A New Paradigm on Semihypergraph

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ABSTRACT
The generalization of connected hypergraph is a Semihypergraph (Hs). In graph theory, edge connects exactly two vertices. An edge containing two or more than two vertices is a hypergraph. The definition of a semihypergraph and its characteristics are discussed in this paper. The concept of semihypergraph deals with minimum of three vertices in order to satisfy the condition that it should have end vertex, middle vertex and middle end vertex. Some of its basic definitions like subsemihypergraph, degree, walk, path etc., has been discussed and compared with a real-life problem.

Keywords: Hypergraph, Semihypergraph, End point degree, Adjacent degree, Subsemihypergraph, Spanning Subsemihypergraph, partial semihyperedge.

1. INTRODUCTION

The famous Swiss Mathematician Leonhard Euler [5] is the originator of Graph Theory. In 1736, he wrote a book on Solution of a Problem in the Geometry of Position graph theory to solve the problem of Konigsberg bridge. A graph is a structure made up of vertices (also known as nodes or points) and edges (also known as links or lines). The uses of graph in daily life also help in making analysis. A book on Graph Theory with its applications was published by Bondy. J.A and Murty U.S.R[3], which gives an introduction and some applications to solve real life problems. An edge of graph is one of the connections between the two nodes (or vertices). But in hypergraph model, the edges allow any number of vertices. French Mathematician Claude Berge [2] invented the concept of hypergraph in 1989. His work was the foundation of research in hypergraph and their properties. The studies of hypergraph was applied in coloring, matching, transversals and other fundamental graph-theoretic concepts. Hypergraph Theory was expanded by A.Bretto [4]. Analyzing connectivity and separation patterns in hypergraphs can reveal hidden patterns, clusters and associations in complex data sets was discussed by M. Amin Bahmanian and Mateja Sajna [1]. Megan Dewar, David Pike and John Proos[6] explained Connectivity in hypergraphs. Further investigations into the theory and uses of hypergraphs provide a useful foundation for many aspects of designs, combinatorial probabilities and Ramsey theory for infinite sets. Semigraph and their applications was introduced by E. Sampath Kumar [7]. The structure of semigraph with all its properties are fulfilled by both vertices and edges.

But the theory on semihypergraph gives a better non-pairwise relationship in data structure. Section 2, deals with the mathematical notations that are used throughout this paper. Basic definitions and concepts needed to develop semihypergraph is dealt in Section 3. Section 4 gives a brief description on semihypergraph.

2. Notations

The following mathematical notations are used throughout, in this paper:

$H_s(V, E_h)$ or simply (H_s)	-	Semihypergraph
E_{h_j}	-	semihyperedge
$d(H_s)$	-	degree of semihypergraph
$deg_e V$	-	end point degree of vertex V
$deg_a V$	-	adjacent degree of vertex V
$deg_{ca} V$	-	consecutive adjacent degree of vertex V
H'_s	-	subsemihypergraph
E'_h	-	subsemihyperedge
H_s^e	-	end vertex of H_s
H_s^a	-	adjacent vertex of H_s

H_s^{ca} - consecutive adjacent vertex of Hs

3. Preliminaries

Definition 3.1. [3] A hypergraph H is an ordered pair H = (V, E), with the elements of $V = \{v_1, v_2, ..., v_n\}$ be a non-empty, finite set of vertices and $E = \{E_1, E_2, ..., E_n\}$ be a hyperedge such that

1. $E_i \neq \phi$, j = 1, 2, ..., m and
2. $\bigcup_j E_j = V$, j = 1, 2, ..., m

Definition 3.2. [2] The vertex degree v ∈ V is the number of hyperedges $E_j \in E$ that incident on $v \in E_j$. It is denoted by $deg_H(v)$.

