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On the Certain Properties and Results of Incomplete Generalized Hypergeometric Functions with Matrix Argument

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Abstract

This study aims to assess the generalized matrix transform (M-transform) of various incomplete types of special functions named generalized incomplete hypergeometric functions, incomplete H-functions, incomplete $\overline{H}$-functions, incomplete I-functions, all of which possess a matrix argument. The matrix argument in this case is a real symmetric positive definite matrix of size $k \times k$ having $\frac{k(k+1)}{2}$ variables. Here, we establish the special functions with a matrix argument by extending the existing special functions with a scalar argument. Both scalar and matrix arguments are significant in statistical distribution problems, particularly in scenarios where the null hypothesis is not assumed to be true. Additionally, we derived specific cases by extending the univariate cases.

Keywords: Generalized Incomplete Hypergeometric functions, Incomplete H-functions, Incomplete $\overline{H}$-functions, Incomplete I-functions, M-transform.

1 Introduction

Special functions with a matrix argument have demonstrated their significance since 1950 when Bochner [24] resolved a Lattice point problem utilizing the

Bessel function of matrix argument. Furthermore, Herz [11] established the hypergeometric function of matrix argument in terms of the hypergeometric function by utilizing the Laplace transform, which is an extension of the univariate Laplace transform presented in (Eq. 16, P. 219, [1]). This univariate Laplace transform and its inverse formula aid in defining the hypergeometric function ${}_pF_q$ for all p and q. However, the explicit expression of the hypergeometric function ${}_pF_q$ with a matrix argument remains undefined.

In 1955, Herz [11] derived the hypergeometric function with matrix argument by using the Laplace transform and inductive method starting from ${}_0F_0(A) = e^{tr(A)}$ and defined it by:

$$\begin{aligned} {}_{p+1}F_q\left(a_1,\ldots,a_p,y;b_1,\ldots,b_q;-z^{-1}\right)|z|^{-y} \\ = \frac{1}{\Gamma_k(y)}\int_{\Lambda>0} e^{-tr(\Lambda z)}{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-\Lambda\right)|\Lambda|^{y-\phi}d\Lambda, \quad (1)\end{aligned}$$

where, $\mathbb{R}(z) > 0, \phi = \frac{k+1}{2}, y = \phi - 1$ and

$$\begin{aligned} {}_pF_{q+1}\left(a_1,\ldots,a_p,b_1,\ldots,b_q,y;-\Lambda\right)|\Lambda|^{y-\phi} = \Gamma_k(y)\frac{1}{(2\pi i)^{k(k+1)/2}}\times \\ \int_{R(z)=X_0>0} e^{tr(\Lambda z)}{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-z^{-1}\right)|z|^{-y}dz, \quad \mathbb{R}(\Lambda) > 0. \quad (2)\end{aligned}$$

Further, Mathai [6, 8, 9] introduced the **generalized matrix transform (M-transform)** defined an integral over the $k \times k$ positive symmetric definite matrix A as follows:

$$M(f) = \int_{A>0} |A|^{s-\frac{k+1}{2}} f(A)dA. \quad (3)$$

This integral exists for $\mathbb{R}(s) > \frac{k+1}{2} - 1$, where $R(.)$ is the real part of (.). For $f(A) = e^{-trA}$ the M-transform will be $M(f) = \Gamma_k(s)$ (real matrix-variate gamma function).

Real matrix-variate gamma function $\Gamma_k(s)$ is defined as follows:

$$\Gamma_k(s) = \pi^{k(k-1)/4}\Gamma(s)\Gamma(s-\frac{1}{2})\Gamma(s-1)\ldots\Gamma(s-\frac{k-1}{2}), \quad \mathbb{R}(s) > \frac{k-1}{2}. \quad (4)$$

The M-transform of the hypergeometric function of $k \times k$ real symmetric positive definite matrix argument by the integral

$$\begin{aligned} \int_{Z>0} |Z|^{s-\frac{k+1}{2}}{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-Z\right)dZ \\ = \frac{\prod_{j=1}^q \Gamma_k(b_j)\prod_{j=1}^p \Gamma_k(a_j - s)}{\prod_{j=1}^p \Gamma_k(a_j)\prod_{j=1}^q \Gamma_k(b_j - s)}\Gamma_k(s), \quad (5)\end{aligned}$$

provided the left-hand side integral exists and it is equal to the gamma products on the right-hand side.

2

Application of hypergeometric functions of matrix argument in the field of statistical distributions developed by Mathai [10].
Progressively, Mathai [7] figure out the Fox's H-function $H(Z)$ of $k \times k$ real symmetric positive definite matrix argument z satisfies the integral equation:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} H(Z) dZ = \frac{\prod_{j=1}^{m} \Gamma_k(b_j + B_j s) \prod_{j=1}^{n} \Gamma_k(\frac{k+1}{2} - a_j - A_j s)}{\prod_{j=m+1}^{q} \Gamma_k(\frac{k+1}{2} - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma_k(a_j + A_j s)}, \quad (6)$$

whenever the left-hand side integral exists, it is equal to the gamma products on the right side and for more conditions (see [7]). Result (6) can transform to two known results:

1. By putting $k = 1$, matrix argument converts to scalar argument and
2. By putting $A_j \ (j = 1, \ldots, p) = B_j \ (j = 1, \ldots, q) = 1$, Fox's H-function of matrix argument convert to Meijer's G-function of matrix argument detail literature available in [5].

Special functions with a matrix argument are employed to address fading issues in wireless communication. Several authors have explored the applications of special functions with scalar and matrix argument, including [3, 20, 16, 17, 28, 21, 26, 22, 29, 27].

2 Some Definitions and Preliminary Results

In this section, we discuss a few more elementary definitions and preliminary results which we use to derive main theorems.

2.1 Incomplete Gamma Functions

The incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ for $x = 1$ was introduced by Prym [13] in 1877. Systematically, the incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ defined by

$$\gamma(s, y) = \int_0^y t^{s-1} e^{-t} dt, \quad (\mathbb{R}(s) > 0; \ y \geq 0), \quad (7)$$

and

$$\Gamma(s, y) = \int_y^\infty t^{s-1} e^{-t} dt, \quad (y \geq 0; \ \mathbb{R}(s) > 0 \ when \ y = 0), \quad (8)$$

respectively. The incomplete gamma functions holds the decomposition formula $\gamma(s, y) + \Gamma(s, y) = \Gamma(s)$, here $\Gamma(.)$ is the well known gamma function given by $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$, $\mathbb{R}(s) > 0$.

