate the undesirable variable by implicit differentiation. In this optic, we differentiate (12) with respect to u_{n+1} (keeping Ω fixed) and viewing u_{n+2} as a function of $u_n, u_{n+1}, \ldots, u_{n+k}$ and Ω, that is, we act the operator

$$L = \frac{\partial}{\partial u_{n+1}} + \frac{\partial u_{n+2}}{\partial u_{n+1}} \frac{\partial}{\partial u_{n+2}} = \frac{\partial}{\partial u_{n+1}} - \frac{\Omega_{,u_{n+1}}}{\Omega_{,u_{n+2}}} \frac{\partial}{\partial u_{n+2}} \tag{13}$$

on (12). This yields

$$\begin{aligned} &- \Omega_{,u_{n+1}} Q'(n+1, u_{n+1}) + \Omega_{,u_{n+1}} Q'(n+2, u_{n+2}) \\ &- \sum_{i=0}^{k} \left[\Omega_{,u_{n+i}u_{n+1}} - \frac{\Omega_{,u_{n+1}}}{\Omega_{,u_{n+2}}} \Omega_{,u_{n+i}u_{n+2}} \right] Q(n+i, u_{n+i}) = 0 \end{aligned} \tag{14}$$

which simplifies to

$$\begin{aligned} &- u_{n+1}u_{n+2}Q'(n+2, u_{n+2}) + u_{n+1}u_{n+2}Q'(n+1, u_{n+1}) - u_{n+2}Q(n+1, u_{n+1}) \\ &+ u_{n+1}Q(n+2, u_{n+2}) = 0 \end{aligned} \tag{15}$$

after a set of rather long calculations. Note that $'$ stands for the derivative with respect to the continuous variable. The differentiation of (15) with respect to u_{n+1} twice (keeping u_{n+2} fixed) leads to

$$[u_{n+1}Q'(n+1, u_{n+1}) - Q(n+1, u_{n+1})]'' = 0 \tag{16}$$

after simplification. The solution of (16) is given by

$$Q(n, u_n) = a_n u_n + b_n u_n \ln u_n + c_n \tag{17}$$

for some functions a_n, b_n and c_n of n. These functions are obtained by substituting (17) in (12) and by splitting the resulting equations with respect to product of shifts of u_n, since they are functions of n only. It turns out that $b_n = c_n = 0$ and we are left with the following reduced system:

$$1 \quad : \quad a_{n+k+1} - a_n = 0 \tag{18a}$$

$$u_n \ldots u_{n+k} \quad : \quad a_{n+1} + a_{n+2} + \cdots + a_{n+k} + a_{n+k+1} = 0, \tag{18b}$$

or equivalently

$$a_n + a_{n+1} + a_{n+2} + \cdots + a_{n+k} = 0. \tag{19}$$

We have found that

$$a_n = \exp\left(\frac{2\pi ns}{k+1} i\right), \quad 1 \leq s \leq k. \tag{20}$$

Thus, the k infinitesimal generators are given by

$$X_s = \exp\left(\frac{2\pi ns}{k+1} i\right) u_n \frac{\partial}{\partial u_n}, \quad 1 \leq s \leq k. \tag{21}$$

3.2 Reduction and exact solutions

Let

$$\theta_s = \exp\left(\frac{2\pi s}{k+1} i\right) \qquad \text{and} \qquad Q_s(n, u_n) = (\theta_s)^n u_n. \tag{22}$$

To lower the order of (2), we introduce the canonical coordinate defined in (11). We have

$$S_n = \int \frac{du_n}{Q_s(n, u_n)} = \frac{1}{(\theta_s)^n} \ln |u_n|. \tag{23}$$

Thanks to (19), we have proved that

$$X_s \left[(\theta_s)^n S_n + (\theta_s)^{n+1} S_{n+1} + \cdots + (\theta_s)^{n+k} S_{n+k}\right] = 0, \quad 1 \leq s \leq k. \tag{24}$$