Definition 3.3. [1],[6] Let H = (V, E) be a hypergraph with $(v_i, v_j) \in V$ for i ≠ j. A walk with length n in H is a sequence $v_1E_1v_2E_2...v_n$ of vertices and hyperedge such that $(v_1, v_2, ..., v_n) \in V$, $(E_1, E_2, ..., E_j) \in E$ and the vertices v_{i-1} and v_i are adjacent in H via the hyperedge E_j. If $W = v_1E_1v_2E_2v_3...v_n$ is a walk in H, thenvertices v_1 and v_n are called the endpoints of W and $(v_2, v_3, ..., v_{n-1})$ are the internal vertices of W.
If the walk of H is $(E_1, E_2, ..., E_j)$ pairwise different, then W is called a trail.
If both the vertices $(v_1, v_2, ...v_n)$ and the hyperedges $(E_1, E_2, ..., E_j)$ are distinct pairwise, then W is said to be a path.
A cycle is a closed path with distinct vertices and hyperedges such that $v_1 = v_n$.

Definition 3.4. [7] A Semigraph G is a pair (V, E) where V is a non-empty set whose elements are called vertices and E is a set of ordered n-tuples, called edges of distinct vertices for n ≥ 2, satisfying the following conditions:

- SG1: Any two edges have atmost one vertex in common.
- SG2: Two edges $E_1 = (u_1, u_2, ..., u_m)$ and $E_2 = (v_1, v_2, ..., v_n)$ are equal if and only if (1) m = n and (2) either $u_i = v_i$ or $u_i = v_{n-i+1}$, for 1 ≤ i ≤ n.

Thus, the edges E_1 and E_2 are same.

Definition 3.5. [7] In a semigraph G, a walk is an alternating sequence of vertices and subedges $v_1E_1v_2E_2...v_{m-1}E_mv_m$ with initial and final vertices, such that v_1 and v_m are the end vertices of the subedge E_j, 1 ≤ j ≤ m.
A walk is said to be a trail, if there exists disjoint subedges. In trail, vertices may be repeated.
Any trial with different vertices is called a path.
A closed path is called cycle.
A path is known as strong path(s-path), if all the subedges are partial edges. Else, it is a weak path (w-path). Similarly, s-cycle and a w-cycle can be defined.

4. Semi hyper graph

4.1 Basic Definitions

Definition 4.1. A *Semihypergraph* is a connected hypergraph Hs = (V, E_h, <) where V ={v_i / i = 1, 2, ... , n} be a non-empty, vertex order preserving finite set and $E_h = \{E_{h_1}, E_{h_2}, \ldots, E_{h_p}\}$ such that E_{h_j} , j = 1, 2, ..., p is a subset of V, with minimum of three vertices satisfying the following conditions:

- $E_{h_j} \neq \emptyset$ and $\bigcup E_{h_j}$= V, 1 ≤ j ≤ p.
- A minimum of one vertex unites any two hyperedges
- Any pair of hyperedges E_{h_m} = {$u_1, u_2, ..., u_m$} and E_{h_n} = {$v_1, v_2, ..., v_n$} with ascending indices are equal if and only if
  (a) n = m and
  (b) either $u_i = v_{n-i+1}$ or $u_i = v_i$, for 1 ≤ i ≤ n.

Example 4.1.1.
Figure 1, depicts the Suburban Rail route map of the Bengaluru City. While compared to metro, suburban rail route is faster. Here, stations are called vertices and the routes are called hyperedges.

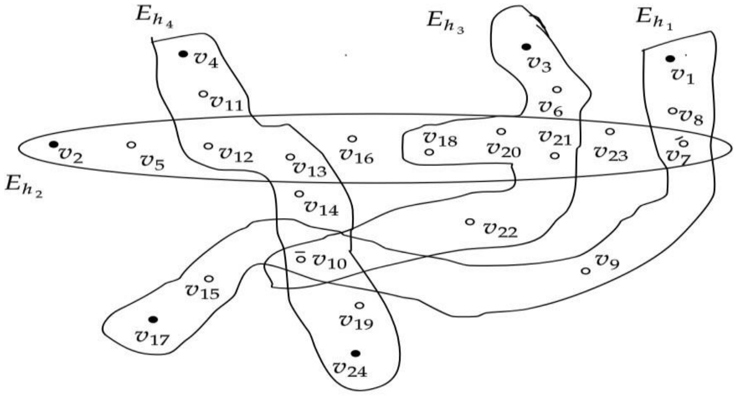

Figure 1: Semihypergraph

●	End vertices	:	v_1, v_2, v_3, v_4, v_{17} and v_{24}
o	Middle vertices or (m- vertices)	:	v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22} and v_{23}
o̲	Middle-end vertices	:	v_7 and v_{10}