2.2 Incomplete Pochhammer Symbols

In terms of incomplete gamma functions $\gamma(s, y)$ and $\Gamma(s, y)$ defined in (7) and (8) Srivastava et al. [14] introduced incomplete Pochhammer symbols $(\nu; x)_\lambda$ and $[\nu; x]_\lambda$ as follows:

$$(\nu; x)_\lambda = \frac{\gamma(\nu+\lambda, s)}{\Gamma(\nu)} \quad \text{and} \quad [\nu; x]_\lambda = \frac{\Gamma(\nu+\lambda, s)}{\Gamma(\nu)}, \tag{9}$$

here $\nu, \lambda \in \mathbb{C},\ x \geq 0$. These incomplete Pochhammer symbols $(\nu; x)_\lambda$ and $[\nu; x]_\lambda$ given in (9) holds the decomposition formula as:

$$(\nu; x)_\lambda + [\nu; x]_\lambda = (\nu)_\lambda \quad (\nu, \lambda \in \mathbb{C},\ x \geq 0),$$

where well known Pochhammer symbol $(\nu)_\lambda = \frac{\Gamma(\nu+\lambda)}{\Gamma(\lambda)},\ \nu \in \mathbb{C} \backslash Z_0^-$.

2.3 Generalized Incomplete Hypergeometric Functions

The incomplete Pochhammer symbols are the backbone of the incomplete form of special functions defined in this section. For $(|arg(-z)| < \pi)$, Srivastava et al. [14] introduced generalized incomplete hypergeometric functions along with Mellin-Barnes integral in terms of incomplete Pochhammer symbols as follows:

$$\begin{aligned} {}_p\gamma_q \left[\begin{array}{l} (\alpha_1, x), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} z \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1; x)_n (\alpha_2)_n, \ldots, (\alpha_p)_n}{(\delta_1)_n, \ldots, (\delta_q)_n} \frac{z^n}{n!} \\ = \frac{1}{2\pi i} \frac{\Gamma(\delta_1) \ldots \Gamma(\delta_q)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)} &\int_L \frac{\gamma(\alpha_1 + s, x) \Gamma(\alpha_2 + s) \ldots \Gamma(\alpha_p + s)}{\Gamma(\delta_1 + s) \ldots \Gamma(\delta_q + s)} \Gamma(-s)(-z)^s ds, \end{aligned} \tag{10}$$

and

$$\begin{aligned} {}_p\Gamma_q \left[\begin{array}{l} (\alpha_1, x), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} z \right] &= \sum_{n=0}^{\infty} \frac{[\alpha_1; x]_n (\alpha_2)_n, \ldots, (\alpha_p)_n}{(\delta_1)_n, \ldots, (\delta_q)_n} \frac{z^n}{n!} \\ = \frac{1}{2\pi i} \frac{\Gamma(\delta_1) \ldots \Gamma(\delta_q)}{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)} &\int_L \frac{\Gamma(\alpha_1 + s, x) \Gamma(\alpha_2 + s) \ldots \Gamma(\alpha_p + s)}{\Gamma(\delta_1 + s) \ldots \Gamma(\delta_q + s)} \Gamma(-s)(-z)^s ds. \end{aligned} \tag{11}$$

Let $L = L_{(\sigma; \mp i\infty)}$ be a MellinBarnes-type contour from $\sigma - i\infty$ to $\sigma + i\infty$ $(\sigma \in \mathbb{R})$ with the usual indentations to separate one set of poles from the other set of poles of the integrand.
Further, we have the following decomposition formula in terms of the well-known generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N})$ as follows:

$$\begin{aligned} {}_p\gamma_q \left[\begin{array}{l} (\alpha_1, x), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} z \right] + {}_p\Gamma_q \left[\begin{array}{l} (\alpha_1, x), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} z \right] \\ = {}_pF_q \left[\begin{array}{l} \alpha_1, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} z \right]. \end{aligned}$$

2.4 Incomplete H-Functions

The incomplete H-functions introduced by Srivastava et al. [15] in terms of incomplete gamma functions $\gamma(s,y)$ and $\Gamma(s,y)$ as follows:

$$\gamma_{P,Q}^{M,N}(z)=\gamma_{P,Q}^{M,N}\left[z\,\middle|\,\begin{matrix}(\mathfrak{f}_1,\mathfrak{F}_1,t),(\mathfrak{f}_j,\mathfrak{F}_j)_{2,P}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,Q}\end{matrix}\right]:=\frac{1}{2\pi i}\int_L\varphi(s,t)z^{-s}ds, \tag{12}$$

where

$$\varphi(s,t)=\frac{\gamma(1-\mathfrak{f}_1-\mathfrak{F}_1s,t)\prod_{j=1}^{M}\Gamma(\mathfrak{w}_j+\mathfrak{W}_js)\prod_{j=2}^{N}\Gamma(1-\mathfrak{f}_j-\mathfrak{F}_js)}{\prod_{j=M+1}^{Q}\Gamma(1-\mathfrak{w}_j-\mathfrak{W}_js)\prod_{j=N+1}^{P}\Gamma(\mathfrak{f}_j+\mathfrak{F}_js)}, \tag{13}$$

and

$$\Gamma_{P,Q}^{M,N}(z)=\Gamma_{P,Q}^{M,N}\left[z\,\middle|\,\begin{matrix}(\mathfrak{f}_1,\mathfrak{F}_1,t),(\mathfrak{f}_j,\mathfrak{F}_j)_{2,P}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,Q}\end{matrix}\right]:=\frac{1}{2\pi i}\int_L\phi(s,t)z^{-s}ds, \tag{14}$$

where

$$\phi(s,t)=\frac{\Gamma(1-\mathfrak{f}_1-\mathfrak{F}_1s,t)\prod_{j=1}^{M}\Gamma(\mathfrak{w}_j+\mathfrak{W}_js)\prod_{j=2}^{N}\Gamma(1-\mathfrak{f}_j-\mathfrak{F}_js)}{\prod_{j=M+1}^{Q}\Gamma(1-\mathfrak{w}_j-\mathfrak{W}_js)\prod_{j=N+1}^{P}\Gamma(\mathfrak{f}_j+\mathfrak{F}_js)}. \tag{15}$$

The incomplete H-functions $\gamma_{P,Q}^{M,N}(z)$ and $\Gamma_{P,Q}^{M,N}(z)$ are exist for all $t\geq 0$ and for more existing conditions (see [12], [15]).

