NECESSARY AND SUFFICIENT CONDITIONS OF FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS

ABHAY KUMAR SETHI[1*] AND JUNG RYE LEE[2*]

ABSTRACT. In this work, we establish the necessary and sufficient conditions for oscillation of a class of functional differential equations of the form

$$((x(t)+p(t)x(t-\sigma))'+q(t)\phi(x(t-\tau))+v(t)\psi(x(t-\eta))=0$$

of a neutral coefficient $p(t)$, by using the Knaster-Tarski fixed point theorem and Banach's fixed point theorem.

1. Introduction

Consider a class of first-order nonlinear neutral differential equations of the form

$$((x(t)+p(t)x(t-\sigma)))'+q(t)\phi(x(t-\tau))+v(t)\psi(x(t-\eta))=0, \tag{1.1}$$

where $r,\ q,\ v,\ \tau,\ \sigma,\eta\in C(\mathbb{R}_+,\mathbb{R}_+),\ p\in C(\mathbb{R}_+,\mathbb{R}),\ \phi\in C(\mathbb{R},\mathbb{R})$ such that $x\phi(x)>0, x\phi(x)>0$ for $x\neq 0$ and $\phi,\ \psi\in C(\mathbb{R},\mathbb{R})$ satisfying the property $x\phi(x)>0,\ \ u\psi(u)>0$ for $x,u\neq 0$.

In this work, our objective is to establish the necessary and sufficient condition results for oscillation of all solutions of (1.1), where

(A_0) $p\in C([0,\infty),\mathbb{R}),\ f\in C(\mathbb{R},\mathbb{R}),\ q,\tau,\sigma,\eta\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that $t-\tau<t,\ t-\sigma<t$ and $t-\eta<t$;

(A_1) $\phi,\psi\in C(\mathbb{R},\mathbb{R})$ are nondecreasing and satisfy $u\phi(u)>0,\ u\psi(u)>0$ for $u,v\neq 0$.

Fatima *et al.* [1] studied the nonlinear neutral differential equation (NDDE) of the form

$$[r(t)(x(t)+p(t)x(t-\tau))]'+q(t)x(t-\sigma)=0, \tag{1.2}$$

where $p\in C[[t_0,\infty)],\mathbb{R}],\ r,q\in C[[t_0,\infty),\mathbb{R}^+],\ \tau,\sigma^+\in\mathbb{R}^+$, and they obtained new sufficient conditions for all solutions of NDDE (1.2) to be oscillatory.

Graef *et al.* [8] studied the first order neutral delay differential equations of the form

$$[x(t)+p(t)x(t-\tau)]'+q(t)f(x(t-\sigma))=0, \tag{1.3}$$

under the conditions
$(a)\ p\in\mathbb{R},\tau$ *and* σ *are positive constants*;
$(b)\ q:[t_0,\infty)\to\mathbb{R}$ *is a continuous function with* $q(t)>0$;
$(c)\ f:\mathbb{R}\to\mathbb{R}$ *is continuous with* $uf(u)>0$ *for* $u\neq 0$, *and there is a positive constant* M *such that* $\frac{f(u)}{u^\alpha}\geq M>0$, *where* α *is a ratio of odd positive integers*. They established internal conditions for all solutions of nonlinear first order neutral delay differential equations.

Grammatikopoulos *et al.* [9] studied first order neutral delay differential equations of the form

$$[x(t)-p(t)x(t-\tau)]'+Q(t)x(t-\delta))=0, \tag{1.4}$$

2010 *Mathematics Subject Classification.* Primary 34K40, 34K30, 34C10.
Key words and phrases. Oscillation; nonoscillation; nonlinear neutral delay differential equation.
*Corresponding authors: Abhay Kumar Sethi (email: sethiabhaykumar100@gmail.com) and Jung Rye Lee (email: jrlee@daejin.ac.kr).

where $p, Q, \delta \in C([t_0,\infty)], \mathbb{R}^+)$, and $\lim_{t\to\infty}(t-\delta(t)) = \infty$. They established sufficient conditions for oscillation of all solutions of the the neutral delay differential equations.

The motivation of the present work comes from the above studies. In this work, an attempt is made to establish the necessary and sufficient condition for asymptotic behaviour of solutions of (1.1), under various ranges in the neutral coefficient $p(t)$. Clearly, (1.2), (1.3) and (1.4) are special cases of (1.1). However, there are few results to study the oscillation of (1.1). The purpose of this work is to obtain some sufficient condition results for oscillation of (1.1). This work would be interesting than the works of [15, 19] as long as (1.1) is concerned.