So,

$$r_n = (\theta_s)^n S_n + (\theta_s)^{n+1} S_{n+1} + \cdots + (\theta_s)^{n+k} S_{n+k} \tag{25}$$

is an invariant function of $X_s,\ s = 0, 1, 2, \ldots, k$. For convenience, we consider

$$|\tilde{r_n}| = \exp\{-r_n\} = \pm \frac{1}{\prod_{i=0}^{k} u_{n+i}}, \tag{26}$$

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instead. We choose $\tilde{r_n} = 1/\prod_{i=0}^{k} u_{n+i}$ and the reader can readily check that $\tilde{r}_n$ satisfies

$$\tilde{r}_{n+1} = B_n\tilde{r}_n + A_n \tag{27}$$

and that

$$\tilde{r}_n = \tilde{r}_0\left(\prod_{k_1=0}^{n-1} B_{k_1}\right) + \sum_{l=0}^{n-1}\left(A_l \prod_{k_2=l+1}^{n-1} B_{k_2}\right). \tag{28}$$

Thanks to (26) and (2), we have that

$$u_{n+k+1} = \frac{\tilde{r}_n}{\tilde{r}_{n+1}}u_n \tag{29}$$

and thus

$$u_{(k+1)n+j} = u_j \prod_{s=0}^{n-1}\frac{\tilde{r}_{(k+1)s+j}}{\tilde{r}_{(k+1)s+j+1}}, \qquad j = 0, 1, \ldots, k. \tag{30}$$

We have

$$\begin{aligned} u_{(k+1)n+j} =& u_j \prod_{s=0}^{n-1}\frac{\tilde{r}_0\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1}\right) + \sum\limits_{l=0}^{(k+1)s+j-1}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2}\right)}{\tilde{r}_0\left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1}\right) + \sum\limits_{l=0}^{(k+1)s+j}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2}\right)} \\ =& u_j \prod_{s=0}^{n-1}\frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} B_{k_1}\right) + \left(\prod\limits_{i=0}^{k} u_i\right)\sum\limits_{l=0}^{(k+1)s+j-1}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} B_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} B_{k_1}\right) + \left(\prod\limits_{i=0}^{k} u_i\right)\sum\limits_{l=0}^{(k+1)s+j}\left(A_l \prod\limits_{k_2=l+1}^{(k+1)s+j} B_{k_2}\right)} \end{aligned} \tag{31}$$

for $j = 0, 1, \ldots, k$. The solution to the sequence $\{x_n\}$ is then given by

$$x_{(k+1)n+j-k} = x_{j-k}\prod_{s=0}^{n-1}\frac{\left(\prod\limits_{k_1=0}^{(k+1)s+j-1} \beta_{k_1}\right) + \mathcal{P}\sum\limits_{l=0}^{(k+1)s+j-1}\left(\gamma_l \prod\limits_{k_2=l+1}^{(k+1)s+j-1} \beta_{k_2}\right)}{\left(\prod\limits_{k_1=0}^{(k+1)s+j} \beta_{k_1}\right) + \mathcal{P}\sum\limits_{l=0}^{(k+1)s+j}\left(\gamma_l \prod\limits_{k_2=l+1}^{(k+1)s+j} \beta_{k_2}\right)}$$

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where $j = 0, 1, 2, \ldots, k$ and $\mathcal{P} = \prod_{i=0}^{k} x_{-i}$. In the subsequent sections, we investigate solutions to special cases of the difference equations.

4 The case when β_n and γ_n are 1-periodic

In this case, we assume that $\beta_0 = \beta_j$ for all $j \geq 1$ and $\gamma_0 = \gamma_j$ for all $j \geq 1$.