2.5 Incomplete $\overline{H}$-Functions

The incomplete $\overline{H}$-functions $\overline{\gamma}_{P,Q}^{M,N}(z)$ and $\overline{\Gamma}_{P,Q}^{M,N}(z)$ introduced by Srivastava et al. [15] in terms of incomplete gamma functions $\gamma(s,y)$ and $\Gamma(s,y)$ as follows:

$$\overline{\gamma}_{P,Q}^{M,N}\left[z\,\middle|\,\begin{matrix}(\mathfrak{f}_1,\mathfrak{F}_1;\beta_1:t),(\mathfrak{f}_j,\mathfrak{F}_j;\beta_j)_{2,N},(\mathfrak{f}_j,\mathfrak{F}_j)_{N+1,P}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,M},(\mathfrak{w}_j,\mathfrak{W}_j;\alpha_j)_{M+1,Q}\end{matrix}\right]:=\frac{1}{2\pi i}\int_L\overline{\varphi}(s,t)z^{-s}ds, \tag{16}$$

where

$$\overline{\varphi}(s,t)=\frac{[\gamma(1-\mathfrak{f}_1-\mathfrak{F}_1s,t)]^{\beta_1}\prod_{j=1}^{M}\Gamma(\mathfrak{w}_j+\mathfrak{W}_js)\prod_{j=2}^{N}[\Gamma(1-\mathfrak{f}_j-\mathfrak{F}_js)]^{\beta_j}}{\prod_{j=M+1}^{Q}[\Gamma(1-\mathfrak{w}_j-\mathfrak{W}_js)]^{\alpha_j}\prod_{j=N+1}^{P}\Gamma(\mathfrak{f}_j+\mathfrak{F}_js)}, \tag{17}$$

and

$$\overline{\Gamma}_{P,Q}^{M,N}\left[z\,\middle|\,\begin{matrix}(\mathfrak{f}_1,\mathfrak{F}_1;\beta_1:t),(\mathfrak{f}_j,\mathfrak{F}_j;\beta_j)_{2,N},(\mathfrak{f}_j,\mathfrak{F}_j)_{N+1,P}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,M},(\mathfrak{w}_j,\mathfrak{W}_j;\alpha_j)_{M+1,Q}\end{matrix}\right]:=\frac{1}{2\pi i}\int_L\overline{\phi}(s,t)z^{-s}ds, \tag{18}$$

where

$$\overline{\phi}(s,t) = \frac{[\Gamma(1-\mathfrak{f}_1-\mathfrak{F}_1 s,t)]^{\beta_1}\prod_{j=1}^{M}\Gamma(\mathfrak{w}_j+\mathfrak{W}_j s)\prod_{j=2}^{N}[\Gamma(1-\mathfrak{f}_j-\mathfrak{F}_j s)]^{\beta_j}}{\prod_{j=M+1}^{Q}[\Gamma(1-\mathfrak{w}_j-\mathfrak{W}_j s)]^{\alpha_j}\prod_{j=N+1}^{P}\Gamma(\mathfrak{f}_j+\mathfrak{F}_j s)}. \tag{19}$$

The incomplete $\overline{H}$-functions $\overline{\gamma}_{P,Q}^{M,N}(z)$ and $\overline{\Gamma}_{P,Q}^{M,N}(z)$ for conditions (see [15]) are exist for all $t \geq 0$ and for more existing conditions (see, [15]).

2.6 Incomplete I-Functions

The incomplete I-functions ${}^{(\gamma)}I_{P_i.Q_i,R}^{M,N}(z)$ and ${}^{(\Gamma)}I_{P_i.Q_i,R}^{M,N}(z)$ introduced by Bansal et al. [18] in terms of incomplete gamma functions $\gamma(s,y)$ and $\Gamma(s,y)$ as follows:

$$ {}^{(\gamma)}I_{P_i.Q_i,R}^{M,N}\left[z\left|\begin{matrix}(\mathfrak{f}_1,\mathfrak{F}_1,t),(\mathfrak{f}_j,\mathfrak{F}_j)_{2,N},(\mathfrak{f}_{ji},\mathfrak{F}_{ji})_{N+1,P_i}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,M},(\mathfrak{w}_{ji},\mathfrak{W}_{ji})_{M+1,Q_i}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_L \varphi(s,t)z^{-s}ds, \tag{20}$$

where

$$\varphi(s,t) = \frac{\gamma(1-\mathfrak{f}_1-\mathfrak{F}_1 s,t)\prod_{j=1}^{M}\Gamma(\mathfrak{w}_j+\mathfrak{W}_j s)\prod_{j=2}^{N}\Gamma(1-\mathfrak{f}_j-\mathfrak{F}_j s)}{\sum_{i=1}^{R}\left[\prod_{j=M+1}^{Q_i}\Gamma(1-\mathfrak{w}_{ji}-\mathfrak{W}_{ji}s)\prod_{j=N+1}^{P_i}\Gamma(\mathfrak{f}_{ji}+\mathfrak{F}_{ji}s)\right]}, \tag{21}$$

and

$$ {}^{(\Gamma)}I_{P_i.Q_i,R}^{M,N}\left[z\left|\begin{matrix}(\mathfrak{f}_1,\mathfrak{F}_1,t),(\mathfrak{f}_j,\mathfrak{F}_j)_{2,N},(\mathfrak{f}_{ji},\mathfrak{F}_{ji})_{N+1,P_i}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,M},(\mathfrak{w}_{ji},\mathfrak{W}_{ji})_{M+1,Q_i}\end{matrix}\right.\right] := \frac{1}{2\pi i}\int_L \phi(s,t)z^{-s}ds, \tag{22}$$

where

$$\phi(s,t) = \frac{\Gamma(1-\mathfrak{f}_1-\mathfrak{F}_1 s,t)\prod_{j=1}^{M}\Gamma(\mathfrak{w}_j+\mathfrak{W}_j s)\prod_{j=2}^{N}\Gamma(1-\mathfrak{f}_j-\mathfrak{F}_j s)}{\sum_{i=1}^{R}\left[\prod_{j=M+1}^{Q_i}\Gamma(1-\mathfrak{w}_{ji}-\mathfrak{W}_{ji}s)\prod_{j=N+1}^{P_i}\Gamma(\mathfrak{f}_{ji}+\mathfrak{F}_{ji}s)\right]}. \tag{23}$$

The incomplete I-functions ${}^{(\gamma)}I_{P_i,Q_i,R}^{M,N}(z)$ and ${}^{(\Gamma)}I_{P_i,Q_i,R}^{M,N}(z)$ exists for all $t \geq 0$ and for existing conditions (see, [18, 21, 23]). We can easily define the decomposition formula of incomplete form of special functions.

2.7 Jacobians of Matrix Transformations

This section will present a few outcomes on the Jacobians of transformations that we require. For now, we will focus on the prerequisites for formulating the special functions with matrix argument. We define a real symmetric positive definite matrix X as $X = X' > 0$ (where X' is the transpose of matrix X).