Neutral delay differential equations find numerous applications in electric network. For example, they are frequently used for the study of distributed networks containing lossless transmission lines which arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see for example [12]). The problem of obtaining sufficient conditions to ensure the second order differential equations which are special cases of (1.1) is oscillatory has received a great attention. Since the first order equations have the applied applications, there is permanent interest in obtaining new sufficient conditions for oscillation or nonoscillation of solutions of varietal type of the first order equations (see [2–7, 11, 13, 14, 16–18, 20]).

Definition 1.1. *By a solution of (1.1), we mean a continuously differentiable function $x(t)$ which is defined for $t \geq T^* = \min\{(t-\sigma_0),(t-\tau_0),(t-\eta_0)\}$ such that $x(t)$ satisfies (1.1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solution of (1.1) exists on some half line $[t_1,\infty)$, $t_1 \geq t_0$. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.*

2. Oscillation results

This section deals with the oscillation results for necessary and sufficient conditions for oscillation of all solutions of (1.1), Throughout our discussion, we use the following notation

$$z(t) = x(t) + p(t)x(t-\sigma).$$

Lemma 2.1. *[10] Let $p, x, z \in C([0,\infty),\mathbb{R})$ be such that $z(t) = x(t) + p(t)x(t-\sigma)$, $t \geq \tau > 0$, $x(t) > 0$ for $t \geq t_1 > \tau$, $\liminf_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L$ exists. Let $p(t)$ satisfy one of the following conditions:*

$$i)\ 0 \leq p_1 \leq p(t) \leq p_2 < 1,\ ii)\ 1 < p_3 \leq p(t) \leq p_4 < \infty,\ iii)\ -\infty < -p_5 \leq p(t) \leq 0,$$

where $r_i > 0$, $1 \leq i \leq 5$.
Then $L = 0$.

Theorem 2.2. *Assume that (A_0) and (A_1) hold and $0 \leq a_1 \leq p(t) \leq a_2 < 1$ for $t \in \mathbb{R}_+$. Let ϕ, ψ be Lipschitzian on intervals of the form $[\alpha,\beta]$, $0 < \alpha < \beta < \infty$. Then every solution of (1.1) converges to zero as $t \to \infty$ if and only if*

(A_2) $\int_t^\infty [q(s) + v(s)]ds = \infty$.

Proof. Assume that (A_2) holds. Let $x(t)$ be a solution of (1.1) on $[t_x,\infty]$, $t_x \geq 0$. Let $x(t) > 0$ for $t \geq t_x$. Set

$$z(t) = x(t) + p(t)x\big(t-\sigma\big),\ t \geq t_0. \tag{2.1}$$

Then (1.1) becomes

$$z'(t) = -q(t)\phi(x(t-\tau)) - v(t)\psi(x(t-\eta)) < 0, \tag{2.2}$$

and hence $z(t)$ is a decreasing function for $t \geq t_1 > t_0 + \rho$. Since $z(t) > 0$ for $t \geq t_2$, $\lim_{t\to\infty} z(t)$ exists. Consequently, $z(t) > x(t)$ implies that $x(t)$ is bounded. Our aim is to show that $\lim_{t\to\infty} x(t) = 0$. For this, we need to show that $\liminf_{t\to\infty} x(t) = 0$. If $\liminf_{t\to\infty} x(t) \neq 0$, then there exist $t_3 > t_2$ and $\beta > 0$ such that $x(t-\sigma) \geq \beta > 0$ for $t \geq t_3$. Ultimately,

$$\begin{aligned}\int_{t_3}^{t} [\phi(x(t-\tau)) + v(t)\psi(x(t-\eta))]\, dt &\geq \phi(\beta)[q(t)]dt + \psi(\beta)\int_{t_3}^{t}[v(t)]dt \\ &\to +\infty, \;\; as \;\; t \to \infty,\end{aligned}$$

due to (A_2).