4.1 The case when $\beta_0 \neq 1$

The solution becomes

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{\beta_0^{(k+1)s+j} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j}}{1-\beta_0}\gamma_0}{\beta_0^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-\beta_0^{(k+1)s+j+1}}{1-\beta_0}\gamma_0}, \qquad j = 0, 1, 2, \ldots, k.$$

Set $\beta_0 = \gamma_0 = \frac{1}{a}$ where a is a constant. Then the solution reduces to

$$x_{(k+1)n+j-k} = x_{j-k} \prod_{s=0}^{n-1} \frac{(a^{-1})^{(k+1)s+j} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-(a^{-1})^{(k+1)s+j}}{1-a^{-1}}a^{-1}}{(a^{-1})^{(k+1)s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right) \frac{1-(a^{-1})^{(k+1)s+j+1}}{1-a^{-1}}a^{-1}},$$

which is equivalent to

$$x_{(k+1)n+j-k} = x_{j-k} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+j} a^l}.$$

More explicitly, we have

$$x_{(k+1)n-k} = x_{-k} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s} a^l},$$

$$x_{(k+1)n+1-k} = x_{1-k} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+1} a^l},$$

$$x_{(k+1)n+2-k} = x_{2-k} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+2} a^l},$$

$$\vdots$$

$$x_{(k+1)n-1} = x_{-1} a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+k-2} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+k-1} a^l}$$

and

$$x_{(k+1)n} = x_0 a^n \prod_{s=0}^{n-1} \frac{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+k-1} a^l}{1 + \left(\prod_{i=0}^{k} x_{-i}\right) \sum_{l=0}^{(k+1)s+k} a^l}.$$

This solution has appeared in [1].

4.1.1 The special case $\beta = -1$ and k is odd

The solution simplifies to

$$x_{(k+1)n-k} = x_{-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$x_{(k+1)n+1-k} = x_{1-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{n},$$

$$x_{(k+1)n+2-k} = x_{2-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$\vdots$$

$$x_{(k+1)n-1} = x_{-1}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{-n},$$

$$x_{(k+1)n} = x_{j-k}(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0)^{n},$$

as long as the denominator does not vanish.
However, the solution above can be written in compact form as

$$x_{(k+1)n+j-k} = x_{j-k}\left(-1 + (x_{-k}x_{-k+1}x_{-k+2}\dots x_{-1}x_0)\gamma_0\right)^{(-1)^{j+1}n}$$

for $j = 0, 1, \dots, k$.
This solution has appeared in [1] (See Theorem 9).

Remark 4.1 *Note that if* $\gamma_0 \prod_{i=0}^{k} x_{-i} = 2$*, the solution is periodic with period* $k+1$*.*

4.1.2 The special case $\beta = -1$ and k is even

In this case, we have

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\prod_{s=0}^{n-1}\frac{(-1)^{s+j} + \left(\prod_{i=0}^{k} x_{-i}\right)^{\frac{1-(-1)^{s+j}}{2}}\gamma_0}{(-1)^{s+j+1} + \left(\prod_{i=0}^{k} x_{-i}\right)^{\frac{1-(-1)^{s+j+1}}{2}}\gamma_0} \\ &= x_{j-k}\prod_{\substack{s\geq 0,\\ s-j \text{ is even}}}^{n-1}\frac{1}{-1 + \left(\prod_{i=0}^{k} x_{-i}\right)\gamma_0}\prod_{\substack{s\geq 0,\\ s-j \text{ is odd}}}^{n-1}\left(-1 + \left(\prod_{i=0}^{k} x_{-i}\right)\gamma_0\right). \end{aligned}$$

If j is even and n is odd,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1 + \gamma_0\prod_{i=0}^{k} x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1 + \gamma_0\prod_{i=0}^{k} x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor} \\ &= x_{j-k}\left(-1 + \gamma_0\prod_{i=0}^{k} x_{-i}\right)^{-1}. \end{aligned}$$

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If j is odd and n is odd,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right). \end{aligned}$$

If j is even and n is even,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}. \end{aligned}$$