6

We use the notations $X \geq 0$ to represent a positive semi-definite matrix, $X < 0$ for a negative definite matrix, $X \leq 0$ for a negative semi-definite matrix, and indefinite matrices for all other matrices. The notations $\int_{X>0} f(X)dX$ and $\int_{X=0}^{I} g(X)dX$ represent the integration of $f(X)$ over all positive matrices $X = X' > 0$ and the integration of $g(X)$ over all matrices $I - X > 0$ that are positive definite. The symbol dX represents the differential element.
Let us now discuss some elementary results regarding the Jacobians of transformations. Let L be a symmetric matrix of order k. Then L involves $\frac{k(k+1)}{2}$ variables, and its differential element is defined as $dL = dl_{11} \ldots dl_{1k}; dl_{22} \ldots dl_{2k} \ldots dl_{k-1k}; dl_{kk}$. In the case of an asymmetric (non-symmetric) matrix $L = [l]\, ij$ of order k, L involves k^2 variables, and its differential element dL is defined as $dl_{11} \ldots dl_{1k}; dl_{21} \ldots dl_{2k} \ldots dl_{k1} \ldots dl_{kk}$. Transformation of $L = [l]_{ij}$ to $M = [m]_{ij}$ here both are symmetric matrix of order k. Which implies that $\frac{k(k+1)}{2}$ variables of L transform to $\frac{k(k+1)}{2}$ variables of M. Here, we have a few results given in the previous literature.

1. If A and B are $k \times k$ symmetric and X is a $k \times k$ non singular then

$$A = XBX' \implies dA = |X|^{k+1} dB, \tag{24}$$

   where $|X|$ and X' represent the determinant and transpose of X.

2. If $L = [l]_{ij}$ is $k \times k$ symmetric and $M = [m]_{ij}$ is $k \times k$ lower triangular matrices respectively then

$$L = MM' \implies dL = \left[2^k \prod_{i=1}^{k} m_{ii}^{k+1-i} \right] dM. \tag{25}$$

Convolution Property: If M-transform of two symmetric functions $f_1(A)$ and $f_2(A)$ are $G_1(s)$ and $G_2(s)$ respectively, then M.transform of a function $f_3(A) = \int_{\Lambda>0} |\Lambda|^a f_1(A\Lambda) f_2(\Lambda) d\Lambda$ is defiend by

$$M(f_3) = G_1(s) G_2 \left(\frac{k+1}{2} + a - s \right). \tag{26}$$

From the (3) we observe that $M(f)$ is a function of s (univariate), although $f(\Lambda)$ is a multivariate we need not have uniqueness for $f(\Lambda)$.
Real matrix-variate Beta function $B_k(s_1, s_2)$ define as follows:

$$B_k(s_1, s_2) = \frac{\Gamma_k(s_1)\Gamma_k(s_2)}{\Gamma_k(s_1 + s_2)}, \quad \mathbb{R}(s_1) > \frac{k-1}{2},\ \mathbb{R}(s_2) > \frac{k-1}{2}.$$

The integral representation of the Real Matrix-variate Beta function is defined as follows:

$$B_k(s_1, s_2) = \int_X |X|^{s_1 - \frac{k+1}{2}} |I - X|^{s_2 - \frac{k+1}{2}} dX, \tag{27}$$

here, $X > 0,\ 0 < X < I \implies I - X > 0$ and $\mathbb{R}(s_1) > \frac{k-1}{2},\ \mathbb{R}(s_2) > \frac{k-1}{2}$. Eigen values of X i.e. $\lambda_1, \lambda_2, \ldots, \lambda_k$ are in the interval of $(0, 1)$.
We can extend more univariate integrals to matrix cases by using convolution property (26) as follows:

1. Taking $a = \alpha - \frac{k+1}{2},\ f_1(A) = e^{-trA}$ and $f_2(A) = |I - A|^{\beta - \frac{k+1}{2}}$ in (26), we get

$$\int_0^I |A|^{\alpha-\frac{k+1}{2}} |I - A|^{\beta-\frac{k+1}{2}} e^{-tr\Lambda A} dA = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} {}_1F_1(\alpha; \alpha+\beta; -\Lambda), \quad \mathbb{R}(\alpha),\ \mathbb{R}(\beta) > \frac{k+1}{2} - 1. \tag{28}$$

2. Putting $a = \alpha - \frac{k+1}{2},\ f_1(A) = |I - A|^{-\beta}$ and $f_2(A) = |I - A|^{\gamma-\alpha-\frac{k+1}{2}}$ in (26), we get

$$\int_0^I |I - \Lambda A|^{-\beta} |\Lambda|^{\alpha-\frac{k+1}{2}} |I - \Lambda|^{\gamma-\alpha-\frac{k+1}{2}} d\Lambda = \frac{\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}{\Gamma_k(\gamma)} \times {}_2F_1(\alpha, \beta; \gamma; A), \quad \mathbb{R}(\beta) > 0,\ \mathbb{R}(\alpha),\ \mathbb{R}(\gamma-\alpha) > \frac{k+1}{2} - 1. \tag{29}$$

3. Another extension of univariate integral to matrix case as follows:

$$\int_{\Lambda>0} |\Lambda|^{\alpha-\frac{k+1}{2}} |I + A\Lambda|^{-\mu} d\Lambda = \frac{\Gamma_k(\alpha)\Gamma_k(\mu-\alpha)}{\Gamma_k(\mu)} |A|^{-\alpha}, \quad A > 0,\ \mathbb{R}(\alpha),\ \mathbb{R}(\mu-\alpha) > \frac{k+1}{2} - 1. \tag{30}$$

   Here, we substitute $V = A^{1/2}\Lambda A^{1/2}$. Then $dV = |A|^{\frac{k+1}{2}} d\Lambda$ in (30).

4. Further, set $U^{-1} = I + V$ i.e. $dV = |U|^{-(k+1)} dU$ and $0 < U < I$ then LHS of (30) can be written as $|A|^{-\alpha} \int_{V>0} |V|^{\alpha-\frac{k+1}{2}} |I + V|^{-\mu} dV$ and reduces to beta integral defined in (27) and on transforming $CV = A$ in (29), we get a new univariate integral as follows:

$$\int_0^C |I + ZA|^{-\mu} |A|^{\alpha-\frac{k+1}{2}} dA = \frac{\Gamma_k(\alpha)\Gamma_k(\frac{k+1}{2})}{\Gamma_k(\alpha + \frac{k+1}{2})} |C|^{\alpha} \times {}_2F_1(\alpha, \mu; \alpha + \frac{k+1}{2}; -ZC), \quad C > 0,\ \mathbb{R}(\alpha) > \frac{k+1}{2} - 1. \tag{31}$$