4.2 Structural representation of a semihypergraph

- Each semihyperedge of a semihypergraph must have cardinality of atleast three vertices.
- In a semihyperedge of Hs, vertices are specified by three types. Semihyperedge of initial vertex v_i and final vertex v_n are end vertices. Then v_i's are said to be m-vertices (middle vertices) of a semihyperedge, for i = 2, 3, n-1.
- A vertex is known as a middle-end vertex if any semihyperedge of H_s contains a middle vertex from one semihyperedge which is an end vertex of another semihyperedge.
- If a hypergraph is linear, then the intersection of any two hyperedges has atmost one vertex in common and it results in a semigraph.
- The exact position of vertices and semihyperedges depends on many aspects, including the ordered arrangement of vertices.

4.3 Degree of a Semihypergraph

The Degree of Hs is denoted as $d(H_s)$, the incidence of E_{h_j} containing a vertex v ∈ V. The following represents types of degrees of V ∈ Hs.

- The incidence of semihyperedges containing end vertex of H_s is called End point degree and is represented as deg_eV.
- The incidence of semihyperedges containing adjacent vertices of H_s is called Adjacent degree. It is denoted by deg_aV.
- The incidence of semihyperedges containing consecutively adjacent vertices of H_s is called Consecutive Adjacent degree. It is denoted by $deg_{ca}V$.

The above said degrees are calculated below by considering example 4.1.1.

Table 1. Types of Degrees in a semihypergraph

Vertices	$d(H_s)$	deg_eV	deg_aV	$deg_{ca}V$
v_1	1	1	6	1
v_2	1	1	9	1
v_3	1	1	6	1
v_4	1	1	7	1
v_5	1	0	9	2
v_6	1	0	6	2
v_7	2	1	15	3
v_8	1	0	6	2
v_9	1	0	6	2

v_{10}	3	1	19	5
v_{11}	1	0	7	2
v_{12}	2	0	15	3
v_{13}	2	0	15	3
v_{14}	1	0	7	2
v_{15}	1	0	6	2
v_{16}	1	0	9	2
v_{17}	1	1	6	1
v_{18}	2	0	13	3
v_{19}	1	0	7	2
v_{20}	2	0	13	2
v_{21}	2	0	13	3
v_{22}	1	0	6	2
v_{23}	1	0	9	1
v_{24}	1	1	7	1

4.4 Characteristics of Semihypergraph

Definition 4.2. A subsemihypergraph H'_s is a semihypergraph defined by $H_s^{'} = \left\{V', E_h^{'} = \left(E_{h_j}\right) \ni E_h^{'} \subseteq V' \text{ and } E_h^{'} \neq \phi\right\}$, $\forall E_{h_j} \in E'_h$ where j = 1, 2, ..., p.

Definition 4.3. A subsemihypergraph $H_s^{'} = \left(V', E_h^{'}\right)$ with the hyperedges $E_h^{'} = \left\{V\left(E_{h_j}\right) \cap V' \neq \phi : E_{h_j} \text{ is a loop or } \left|V\left(E_{h_j}\right) \cap V'\right| \geq 3\right\}$, where V′ induced by the vertices in V′ ⊆ V is said to be an induced subsemihypergraph.

Definition 4.4. If $H' = \left(V', E_{h_j}\right)$, where $\bigcup_{j=1}^{p} E_{h_j} \subseteq V'$ are partial semihypergraph of a semihypergraph H_s then it is called a partial subsemihypergraph.

Definition 4.5. If subsemihypergraph H'_s have all the vertices of a semihypergraph H_s then $H_s^{'} = \left(V', E_h^{'}\right)$ is called a spanning subsemihypergraph.