If j is odd and n is even,

$$\begin{aligned} x_{(k+1)n+j-k} &= x_{j-k}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{-\lfloor\frac{n-1}{2}\rfloor-1}\left(-1+\gamma_0\prod_{i=0}^{k}x_{-i}\right)^{\lfloor\frac{n-1}{2}\rfloor+1} \\ &= x_{j-k}. \end{aligned}$$

In summary, and more compactly, the solution is

$$x_{(k+1)n+j-k} = \begin{cases} x_{j-k}\left(-1+\gamma_0\prod\limits_{i=0}^{k}x_{-i}\right)^{(-1)^{j+1}}, & \text{if } n \text{ is odd} \\ x_{j-k}, & \text{if } n \text{ is even.} \end{cases}$$

This solution has appeared in [1] (See Theorem 8).

4.1.3 The case when $\beta_0 = 1$

The solution is given by

$$x_{(k+1)n+j-k} = x_{j-k}\prod_{s=0}^{n-1}\frac{1+\left(\prod\limits_{i=0}^{k}x_{-i}\right)((k+1)s+j)\gamma_0}{1+\left(\prod\limits_{i=0}^{k}x_{-i}\right)((k+1)s+j+1)\gamma_0}, \qquad j=0,1,2,\ldots,k.$$

5 Conclusion

We have utilized symmetry analysis to find point symmetries for certain $(k+1)$ th-order difference equations. We performed the group reduction of the equations using one of these symmetries and solutions were given in a unified manner. Our results generalise those in [1] in the sense that (a) α, β and γ need not necessarily be non-negative integers and (b) the constants can be replaced with sequences (variable constants).

References

[1] E.M. Elsayed, E.M Elabbasy and H. El-Metwally, On the difference equation $x_{n+1} = \alpha x_{n-k}/(\beta + \gamma \prod_{i=0}^{k} x_{n-i})$, *J. Concrete and Applicable Mathematics*, **5:2**, 101 - 113 (2007).

[2] M. Folly-Gbetoula and D. Nyirenda, On Some Rational Difference Equations of Order Eight, *International Journal of Contemporary Mathematical Sciences*, **13:6**, 239–254 (2018).

[3] M. Folly-Gbetoula and D. Nyirenda, A generalised two-dimensional system of higher order recursive sequences, *Journal of Difference Equations and Applications* **26:2**, 244–260 (2020).

[4] M. Folly-Gbetoula and D. Nyirenda, Lie Symmetry Analysis and Explicit Formulas for Solutions of some Third-order Difference Equations, *Quaestiones Mathematicae*, **42:7**, 907–917 (2019).

[5] M. Folly-Gbetoula and D. Nyirenda, On some sixth-order rational recursive sequences, *Journal of computational analysis and applications*, **27:6**, 1057–1069 (2019).

[6] P. E. Hydon, *Difference Equations by Differential Equation Methods*, *Cambridge University Press*, Cambridge, 2014.

[7] P. E. Hydon, Symmetries and first integrals of ordinary difference equations, *Proc. Roy. Soc. Lond. A*, **456**, 2835-2855 (2000).

[8] N. Joshi and P. Vassiliou, The existence of Lie Symmetries for First-Order Analytic Discrete Dynamical Systems, *J. Math. Anal. Appl.*, *195*, 872-887 (1995).

[9] S. Maeda, The similarity method for difference equations, IMA *J. Appl. Math.* **38**, 129-134 (1987).

[10] N. Mnguni, D. Nyirenda and M. Folly-Gbetoula, Symmetry Lie Algebra and Exact Solutions of Some fourth-order Difference Equation, *Journal of Nonlinear Sciences and Applications*, **11:11**, 1262–1270 (2018).

[11] N. Mnguni and M. Folly-Gbetoula, Invariance analysis of a third-order difference equation with variable coefficients, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms*, **25**, 63-73(2018).