5. In (31), making the transformations $V = I + \Lambda,\ U = V^{-1}$ and then use ${}_2F_1(\alpha, \beta; \gamma; A) = |(I - A)|^{-\beta} {}_2F_1(\gamma - \alpha, \beta; \gamma; -A(I - A)^{-1})$. We have

$$\int_{\Lambda>0} |\Lambda|^{\alpha-\frac{k+1}{2}} |I + \Lambda|^{\mu} |I + Z\Lambda|^{\nu} d\Lambda = \frac{\Gamma_k(\alpha)\Gamma_k(\nu+\mu-\alpha)}{\Gamma_k(-\nu-\mu)} \times {}_2F_1(-\nu, \alpha; -\nu-\mu; I - Z),\ -\mathbb{R}(\nu+\mu) > \mathbb{R}(\alpha) > \frac{k+1}{2} - 1. \tag{32}$$

6. Making the transformation $V = A^{-1/2}\Lambda A^{-1/2}$ in (28). We get

$$\int_0^Z |\Lambda|^{\alpha-\frac{k+1}{2}} e^{-trC\Lambda} d\Lambda = |Z|^{\alpha} \frac{\Gamma_k(\alpha)\Gamma_k(\frac{k+1}{2})}{\Gamma_k(\alpha+\frac{k+1}{2})} \times {}_1F_1(\alpha; \alpha+\frac{k+1}{2}; -ZC), \quad \mathbb{R}(\alpha) > \frac{k+1}{2} - 1. \quad (33)$$

It is important to note that **the incomplete gamma function** can be generalized using $C = I$ in (33). The incomplete gamma functions for univariate matrix cases can be written as:

$$\begin{aligned} \gamma_k(\alpha, Z) &= \int_{\Lambda=0}^{Z} |\Lambda|^{\alpha-\frac{k+1}{2}} e^{-tr\Lambda} d\Lambda, \\ \text{and } \Gamma_k(\alpha, Z) &= \int_{\Lambda>Z} |\Lambda|^{\alpha-\frac{k+1}{2}} e^{-tr\Lambda} d\Lambda = \Gamma_k(\alpha) - \gamma_k(\alpha, Z). \end{aligned} \quad (34)$$

For multivariate cases $\int_{\Lambda>Z} |\Lambda|^{\alpha-\frac{k+1}{2}} e^{-tr\Lambda} d\Lambda \neq \Gamma_k(\alpha) - \gamma_k(\alpha, Z)$ since $\int_A^B + \int_B^C \neq \int_A^C$ is not valid for all values of Z when Z is a matrix.
There are three approaches to deriving special functions of matrix argument:

1. Bochner [24] and Herz's [11] using Laplace approach,
2. James [2, 3] and Constantine's [4] develop zonal polynomial approach and
3. Mathai's [19, 7] generalized matrix transform (M-transform) method.

In this work, we use the M-transform method to derive the special functions of the matrix argument.

3 Main Results

In this section, we evaluate some results using the M-transform of various incomplete types of special functions like generalized incomplete hypergeometric functions, incomplete H-functions, incomplete $\overline{H}$-functions, incomplete I-functions.
Theorem 1: Let Z be a $k \times k$ real symmetric positive definite matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ and generalized incomplete hypergeometric functions ${}_p\gamma_q(Z)$ and ${}_p\Gamma_q(Z)$ are symmetric functions in the sense ${}_p\gamma_q(Z) = {}_p\gamma_q(lZl')$ and ${}_p\Gamma_q(Z) = {}_p\Gamma_q(lZl')$, $ll' = I$ for all orthogonal matrices. If s is an arbitrary parameter then consider the integral equations:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_p\gamma_q \left[\begin{matrix} (\alpha_1, A), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{matrix} \; -Z \right] dZ = \frac{\prod_{j=1}^q \Gamma_k(\delta_j)}{\prod_{i=1}^p \Gamma_k(\alpha_i)} \frac{\gamma_k(\alpha_1 - s, A) \prod_{i=2}^p \Gamma_k(\alpha_i - s)}{\prod_{j=1}^q \Gamma_k(\delta_j - s)} \Gamma_k(s), \quad (35)$$

9

and

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_p\Gamma_q \left[\begin{array}{l} (\alpha_1, A), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} -Z \right] dZ = \frac{\prod_{j=1}^{q} \Gamma_k(\delta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\Gamma_k(\alpha_1 - s, A) \prod_{i=2}^{p} \Gamma_k(\alpha_i - s)}{\prod_{j=1}^{q} \Gamma_k(\delta_j - s)} \Gamma_k(s), \quad (36)$$

provided these gamma products are defined.

Proof: Here Z is a $k \times k$ real symmetric positive definite matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ and generalized incomplete hypergeometric functions ${}_p\gamma_q(Z)$ and ${}_p\Gamma_q(Z)$ are symmetric functions in the sense that ${}_p\gamma_q(Z) = {}_p\gamma_q(lZl')$ and ${}_p\Gamma_q(Z) = {}_p\Gamma_q(lZl')$, $ll' = I$ for all orthogonal matrices. In this case, we have $f(Z\Lambda) = f(\Lambda Z) = f(\Lambda^{1/2} Z \Lambda^{1/2})$ whenever $\Lambda^{1/2}$ is defined.
For the positive semi definite matrix Z there exists a lower triangular matrix T such that $Z = TT'$. Now transforming Z to T by using (25) as $dZ = \left[2^k \prod_{i=1}^{k} t_{ii}^{k+1-i}\right] dT$ and $|TT'| = \prod_{i=1}^{k} t_{ii}^2$. After substituting these values in the left-hand of (35) and (36), use (34) and after a bit of simplification, we get the desired result.

Theorem 2: This generalized incomplete hypergeometric functions with matrix argument hold the decomposition formula as follows:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_p\gamma_q \left[\begin{array}{l} (\alpha_1, A), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} -Z \right] dZ + \int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_p\Gamma_q \left[\begin{array}{l} (\alpha_1, A), \alpha_2, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} -Z \right] dZ = \int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_pF_q \left[\begin{array}{l} \alpha_1, \ldots, \alpha_p; \\ \delta_1, \ldots, \delta_q; \end{array} -Z \right] dZ. \quad (37)$$

Proof: We can write left hand side of (37) by using (35) and (36) as follows:

$$\frac{\prod_{j=1}^{q} \Gamma_k(\delta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\gamma_k(\alpha_1 - s, A) \prod_{i=2}^{p} \Gamma_k(\alpha_i - s)}{\prod_{j=1}^{q} \Gamma_k(\delta_j - s)} \Gamma_k(s) + \frac{\prod_{j=1}^{q} \Gamma_k(\delta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\Gamma_k(\alpha_1 - s, A) \prod_{i=2}^{p} \Gamma_k(\alpha_i - s)}{\prod_{j=1}^{q} \Gamma_k(\delta_j - s)} \Gamma_k(s) = \frac{\prod_{j=1}^{q} \Gamma_k(\delta_j)}{\prod_{i=1}^{p} \Gamma_k(\alpha_i)} \frac{\prod_{i=1}^{p} \Gamma_k(\alpha_i - s)}{\prod_{j=1}^{q} \Gamma_k(\delta_j - s)} \Gamma_k(s). \quad (38)$$

The right-hand side of (38) is the same as derived by Mathai (eq. 3.3, [6]).