Definition 4.6. Let $H_s = (V, E_h)$, where $E_h = (v_1, v_2, ..., v_n)$ be a semihyperedge of semihypergraph. A subsemihyperedge of H_s is $E_h^{'} = \left(v_{i_1}, v_{i_2}, \ldots, v_{i_k}\right)$ where k ≥ 3 and $1 \leq v_{i_1} < v_{i_2} < \ldots < v_{i_k} \leq n$ or $1 \leq v_{i_k} < v_{i_{k-1}} < \ldots < v_{i_1} \leq n$. The subsemihyperedge E_{h_j} is induced by the set of vertices $\left(v_{i_1}, v_{i_2}, \ldots, v_{i_k}\right)$. The subsemihyperedge E'_{h_j} of E_h is said to be a partial semihyperedge, if any two consecutive vertices in E'_{h_j} is also consecutive vertices in E_h.

Definition 4.7. Walk begins and ends with a vertex that is formed by changing the sequence of vertices and semihyperedges. It can be specified by a sequence of vertices which are adjacent. In a walk both vertices and semihyperedges are repeated.
A walk in a semihypergraph can also be defined as follows:
Vertex traversal from the starting vertex to a semihyperedges that contains the current vertex then moving to a different vertex of the same semihyperedge or any other semihyperedges.

The length of walk in H_s is $v_1E_{h_1}v_2E_{h_2}\ldots E_{h_p}v_n$ of vertices and semihyperedges such that $(v_1, v_2, \ldots, v_n) \in V$, $(E_{h_1}, E_{h_2}, \ldots, E_{h_p}) \in E_{h_j} \subseteq E_h$, i = {1, 2, ..., n} and j = {1, 2, ..., p}.

If $W = v_1E_{h_1}v_2E_{h_2}\ldots E_{h_p}v_n$ is a walk in H_s then v_1 and v_n are called end vertices of W and $(v_2, \ldots, v_{n-1})$ are middle vertices of W. Consider example 4.1, to categorise vertices of W:

end vertex of W = $v_4E_{h_4}v_{11}E_{h_4}v_{12}E_{h_2}v_5E_{h_2}v_2$

middle vertex of W = $v_{11}E_{h_4}v_{12}E_{h_2}v_{13}E_{h_2}v_{16}E_{h_2}v_2E_{h_2}v_{18}E_{h_2}v_{22}$

middle-end vertex of W = $v_8E_{h_1}v_7E_{h_2}v_9E_{h_1}v_{10}E_{h_3}v_{14}$

Definition 4.8. A trail is a walk, that doesn't contain repeated semihyperedges. It consists of an ordered collection of vertices and semihyperedges, where each semihyperedges are visited atmost once. Trail of Bengaluru suburban rail route is T = $v_4E_{h_4}v_{16}E_{h_2}v_{18}E_{h_3}v_{20}$, in example 4.1.

Definition 4.9. A path in a semihypergraph is a trail, that visits distinct vertices and semihyperedges atmost once. If any two consecutive vertices of semihyperedge appears in the path, then it is called strong path (P_s). Otherwise, it is called weak path (P_w). In example 4.1, the path P = $v_{12}E_{h_4}v_{16}E_{h_2}v_{18}E_{h_3}v_{20}$ is a strong path (P_s).

Definition 4.10. A cycle is a closed path that does not contain any repeating vertices except for the first vertex. It is also defined as a sequence of vertices $(v_1, v_2, \ldots, v_n)$ such that

- $n \geq 3$ i.e., the cycle must have atleast three vertices with middle/middle-end vertices.
- $v_i \neq v_j$ for $2 \leq i \leq j \leq (n-1)$ (except for the first and last vertices, no other vertices are repeated in the cycle).

If any two consecutive vertices of semihyperedges appears in the cycle, then it is called strong cycle (C_s). Otherwise, it is called weak cycle (C_w).

Definition 4.11. Assume that $P = v_1E_{h_1}v_2E_{h_2}\ldots E_{h_p}v_n$ is a path of semihyperedge $E_h = (v_1, v_2, \ldots, v_n)$. Consider that $E'_h = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ as a subsemihyperedge of E_h in P.