On the other hand, we integrate (2.2) from t_3 to $t(> t_3)$ to obtain

$$\begin{aligned}\int_{t_3}^{t} [q(t)\phi(x(t-\tau)) + v(t)\psi(x(t-\eta))]\, dt &\leq -z(t) + z(t_3) \\ &< \infty, \;\; as \;\; t \to \infty,\end{aligned}$$

which is a contradiction. Therefore, $\liminf_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$ due to Lemma 2.1. Thus we obtain

$$\begin{aligned}0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}(x(t) + p(t)x(t-\sigma)) \\ &\geq \limsup_{t\to\infty} x(t),\end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

Assume that $x(t) < 0$ for $t \geq t_0$. Setting $y(t) = -x(t)$ for $t \geq t_0$ in (1.1), we obtain

$$((y(t) + p(t)y(t-\sigma)))' + q(t)\phi(y(t-\tau)) + v(t)\psi(y(t-\eta)) = 0,$$

and proceeding as above it is easy to prove that $\lim_{t\to\infty} y(t) = 0$.

In order to prove the condition (A_2) is necessary, we suppose that

$$\int_{t}^{\infty} [q(s) + v(s)]\, ds < \infty \tag{2.3}$$

and we need to show that the equation (1.1) admits a nonoscillatory solution which does not tend to zero as $t \to \infty$ when the limit exists. If possible, let there exist $t_1 > 0$ such that

$$\int_{t}^{\infty} [q(s) + v(s)]\, ds < \frac{1-a_1}{10c},$$

where $C = \max\{C_1, \frac{C_2}{L}, \phi(1), \psi(1)\}$, C_1 is the Lipschitz constant of ϕ and C_2 is the Lipschitz constant of ψ on $\left[\frac{2(1-a_1)}{5}, 1\right]$. For $t_2 > t_1$, set $Y = BC([t_2,\infty),\mathbb{R})$, the space of real valued bounded continuous functions on $[t_2,\infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||Y|| = \sup\{|Y(t)| : t \geq t_2\}.$$

Let's define

$$S = \left\{u \in Y : \; \frac{2(1-a_1)}{5} \leq u(t) \leq 1, \; t \geq t_2\right\}.$$

Clearly, S is a closed and convex subspace of Y. Let $T : S \to S$ be defined by

$$Ty(t) = \begin{cases} Ty(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)y(t-\sigma) + \frac{2+3a_1}{5} + \int_t^\infty [q(s)\phi(y(t-\tau)) + v(s)\psi(y(t-\eta))]\, ds, t \geq t_2+\rho. \end{cases}$$

For every $y \in S$,

$$\begin{aligned} Ty(t) &\leq \frac{2+3a_1}{5} + \phi(1)\int_t^\infty [q(s)]ds + \psi(1)\int_t^\infty [v(s)]ds \\ &< \frac{2+3a_1}{5} + \frac{1-a_1}{10} = \frac{1+a_1}{2} < 1 \end{aligned}$$

and

$$\begin{aligned} Ty(t) &\geq -p(t)y(t-\tau) + \frac{2+3a_1}{5} \\ &\geq -a_1 + \frac{2+3a_1}{5} = \frac{2(1-a_1)}{5} \end{aligned}$$

which imply that $Ty \in S$. Now, for $y_1, y_2 \in S$,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq |p(t)||y_1(t-\tau) - y_2(t-\tau)| \\ &+ C_1 \int_t^\infty q(s)|y_1(s-\sigma) - y_2(s-\sigma)|ds + C_2 \int_t^\infty v(s)|y_1(s-\eta) - y_2(s-\eta)|ds, \end{aligned}$$

that is,

$$\begin{aligned} |Ty_1(t) - Ty_2(t)| &\leq a_2||y_1 - y_2|| + C_1||y_1-y_2||\int_t^\infty [q(s)]ds + C_2||y_1-y_2||\int_t^\infty [v(s)]ds \\ &< \left(a_1 + \frac{1-a_1}{10}\right)||y_1-y_2||, \end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \leq \mu||y_1-y_2||,$$

that is, T is a contraction mapping, where $\mu = a_1 + \frac{1-a_1}{10} = \frac{1+9a_1}{10} < 1$. Since S is complete and T is a contraction on S, by the Banach's fixed point theorem, T has a unique fixed point on $\left[\frac{2(1-a_1)}{5}, 1\right]$. Hence $Ty = y$ and

$$y(t) = \begin{cases} y(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)y(t-\sigma) + \frac{2+3a_1}{5}\left[\int_t^\infty q(s)\phi(y(s-\tau)) + \int_t^\infty v(s)\psi(y(s-\eta)))\right] ds, t \geq t_2+\rho \end{cases}$$

is a nonoscillatory solution of (1.1). Therefore, (A_2) is necessary. This completes the proof of the theorem. □

Theorem 2.3. *Assume that (A_0) and (A_1) hold and $1 < a_3 \leq p(t) \leq a_4 < \infty$ such that $a_3^2 > a_4$ for $t \in \mathbb{R}_+$. Let ϕ, ψ be Lipschitzian on intervals of the form $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. Then every solution of (1.1) converges to zero as $t \to \infty$ if and only if (A_2) holds.*

Proof. The sufficient part is the same as in the proof of Theorem 2.2.