Theorem 3: Let Z be a $k \times k$ real symmetric positive definite matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ and the incomplete H-functions $\gamma_{P,Q}^{M,N}(Z)$

10

and $\Gamma_{P,Q}^{M,N}(Z)$ be a symmetric functions in the sense $\gamma_{P,Q}^{M,N}(Z) = \gamma_{P,Q}^{M,N}(lZl')$ and $\Gamma_{P,Q}^{M,N}(Z) = \Gamma_{P,Q}^{M,N}(lZl')$, $ll' = I$ for all orthogonal matrices. If s is an arbitrary parameter then consider the integral equations for incomplete H-functions as follows:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \gamma_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array}\right.\right] dZ$$
$$= \frac{\gamma_k(\frac{k+1}{2} - \mathfrak{f}_1 - \mathfrak{F}_1 s, A) \prod_{j=1}^{M} \Gamma_k(\mathfrak{w}_j + \mathfrak{W}_j s) \prod_{j=2}^{N} \Gamma_k(\frac{k+1}{2} - \mathfrak{f}_j - \mathfrak{F}_j s)}{\prod_{j=M+1}^{Q} \Gamma_k(\frac{k+1}{2} - \mathfrak{w}_j - \mathfrak{W}_j s) \prod_{j=N+1}^{P} \Gamma_k(\mathfrak{f}_j + \mathfrak{F}_j s)}, \quad (39)$$

and

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \Gamma_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array}\right.\right] dZ$$
$$= \frac{\Gamma_k(\frac{k+1}{2} - \mathfrak{f}_1 - \mathfrak{F}_1 s, A) \prod_{j=1}^{M} \Gamma_k(\mathfrak{w}_j + \mathfrak{W}_j s) \prod_{j=2}^{N} \Gamma_k(\frac{k+1}{2} - \mathfrak{f}_j - \mathfrak{F}_j s)}{\prod_{j=M+1}^{Q} \Gamma_k(\frac{k+1}{2} - \mathfrak{w}_j - \mathfrak{W}_j s) \prod_{j=N+1}^{P} \Gamma_k(\mathfrak{f}_j + \mathfrak{F}_j s)}. \quad (40)$$

provided these gamma products are defined.

Proof: Here Z is a $k \times k$ real symmetric positive definite matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ and generalized incomplete hypergeometric functions ${}_p\gamma_q(Z)$ and ${}_p\Gamma_q(Z)$ are symmetric functions in the sense that ${}_p\gamma_q(Z) = {}_p\gamma_q(lZl')$ and ${}_p\Gamma_q(Z) = {}_p\Gamma_q(lZl')$, $ll' = I$ for all orthogonal matrices. In this case, we have $f(Z\Lambda) = f(\Lambda Z) = f(\Lambda^{1/2} Z \Lambda^{1/2})$ whenever $\Lambda^{1/2}$ is defined.
For the positive semi definite matrix Z there exists a lower triangular matrix T such that $Z = TT'$. Now transforming Z to T by using (25) as $dZ = \left[2^k \prod_{i=1}^{k} t_{ii}^{k+1-i}\right] dT$ and $|TT'| = \prod_{i=1}^{k} t_{ii}^2$. After substituting these values in the left-hand of (35) and (36), use (34) and after a bit of simplification, we get the desired result.

Theorem 4: The incomplete H-functions with matrix argument also hold the decomposition formula as follows:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \gamma_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array}\right.\right] dZ +$$
$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \Gamma_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array}\right.\right] dZ$$
$$= \int_{Z>0} |Z|^{s-\frac{k+1}{2}} H_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_j, \mathfrak{F}_j)_{1,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array}\right.\right] dZ. \quad (41)$$

Proof: We can write left hand side of (41) as

$$\frac{\gamma_k(\frac{k+1}{2}-\mathfrak{f}_1-\mathfrak{F}_1 s, A)\prod_{j=1}^{M}\Gamma_k(\mathfrak{w}_j+\mathfrak{W}_j s)\prod_{j=2}^{N}\Gamma_k(\frac{k+1}{2}-\mathfrak{f}_j-\mathfrak{F}_j s)}{\prod_{j=M+1}^{Q}\Gamma_k(\frac{k+1}{2}-\mathfrak{w}_j-\mathfrak{W}_j s)\prod_{j=N+1}^{P}\Gamma_k(\mathfrak{f}_j+\mathfrak{F}_j s)}+$$
$$\frac{\Gamma_k(\frac{k+1}{2}-\mathfrak{f}_1-\mathfrak{F}_1 s, A)\prod_{j=1}^{M}\Gamma_k(\mathfrak{w}_j+\mathfrak{W}_j s)\prod_{j=2}^{N}\Gamma_k(\frac{k+1}{2}-\mathfrak{f}_j-\mathfrak{F}_j s)}{\prod_{j=M+1}^{Q}\Gamma_k(\frac{k+1}{2}-\mathfrak{w}_j-\mathfrak{W}_j s)\prod_{j=N+1}^{P}\Gamma_k(\mathfrak{f}_j+\mathfrak{F}_j s)}$$
$$=\frac{\prod_{j=1}^{M}\Gamma_k(\mathfrak{w}_j+\mathfrak{W}_j s)\prod_{j=1}^{N}\Gamma_k(\frac{k+1}{2}-\mathfrak{f}_j-\mathfrak{F}_j s)}{\prod_{j=M+1}^{Q}\Gamma_k(\frac{k+1}{2}-\mathfrak{w}_j-\mathfrak{W}_j s)\prod_{j=N+1}^{P}\Gamma_k(\mathfrak{f}_j+\mathfrak{F}_j s)}. \quad (42)$$

The right-hand side of (42) is the same as derived by Mathai (eq. 3.1, [7]).

Corollary 1: Let Z be a $k\times k$ real symmetric positive definite matrix. Then by using the definition of incomplete H-function $\Gamma_{P,Q}^{M,N}(Z)$. We have

$$|Z|^{\omega}\Gamma_{P,Q}^{M,N}\left[Z\left|\begin{array}{l}(\mathfrak{f}_1,\mathfrak{F}_1,A),(\mathfrak{f}_j,\mathfrak{F}_j)_{2,P}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,Q}\end{array}\right.\right]$$
$$=\Gamma_{P,Q}^{M,N}\left[Z\left|\begin{array}{l}(\mathfrak{f}_1+\omega\mathfrak{F}_1,\mathfrak{F}_1,A),(\mathfrak{f}_j+\omega\mathfrak{F}_j,\mathfrak{F}_j)_{2,P}\\(\mathfrak{w}_j+\omega\mathfrak{W}_j,\mathfrak{W}_j)_{1,Q}\end{array}\right.\right]. \quad (43)$$

Proof: By using a similar matrix argument, we can obtain property (43) if we substitute $s+\omega=\mathfrak{s}; (\omega>0)$ and $ds=d\mathfrak{s}$. This will give us the desired result.