If P travels through two subsemihyperedges $E_{h_{i-1}}$ and E_{h_i} in r-direction if the vertices are in the relation $i_1 < i_2 < \ldots < i_k$ and l-direction if the vertices are in the relation $i_1 > i_2 > \ldots > i_k$. If P traverses through both $E_{h_{i-1}}$ and E_{h_i} in either r-direction or l-direction, then P travels through two subsemihyperedges $E_{h_{i-1}}$ and E_{h_i} of E_h in the same direction.

If $E_{h_{i-1}}$ and E_{h_i} are subsemihyperedges of the same semihyperedge and are traversed in the opposite direction, the ordered pairs of semihyperedges $E_{h_{i-1}}$, E_{h_i} may be counted. This is defined as Length of the path. Similarly, Length of the Cycle can also be defined.

Definition 4.12. The length of the minimum s-path between the vertices v_{i-1} and v_i where i = (1, 2, ..., n) is called s-distance. Similarly, the length of the minimum w-path between the vertices v_{i-1} and v_i where i = (1, 2, ...n) is called w-distance.

Definition 4.13. In a semihypergraph $H_s = (V, E_h)$, the strong deletion of vertex v_i results in $H''_s = (V'', E'')$ where $V'' = V \setminus \{v_i\}$ with $|V| \geq 4$ i.e., when v_i is strongly deleted, all of its incident semihyperedges are also deleted.

Example 4.4.1.

If v_2 is deleted from H_s, then it results in $H''_s = H_s \setminus \{v_2\}$

The following pictorial representation explains the concept of definition 4.13.

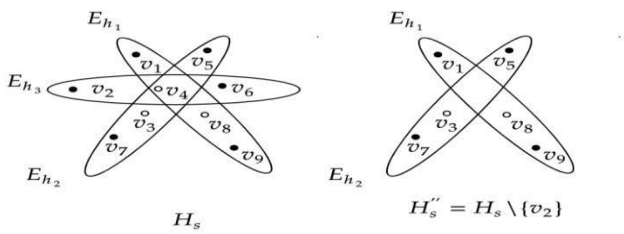

Figure 2: Strong deletion of vertex

Note: If v_4 is deleted, then there exists a null H_s, since all semihyperedges are incident in v_4.

Definition 4.14. In a semihypergraph $H_s = (V, E_h)$ the weak deletion of vertex v_i, removes v_i, from V and all its occurrences from E_h of H_s, denoted as $H''_s = (V'', E''_h)$.

Example 4.4.2.
In example 4.4.1., removing v_2 from Hs results in $H''_s = H_s \setminus \{v_2\}$

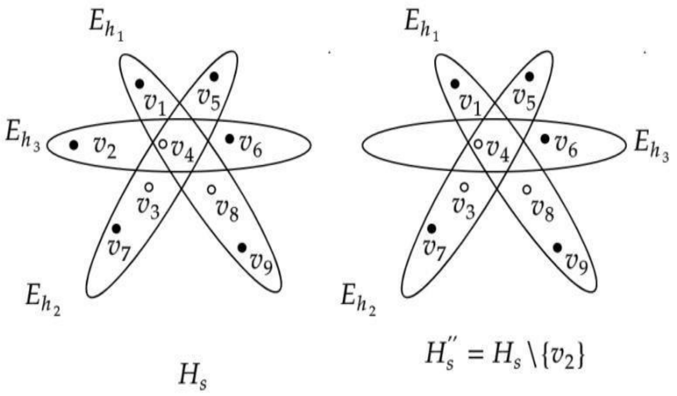

Figure 3: Weak deletion of vertex

Definition 4.15. In $H_s = (V, E_h)$, the strong deletion of semihyperedge is defined as removing a semihyperedge E_{h_j} from E_h deletes all the vertices incident with E_{h_j}. i.e., $H''_s = (V'', E''_h)$, $V'' = V \setminus \text{supp}(E_{h_j})$, $E''_h = (E''_{h_j})_{j \in J'}$, $J' = J \setminus \{j\}$ and $E''_{h_j} = E_{h_j} \setminus \{v_i \in E_{h_j}\}$ for $i \neq j$.