For the necessary part, we suppose that (2.2) holds. It is possible to find a $t_1 > 0$ such that

$$\int_t^\infty [q(s) + v(s)]\, ds < \frac{a_3 - 1}{2K},$$

where $K=\max\{K_1,\frac{K_2}{L}\}$, K_1, K_2 are Lipschitz constants of ϕ and ψ on $[a,b]$ and $K_2=\phi(a),\psi(b)$ such that

$$a=\frac{2\lambda(a_3{}^2-a_4)-a_4(a_3-1)}{2a_3{}^2a_4},$$

$$b=\frac{a_3-1+2\lambda}{2a_3}, \qquad \lambda>\frac{a_4(a_3-1)}{2(a_3{}^2-a_4)}>0.$$

Let $Y=BC([t_2,\infty),\mathbb{R})$ be the space of real valued bounded continuous functions on $[t_2,\infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y||=\sup\{|y(t)|:t\geq t_2\}.$$

Define

$$S=\{u\in Y:\ a\leq u(t)\leq b,\ t\geq t_2\}.$$

It is easy to see that S is a closed convex subspace of Y. Let $T:S\to S$ be such that

$$Tx(t)=\begin{cases}Tx(t_2+\rho), & t\in[t_2,t_2+\rho]\\ -\frac{x\big(t+\sigma\big)}{p\big(t+\sigma\big)}+\frac{\lambda}{p\big(t+\sigma\big)}+\frac{1}{p\big(t+\sigma\big)}\Big[\int_{s+\sigma}^{\infty}q(s)\phi(x(s-\tau))ds+\int_{s+\sigma}^{\infty}v(s)\psi(x(s-\eta)))ds\Big], & t\geq t_2+\rho.\end{cases}$$

For every $x\in S$,

$$\begin{aligned}Tx(t)&\leq\frac{\phi(b)}{p\big(t+\sigma\big)\big)}\left[\int_{s+\sigma}^{\infty}q(s)ds+\frac{\psi(b)}{p\big(t+\sigma\big)\big)}\int_{s+\sigma}^{\infty}v(s)ds\right]+\frac{\lambda}{p\big(s+\sigma\big)\big)}\\&\leq\frac{1}{a_3}\left[\frac{a_3-1}{2}+\lambda\right]=b\end{aligned}$$

and

$$\begin{aligned}Tx(t)&\geq-\frac{x\big(t+\tau\big)\big)}{p\big(t+\tau\big)\big)}+\frac{\lambda}{p\big(t+\tau\big)\big)}\\&>-\frac{b}{a_3}+\frac{\lambda}{a_4}\\&=-\frac{a_3-1+2\lambda}{2a_3^2}+\frac{\lambda}{a_4}\\&=\frac{2\lambda(a_3{}^2-a_4)-a_4(a_3-1)}{2a_3{}^2a_4}=a,\end{aligned}$$

which imply that $Tx\in S$. For $y_1,y_2\in S$,

$$\begin{aligned}|Ty_1(t)-Ty_2(t)|\leq&\frac{1}{|p(t+\sigma)|}|y_1(t+\sigma)-y_2(t+\sigma)|\\&+\frac{K_1}{|p(t+\sigma)|}\left[\int_{s+\sigma}^{\infty}q(s)|y_1(s-\tau)-y_2(s-\tau)|\right]ds\\&+\frac{K_2}{|p(t+\sigma)|}\left[\int_{s+\sigma}^{\infty}v(s)|y_1(s-\eta)-y_2(s-\eta)|\right]ds,\end{aligned}$$

that is,

$$\begin{aligned}|Ty_1(t)-Ty_2(t)|&\leq\frac{1}{p_3}||y_1-y_2||+\frac{K_1}{a_3}||y_1-y_2||\int_T^{\infty}q(s)ds+\frac{K_2}{a_3}||y_1-y_2||\int_T^{\infty}v(s)ds\\&<\left(\frac{1}{a_3}+\frac{a_3-1}{2a_3}\right)||y_1-y_2||,\end{aligned}$$

which implies that

$$||Ty_1 - Ty_2|| \leq \mu||y_1 - y_2||,$$

that is, T is a contraction, where $\mu = \left(\frac{1}{a_3} + \frac{a_3-1}{2a_3}\right) < 1$. Hence by the Banach's fixed point theorem, T has a unique fixed point which is a nonoscillatory solution of (1.1) on $[a, b]$. Thus the proof of the theorem is complete. □