Corollary 2: Let Z be a $k\times k$ real symmetric positive definite matrix. Then by using the definition of incomplete H-function $\Gamma_{P,Q}^{M,N}(Z)$. We have the following result

$$\Gamma_{P,Q}^{M,N}\left[Z^{-1}\left|\begin{array}{l}(\mathfrak{f}_1,\mathfrak{F}_1,A),(\mathfrak{f}_j,\mathfrak{F}_j)_{2,P}\\(\mathfrak{w}_j,\mathfrak{W}_j)_{1,Q}\end{array}\right.\right]$$
$$=\Gamma_{Q,P}^{N,M}\left[Z\left|\begin{array}{l}(\frac{k+1}{2}-\mathfrak{w}_j,\mathfrak{W}_j)_{1,Q}\\(\frac{k+1}{2}-\mathfrak{f}_1,\mathfrak{F}_1,A),(\frac{k+1}{2}-\mathfrak{f}_j,\mathfrak{F}_j)_{2,P}\end{array}\right.\right]. \quad (44)$$

Proof: This result is obtained by using equation (40), and applying the transformation $L=Z^{-1}$ while noting that $dZ=|L|^{-(k+1)}dL$.

Theorem 5: Let Z be a $k\times k$ real symmetric positive definite matrix with eigenvalues $\lambda_1>\lambda_2>\cdots>\lambda_k>0$ and the incomplete $\overline{H}$-functions $\overline{\gamma}_{P,Q}^{M,N}(z)$ and $\overline{\Gamma}_{P,Q}^{M,N}(z)$ be a symmetric functions in the sense $\overline{\gamma}_{P,Q}^{M,N}(Z)=\overline{\gamma}_{P,Q}^{M,N}(lZl')$ and $\overline{\Gamma}_{P,Q}^{M,N}(Z)=\overline{\Gamma}_{P,Q}^{M,N}(lZl')$, $ll'=I$ for all orthogonal matrices. If s is an arbitrary parameter then consider the integral equations:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \overline{\gamma}_{P,Q}^{M,N} \left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1; \beta_1 : A), (\mathfrak{f}_j, \mathfrak{F}_j; \beta_j)_{2,N}, (\mathfrak{f}_j, \mathfrak{F}_j)_{N+1,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,M}, (\mathfrak{w}_j, \mathfrak{W}_j; \alpha_j)_{M+1,Q} \end{array} \right. \right] dZ$$
$$= \frac{[\gamma_k(\frac{k+1}{2} - \mathfrak{f}_1 - \mathfrak{F}_1 s, A)]^{\beta_1} \prod_{j=1}^{M} \Gamma_k(\mathfrak{w}_j + \mathfrak{W}_j s) \prod_{j=2}^{N} [\Gamma_k(\frac{k+1}{2} - \mathfrak{f}_j - \mathfrak{F}_j s)]^{\beta_j}}{\prod_{j=M+1}^{Q} [\Gamma_k(\frac{k+1}{2} - \mathfrak{w}_j - \mathfrak{W}_j s)]^{\alpha_j} \prod_{j=N+1}^{P} \Gamma_k(\mathfrak{f}_j + \mathfrak{F}_j s)}, \tag{45}$$

and

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \overline{\Gamma}_{P,Q}^{M,N} \left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1; \beta_1 : A), (\mathfrak{f}_j, \mathfrak{F}_j; \beta_j)_{2,N}, (\mathfrak{f}_j, \mathfrak{F}_j)_{N+1,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,M}, (\mathfrak{w}_j, \mathfrak{W}_j; \alpha_j)_{M+1,Q} \end{array} \right. \right] dZ$$
$$= \frac{[\Gamma_k(\frac{k+1}{2} - \mathfrak{f}_1 - \mathfrak{F}_1 s, A)]^{\beta_1} \prod_{j=1}^{M} \Gamma_k(\mathfrak{w}_j + \mathfrak{W}_j s) \prod_{j=2}^{N} [\Gamma_k(\frac{k+1}{2} - \mathfrak{f}_j - \mathfrak{F}_j s)]^{\beta_j}}{\prod_{j=M+1}^{Q} [\Gamma_k(\frac{k+1}{2} - \mathfrak{w}_j - \mathfrak{W}_j s)]^{\alpha_j} \prod_{j=N+1}^{P} \Gamma_k(\mathfrak{f}_j + \mathfrak{F}_j s)}. \tag{46}$$

Proof: We can prove this Theorem by follow same steps as in Theorem 1.

Theorem 6: Let Z be a $k \times k$ real symmetric positive definite matrix with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ and the incomplete I-functions ${}^{(\gamma)}I_{P_i.Q_i,R}^{M,N}(z)$ and ${}^{(\Gamma)}I_{P_i.Q_i,R}^{M,N}(z)$ be a symmetric functions ${}^{(\gamma)}I_{P_i.Q_i,R}^{M,N}(z) = {}^{(\gamma)}I_{P_i.Q_i,R}^{M,N}(z)(lZl')$ and ${}^{(\Gamma)}I_{P_i.Q_i,R}^{M,N}(z) = {}^{(\Gamma)}I_{P_i.Q_i,R}^{M,N}(z)(lZl')$, $ll' = I$ for all orthogonal matrices. If s is an arbitrary parameter then consider the integral equations:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}^{(\gamma)}I_{P_i.Q_i,R}^{M,N} \left[z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,N}, (\mathfrak{f}_{ji}, \mathfrak{F}_{ji})_{N+1,P_i} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,M}, (\mathfrak{w}_{ji}, \mathfrak{W}_{ji})_{M+1,Q_i} \end{array} \right. \right]$$
$$= \frac{\gamma_k(\frac{k+1}{2} - \mathfrak{f}_1 - \mathfrak{F}_1 s, A) \prod_{j=1}^{M} \Gamma_k(\mathfrak{w}_j + \mathfrak{W}_j s) \prod_{j=2}^{N} \Gamma_k(\frac{k+1}{2} - \mathfrak{f}_j - \mathfrak{F}_j s)}{\sum_{i=1}^{R} \left[\prod_{j=M+1}^{Q_i} \Gamma_k(\frac{k+1}{2} - \mathfrak{w}_{ji} - \mathfrak{W}_{ji} s) \prod_{j=N+1}^{P_i} \Gamma_k(\mathfrak{f}_{ji} + \mathfrak{F}_{ji} s) \right]}, \tag{47}$$