Example 4.4.3.
If E_{h_3} is deleted from the given H_s below, then $H''_s = H_s \setminus \{E_{h_3}\}$
i.e., $H''_s = H_s \setminus \{v_1, v_3, v_4\}$

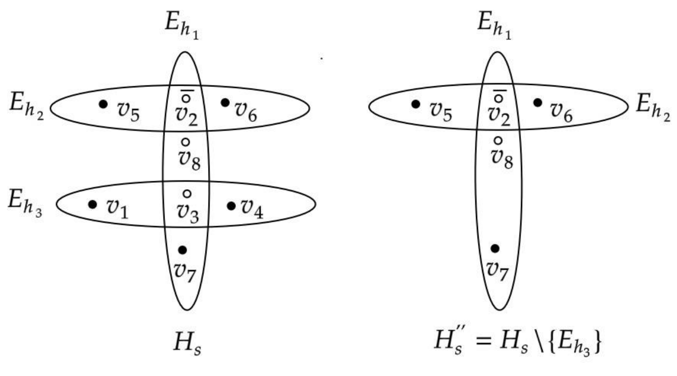

Figure 4: Strong deletion of semihyperedge

Definition 4.16. The weak deletion of semihyperedge simply removes the semihyperedge without affecting the entire H_s i.e., $H''_s = (V'', E''_h)$, $E''_h = (E''_{h_j})_{j \in J'}$, $J' = J \setminus \{j\}$.

Example 4.4.4.
From example 4.4.1., removing E_{h_2} is deleted from E_h results in $E''_h = E_h \setminus \{E_{h_2}\}$.

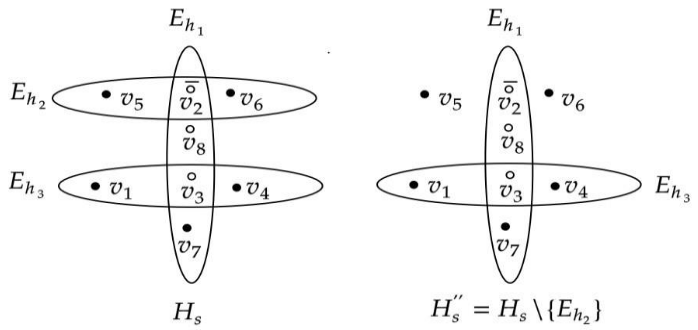

Figure 5: Weak deletion of semihyperedge

Definition 4.17. A path connecting each pair of distinct vertices in a semihypergraph $H_s = (V, E_h)$, then it is said to be connected.
Otherwise, the semihypergraph is said to be disconnected.

4.4 Hypergraph associated with Semihypergraph

Consider a semihypergraph $H_s = (V, E_h)$, then a hypergraph associated with semihypergraph are distinguished as follows:

- End vertex of H_s
- Adjacent vertex of H_s
- Consecutive Adjacent vertex of H_s

Definition 4.18. End Vertex of H_s

In each semihyperedge there exists two end vertices. The end vertices of H_s exists if and only if these two vertices are adjacent.

Example 4.5.1.

In example 4.1.1, consider the vertices of corresponding semihyperedges are adjacent, the end vertex of H_s are represented below graphically.

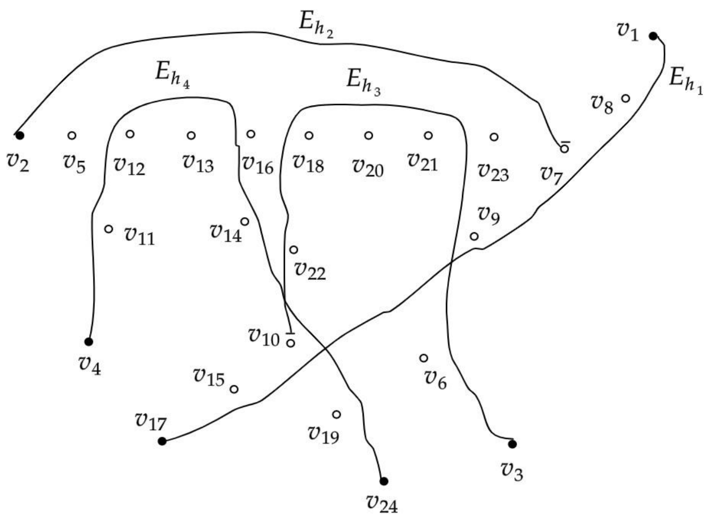

Figure 6: End vertex of Hs

Definition 4.19. Adjacent vertex of H_s

Two vertices are said to be adjacent in semihyperedge if and only if they are adjacent in semihypergraph.