Theorem 2.4. *Assume that* (A_0) *and* (A_1) *hold and* $-1 < -a_5 \leq p(t) \leq 0$, $a_5 > 0$ *for* $t \in \mathbb{R}_+$. *Then every solution of (1.1) converges to zero as* $t \to \infty$ *if and only if* (A_2) *holds.*

Proof. Proceeding as in the proof of Theorem 2.2, we obtain (2.2). Hence $r(t)z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Let $z(t) > 0$ for $t \geq t_2$. Then $\lim_{t\to\infty} z(t)$ exists. Let $z(t) < 0$ for $t \geq t_2$. We claim that $x(t)$ is bounded. If not, there exists $\{\eta_n\}$ such that $\tau(\eta_n) \leq \tau_n$ and $\eta_n \to \infty$ as $n \to \infty$, $x(\eta_n) \to \infty$ as $n \to \infty$ and

$$x(\eta_n) = \max\{x(s) : t_2 \leq s \leq \eta_n\}.$$

Therefore,

$$\begin{aligned} z(\eta_n) &= x(\eta_n) + p(\eta_n)x(\eta_n - \sigma) \\ &\geq (1 - a_5)x(\eta_n) \\ &\to +\infty, \quad as \;\; n \to \infty, \end{aligned}$$

which is a contradiction to the fact $z(t) > 0$. So our claim holds. Consequently, $z(t) \leq x(t)$ implies that $\lim_{t\to\infty} z(t)$ exists. Hence for any $z(t)$, $x(t)$ is bounded. Using the same type of argument as in the proof of Theorem 2.2, it is easy to show that $\liminf_{t\to\infty} x(t) = 0$ and by Lemma 2.1, $\lim_{t\to\infty} z(t) = 0$. Indeed,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}\big(x(t) + p(t)x(t - \sigma))\big) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-a_5 x(t - \sigma)\big) \\ &= (1 - a_5)\limsup_{t\to\infty} x(t) \end{aligned}$$

which implies that $\limsup_{t\to\infty} x(t) = 0$. The rest of the proof follows from Theorem 2.2.

Next, we suppose that (2.2) holds. Then there exists $t_1 > 0$ such that

$$\int_s^\infty [q(s) + v(s)]\, ds < \frac{1 - a_5}{5\phi(1)\psi(1)}, \quad t \geq t_1.$$

For $t_2 > t_1$, let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of all real valued bounded continuous functions defined on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Let $K = \{y \in Y : y(t) \geq 0,\ t \geq t_2\}$. Then Y is a partially ordered Banach space (see [8]). For $u, v \in Y$, we define $u \leq v$ if and only if $u - v \in K$. Let

$$S = \left\{X \in Y : \ \frac{1 - p_5}{5} \leq x(t) \leq 1,\ t \geq t_2\right\}.$$

If $x_0(t) = \frac{1-a_5}{5}$, then $x_0 \in S$ and $x_0 =$ g.l.b S. Further, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{x \in Y : \ l_1 \leq x(t) \leq l_2,\ \frac{1 - a_5}{5} \leq l_1,\ l_2 \leq 1\right\}.$$

Let $v_0(t) = l_2'$, $t \geq t_3$, where $l_2' = \sup\{l_2 : \frac{1-a_5}{5} \leq l_2 \leq 1\}$. Then $v_0 \in S$ and $v_0 = $ l.u.b S^*. For $t_3 = t_2 + \rho$, define $T : S \to S$ by

$$Tx(t) = \begin{cases} Tx(t_3), & t \in [t_2, t_3] \\ -p(t)x(t-\sigma) + \frac{1-a_5}{5}\left[\int_s^\infty q(\eta)\phi(x(s-\tau))ds + \int_s^\infty v(s)\psi(x(s-\eta))ds\right], & t \geq t_3. \end{cases}$$