and

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}^{(\Gamma)}I_{P_i.Q_i,R}^{M,N} \left[z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,N}, (\mathfrak{f}_{ji}, \mathfrak{F}_{ji})_{N+1,P_i} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,M}, (\mathfrak{w}_{ji}, \mathfrak{W}_{ji})_{M+1,Q_i} \end{array} \right. \right]$$
$$= \frac{\Gamma_k(\frac{k+1}{2} - \mathfrak{f}_1 - \mathfrak{F}_1 s, A) \prod_{j=1}^{M} \Gamma_k(\mathfrak{w}_j + \mathfrak{W}_j s) \prod_{j=2}^{N} \Gamma_k(\frac{k+1}{2} - \mathfrak{f}_j - \mathfrak{F}_j s)}{\sum_{i=1}^{R} \left[\prod_{j=M+1}^{Q_i} \Gamma_k(\frac{k+1}{2} - \mathfrak{w}_{ji} - \mathfrak{W}_{ji} s) \prod_{j=N+1}^{P_i} \Gamma_k(\mathfrak{f}_{ji} + \mathfrak{F}_{ji} s) \right]}. \tag{48}$$

Proof: We can prove this Theorem by follow same steps as in Theorem 1.

We can formulate two additional theorems for the decomposition formula of the incomplete $\overline{H}$-functions and the incomplete I-functions, similar to what we did in Theorems 2 and 4.

4 Particular Cases

This section delves into the analysis of specific cases that arise from our main findings. When considering a matrix argument Z (which is a real symmetric positive definite matrix of size $k \times k$), we can identify the following particular cases:

1. When $\mathfrak{F}_j = \mathfrak{W}_j = 1$, it is possible to demonstrate that the M-transform of incomplete H-functions, which have a matrix argument defined in (40), satisfy the M-transform of incomplete Meijer ${}^{(\Gamma)}G$-function (see [25]) with a matrix argument given by:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \Gamma_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, 1, A), (\mathfrak{f}_j, 1)_{2,P} \\ (\mathfrak{w}_j, 1)_{1,Q} \end{array} \right.\right] dZ$$
$$= \int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}^{(\Gamma)}G_{P,Q}^{M,N}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, A), (\mathfrak{f}_j)_{2,P} \\ (\mathfrak{w}_j)_{1,Q} \end{array} \right.\right] dz$$
$$= \frac{\Gamma_k(1-\mathfrak{f}_1-s, A)\prod_{j=1}^{M}\Gamma_k(\mathfrak{w}_j+s)\prod_{j=2}^{N}\Gamma_k(1-\mathfrak{f}_j-s)}{\prod_{j=M+1}^{Q}\Gamma_k(1-\mathfrak{w}_j-s)\prod_{j=N+1}^{P}\Gamma_k(\mathfrak{f}_j+s)}. \quad (49)$$

2. When $M = 1$, $N = P$, and we replace Q with $Q+1$, we can obtain incomplete H-functions with matrix arguments (39) and (40) by appropriately choosing parameters. Specifically, we can set $Z = -Z$ and $\mathfrak{f}_j \to (1-\mathfrak{f}_j)$ for $j = 1, \ldots, P$, and $\mathfrak{w}_j \to (1-\mathfrak{w}_j)$ for $j = 1, \ldots, Q$. With these choices, the incomplete H-functions can be transformed into incomplete Fox-Wright functions with matrix arguments ${}_P\psi_Q{}^{(\gamma)}(Z)$ and ${}_P\psi_Q{}^{(\Gamma)}(Z)$, as follows:

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \gamma_{P,Q+1}^{1,P}\left[-Z \left| \begin{array}{l} (1-\mathfrak{f}_1, \mathfrak{F}_1, A), (1-\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (0,1), (1-\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array} \right.\right] dZ$$
$$= \int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_P\psi_Q{}^{(\gamma)}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array} \right.\right] dZ$$
$$= \frac{\prod_{j=1}^{Q}\Gamma_k(\mathfrak{w}_j)}{\prod_{j=1}^{P}\Gamma_k(\mathfrak{f}_j)} \times \frac{\gamma_k(\mathfrak{f}_1-s, A)\prod_{j=2}^{P}\Gamma_k(\mathfrak{f}_j-s)}{\prod_{j=1}^{Q}\Gamma_k(\mathfrak{w}_j-s)}\Gamma_k(s), \quad (50)$$

and

$$\int_{Z>0} |Z|^{s-\frac{k+1}{2}} \Gamma_{P,Q+1}^{1,P}\left[-Z \left| \begin{array}{l} (1-\mathfrak{f}_1, \mathfrak{F}_1, A), (1-\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (0,1), (1-\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array} \right.\right] dZ$$
$$= \int_{Z>0} |Z|^{s-\frac{k+1}{2}} {}_P\psi_Q{}^{(\Gamma)}\left[Z \left| \begin{array}{l} (\mathfrak{f}_1, \mathfrak{F}_1, A), (\mathfrak{f}_j, \mathfrak{F}_j)_{2,P} \\ (\mathfrak{w}_j, \mathfrak{W}_j)_{1,Q} \end{array} \right.\right] dZ$$
$$= \frac{\prod_{j=1}^{Q}\Gamma_k(\mathfrak{w}_j)}{\prod_{j=1}^{P}\Gamma_k(\mathfrak{f}_j)} \times \frac{\Gamma_k(\mathfrak{f}_1-s, A)\prod_{j=2}^{P}\Gamma_k(\mathfrak{f}_j-s)}{\prod_{j=1}^{Q}\Gamma_k(\mathfrak{w}_j-s)}\Gamma_k(s). \quad (51)$$

In this context, the M-transform of incomplete Fox-Wright functions with matrix argument (50) and (51) represent particular cases of the M-transform of generalized incomplete hypergeometric functions (35) and (36), respectively (see [15]).

3. When $A = 0$, it is possible to demonstrate that the M-transform of incomplete H-functions with matrix argument defined in (40) satisfies the M-transform of H-functions with matrix argument given in (6).

4. When $A = 0$ and $\mathfrak{F}_j = \mathfrak{W}_j = 1$, it is possible to demonstrate that the M-transform of incomplete H-functions with matrix argument defined in (40) satisfies the M-transform of G-functions with matrix argument given in [5].

5. When $A = 0$, it is possible to demonstrate that the M-transform of generalized incomplete hypergeometric functions with matrix argument defined in (36) satisfies the M-transform of hypergeometric functions with matrix argument given in (5).

5 Conclusions

This study aims to establish the definition of special functions with a matrix argument of a symmetric matrix of size $k \times k$, which involves $k(k+1)/2$ variables. To achieve this, we utilized a generalized matrix transform technique to derive the definition of the special function of the matrix argument. Using Jacobians of transformations and substituting specific values into the derived definition, we can obtain various outcomes based on our findings.

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