Example 4.5.2.
In example 4.1.1, the following graph depicts the adjacent vertex of H_s.

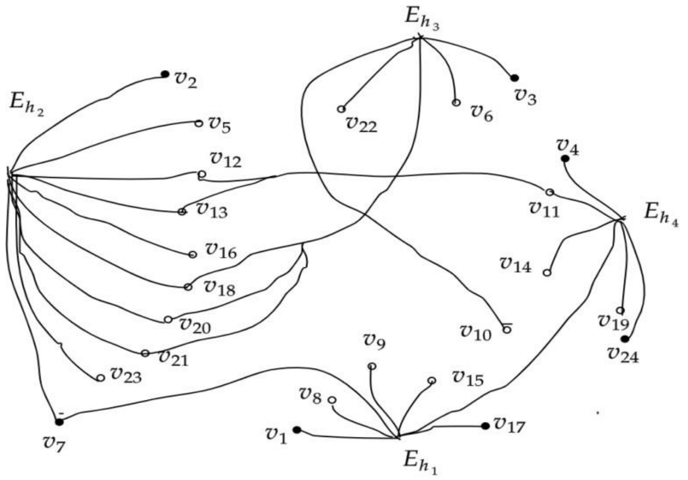

Figure 7: Adjacent vertex of Hs

Note
In a semihypergraph, $d(v_i, v_{i+1})$ between two vertices v_i and v_{i+1} is defined as the distance between them in adjacent vertex of H_s.

Definition 4.20. Consecutive Adjacent vertex of H_s
Two vertices are said to be consecutive adjacent in semihyperedges if and only if they are consecutively adjacent in a semihypergraph.

Example 4.5.3.
Consider example 4.1.1, the following graphical representation explains about the consecutive junctions of Suburban Rail route in Bangalore city.

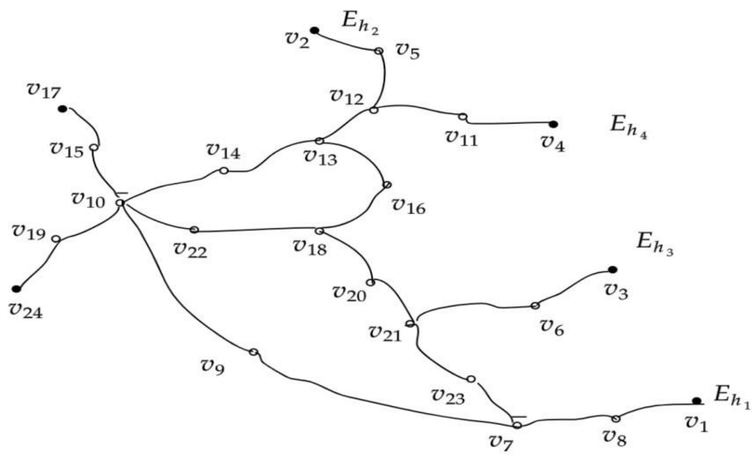

Figure 8: Consecutive Adjacent vertex of Hs

4.5 CONCLUSION

Semihypergraph allows three or more vertices with end, middle and middle-end vertices, which is generalisation of hypergraph, with the condition that the inter section of two semihyperedge has atleast one vertex. The basic concepts like degree, walk, path, cycle has been discussed. This concept has been related with real life application and found some perfect results while comparing with hypergraphs associated with semihypergraph. Further the author has planned to implement algorithms for better results to obtain the nearby perfect paths for passengers based on parameters like the shortest path/minimum travel time and the fewest number of transfers using the same concept.

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