For every $x \in S$, $Tx(t) \geq \frac{1-a_5}{5}$ and

$$\begin{aligned} Tx(t) &\leq a_5 + \frac{1-a_5}{5} + \phi(1)\int_s^\infty [q(s)]ds + \psi(1)\int_s^\infty [v(s)]ds \\ &< \frac{2+3a_5}{5} < 1 \end{aligned}$$

which imply that $Tx \in S$. Now, for $x_1, x_2 \in S$, it is easy to verify that $x_1 \leq x_2$ implies that $Tx_1 \leq Tx_2$. Hence by the Knaster-Tarski fixed point theorem ([8, Theorem 1.7.3]), T has a unique fixed point such that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. □

Theorem 2.5. *Assume that (A_0) and (A_1) hold and $-\infty < -a_6 \leq p(t) \leq -a_7 < -1$, $a_6, a_7 > 0$ for $t \in \mathbb{R}_+$. Let ϕ, ψ be Lipschitzian on intervals of the form $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. Then every bounded solution of (1.1) converges to zero as $t \to \infty$ if and only if (A_2) holds.*

Proof. The proof of the theorem follows from Theorem 2.2. For the necessary part, we need to mention the following:

$$\int_s^\infty [q(s) + v(s)]\, ds < \frac{a_7 - 1}{2K},$$

where $K = \max\{K_1, K_2\}$, K_1, K_2 are Lipschitz constants of ϕ and ψ on $[a, b]$, $K_2 = \phi(a)\psi(b)$ such that

$$a = \frac{2\lambda a_7 - a_6(a_7 - 1)}{2a_6 a_7}, \quad b = \frac{\lambda}{a_7 - 1}$$

for

$$\lambda > \frac{a_6(a_7 - 1)}{2a_7} > 0,$$

and

$$Tx(t) = \begin{cases} Tx(t_2 + \rho), & t \in [t_2, t_2 + \rho] \\ -\frac{x\left(t+\sigma\right)}{p\left(t+\sigma\right)} - \frac{\lambda}{p\left(t+\sigma\right)} + \frac{1}{p\left(t+\sigma\right)}\left[\int_{s+\sigma}^\infty q(s)\phi(x(s-\tau))ds + \int_{s+\sigma}^\infty v(s)\psi(x(s-\eta))ds\right], \end{cases}$$

where $t \geq t_2 + \rho$. This completes the proof of the theorem. □

Remark 2.6. *In the above theorems, ϕ and ψ could be linear, sublinear or superlinear.*

Remark 2.7. *Lemma 2.1 does not include $p(t) \equiv 1$ for all t (see [8]). The present analysis does not allow the case $p(t) \equiv -1$ for all t. Hence in our discussion, a necessary and sufficient condition is established excluding $p(t) = \pm 1$ for all t. It seems that a different approach is necessary to study the case $p(t) = \pm 1$.*

3. An example

Example 3.1. *Consider*

$$((x(t) + x(t-\pi)))' + e^t\phi(x(t-2\pi)) + e^t\psi(x(t-3\pi)) = 0, t \geq 2\pi,$$

where $\phi(x) = \psi(x) = x^3$. Then all the conditions of Theorem 2.2 are satisfied for (1.1). Hence every solution of (1.1) oscillates. In particular, $x(t) = sint$ is one of such solution of (1.1).

Clearly, all the conditions of Theorem 2.2 are satisfied. Hence, by Theorem 2.2 every solutions of (1.1) oscillates.

4. Conclusion

In this work, we established the necessary and sufficient conditions for oscillation of a class of functional differential equations of the form

$$((x(t)+p(t)x(t-\sigma))'+q(t)\phi(x(t-\tau))+v(t)\psi(x(t-\eta))=0$$

of a neutral coefficient $p(t)$, by using the Knaster-Tarski fixed point theorem and Banach's fixed point theorem.

Declarations

Availablity of data and materials
Not applicable.

Human and animal rights
We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest
The authors declare that they have no competing interests.

Fundings
The authors declare that there is no funding available for this paper.

Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

References

[1] F. N. Ahmed, R. R. Ahmad, U. K. S. Din, M. S. M. Noorani, *Oscillation criteria of first order neutral delay differential equations with variable coefficients,* Abstr. Appl. Anal. **2013** (2013), Article ID 489804. https://doi.org/10.1155/2013/489804
[2] F. N. Ahmed, R. R. Ahmad, U. K. S. Din, M. S. M. Noorani, *Oscillations for nonlinear neutral delay differential equations with variable coefficients,* Abstr. Appl. Anal. **2014** (2014), Article ID 179195. https://doi.org/10.1155/2014/179195
[3] B. Baculikova, *Oscillation of even order linear functional differential equations with mixed deviating arguments,* Opuscula Math. **42** (2022), no. 4, 549–560. https://doi.org/10.7494/OpMath.2022.42.4.549
[4] T. Candan, *Existence of non-oscillatory solutions to first-order neutral differential equations,* Electron. J. Differ. Equ. **2016** (2016), Paper No. 39.
[5] P. Das, N. Misra, *A necessary and sufficient condtion for the solutions of a functional differential equation to be oscillatory or tend to zero,* J. Math. Anal. Appl. **204** (1997), 78–87. https://doi.org/10.1006/jmaa.1996.5143
[6] J. Džurina, *Oscillation theorems for second order advanced neutral differential equations*, Tatra Mt. Math. Publ. **48** (2011), 61–71. https://doi.org/10.2478/tatra.v48i0.97
[7] L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation Theory for Functional-differential Equations*, Marcel Dekker, Inc., New York, 1995. https://doi.org/10.1137/1038130
[8] J. R. Graef, R. Savithri, E. Thandapani, *Oscillation of first order neutral delay differential equations,* Proc. Colloq. Qual. Theory Differ. Equ. **7**, No. 12, Electron. J. Qual. Theory Differ. Equ., Szeged, 2004. https://doi.org/10.14232/ejqtde.2003.6.12
[9] M. K. Grammatikopoulos, E. A. Grove, G. Ladas, *Oscillations of first-order neutral delay differential equations,* J. Math. Anal. Appl. **120** (1986), 510–520. https://doi.org/10.1016/0022-247X(86)90172-1

[10] I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[11] J. K. Hale, *Theory of Functional Differential Equations,* Springer, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2

[12] F. Kong, *Existance of nonosillatory solutions of a kind of first order neutral differential equations,* Math. Commun. **22** (2017), no. 2, 151–164.

[13] Q. Li, R. Wang, F. Chen, T. LI, *Oscillation of second order nonlinear delay differential equations with non-positive neutral coefficients*, Adv. Difference Equ. **2015** (2015), Paper No. 35. https://doi.org/10.1186/s13662-015-0377-y

[14] T. Li , Y. V. Rogovchenko, C. Zhang, *Oscillation results for second order nonlinear neutral differential equations*, Adv. Difference Equ. **2013** (2013), Paper No. 336. https://doi.org/10.1186/1687-1847-2013-336

[15] W. T. Li, S. H. Saker, *Oscillation of nonlinear delay differential equations with variable coefficients,* Ann. Polon. Math. **77** (2001), no. 1, 39–51. https://doi.org/10.4064/ap77-1-4

[16] Y. Liu, X. Qi, *Oscillation of solutions of certain linear differential equations*, J. Comput. Anal. Appl. **24** (2018), no. 7, 1366–1374.

[17] A. Raheem, A. Afreen, A. Khatoon, *Some oscillation theorems for nonlinear fractional differential equations with impulsive effect,* Palest. J. Math. **11** (2022), no. 2, 98–107. https://doi.org/10.7494/OpMath.2022.42.6.867

[18] X. H. Tang, X. Lin, *Necessary and sufficient conditions for oscillation of first order nonlinear neutral differential equations,* J. Math. Anal. Appl. **321** (2006), 553–568. https://doi.org/10.1016/j.jmaa.2005.07.078

[19] A. K. Tripathy, K. V. V. S. Rao, *Oscillation properties of a class of nonlinear differential equations of neutral type,* Fasc. Math. **48** (2012), 129–144.

[20] Y. Zhou, Z. Chen, T. Sun, *Oscillation of nth-order nonlinear dynamic equations on time scales*, J. Comput. Anal. Appl. **24** (2018), no. 7, 1270–1285.

Abhay Kumar Sethi
Department of Mathematics, Sambalpur University, Sambalpur-768019, India
E-mail address: sethiabhaykumar100@gmail.com

Jung Rye Lee
Department of Data Science, Daejin University, Kyunggi 11159, Korea
E-mail address: jrlee@daejin.ac.kr