Alpha Power Quantile Transformation Distribution

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Received: 18.04.2024 Revised : 15.05.2024 Accepted: 23.05.2024

ABSTRACT
A more flexible distribution: Alpha power transformation techniques provide great flexibility in the final distributions. The alpha power Weibull quantile exponential distribution, which may have an asymmetric or near-symmetric density, is shown via the innovative approach. The related failure rate function may take on a variety of asymmetric shapes, including falling, rising, L-shaped, near-symmetrical, and right-skewed forms, which improve its tractability for different modeling purposes. There are many critical mathematical properties of the proposed distribution. The most excellent likelihood approach is used to estimate the unknown parameters of the proposed distribution.
Furthermore, many numerical analyses were conducted to assess the accuracy of the estimate's correctness. The utility and flexibility of the suggested distribution are evaluated via an analysis of real-world datasets. The alpha power Weibull quantile exponential distribution that has been suggested may perform better than popular distributions due to its remarkable flexibility.

Keywords: alpha power transformation; moment; order statistics;quantile;

1. INTRODUCTION

Different There has been extensive use of statistical distributions to describe and forecast current trends in many fields, including engineering, mathematics, statistics, demography, biology, environmental sciences, and medicine. Still, in several fields, using these classical distributions is unacceptable because of the restrictions of fitting this data with Several current standard distributions. Some scholars have attempted to enhance the classical distributions to be more versatile when modeling data from different academic subjects. The literature has numerous modified distributions. A common goal in improving the goodness-of-fit of distributions is to develop novel generators for families of distributions. This technology is used to generate extensions of the current standard models. The authors of [1] have devised an all-encompassing procedure known as Alpha power transformation. This innovative and cutting-edge method has alone been created. Any distribution can benefit from the addition of skewness through its usage. A family's may be written as

$$F_{APT}(u) = \begin{cases} \dfrac{\alpha^{W(u)} - 1}{\alpha - 1} & , if \alpha > 0, \alpha \neq 1 \\ W(u) & , if \alpha = 1 \end{cases} \quad (1)$$

$$f_{APT}(u) = \begin{cases} \dfrac{log\alpha}{\alpha - 1} f(u)\alpha^{W(u)}, & if\ \alpha > 1, \alpha \neq 1 \\ f(u) & , \quad if \alpha = 1 \end{cases} \quad (2)$$

The PDF and CDF of the other continuous distribution are denoted by $F(x)$ and $f(x)$.
If an RV X has shape and scale parameters denoted by a and b, respectively, and it obeys the Weibull distribution [2],

$$W(x) = 1 - e^{-\left(\frac{x}{b}\right)^{a}} \quad , x \geq 0 \quad (3)$$

G is an exponential random variable with a CDF. The (WED), a member of the Weibull family [3], was developed, $G(x) = 1 - e^{-\lambda x}, x \geq 0, \lambda > 0$.

$$Let u \sim U(0,1),$$

$$u = G(X) \Rightarrow u = 1 - e^{-\lambda x} \Rightarrow -\lambda x = ln(1 - u)$$

$$x = \frac{-\ln(1-u)}{\lambda} \quad (4)$$

Substituting it into equation (3),

$$W(u) = 1 - e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a}} \quad (5)$$

This study introduces the alpha power Weibull-Quantile exponential distribution (APWQED). This foundation is built around a novel concept approach to distribution development, allowing for better flexibility in portraying accurate data in many domains. Integral to it are several APT methods

2. The Alpha Power Weibull Quantile Exponential Distribution

An RV with a four-parameter APWQED may be described in terms of its CDF in the following way.

$$F(u)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a}}}-1}{\alpha-1} & ,\alpha>0,\alpha\neq 1\\ 1-e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a}} & ,\alpha=1\end{cases} \quad (6)$$

Where $\alpha,\lambda,a,b>0\ and\ 0<u>1$, The corresponding PDF is given as

$$f(u)=\begin{cases}\dfrac{log\alpha}{\alpha-1}\dfrac{a\left(\frac{-ln(1-u}{\lambda b}\right)^{a-1}e^{-\left(\frac{-ln(1-u)}{\lambda b}\right)^{a}}}{\lambda b(1-u)}\alpha^{1-e^{-\left(\frac{-\ln(1-u}{\lambda b}\right)^{a}}} & ,if\ \alpha>0,\alpha\neq 1\\ \dfrac{a\left(\frac{-\ln(1-u}{\lambda b}\right)^{a-1}e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a}}}{\lambda b(1-u)} & ,if\ \alpha=1\end{cases} \quad (7)$$

The survival function, $S(x)$.

$$S(u)=\begin{cases}\dfrac{\alpha}{\alpha-1}\left(1-\alpha^{-e^{-\left(\frac{-ln(1-u)}{\lambda b}\right)^{a}}}\right) & ,if\ \ \alpha>0,\alpha\neq 1\\ e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a}} & ,if\ \ \alpha=1\end{cases} \quad (8)$$

The expression for the APWQED is the hazard rate function, h(x), as follows.

$$h(u)=\begin{cases}log\alpha\left(a\dfrac{\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a-1}}{\lambda b(1-u)}e^{-\left(\frac{-ln(1-u}{\lambda b}\right)^{a}}\right)\dfrac{\alpha^{-e^{-\left(\frac{-ln(1-u)}{\lambda b}\right)^{a}}}}{1-\alpha^{-e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^{a}}}} & ,if\alpha>0,\alpha\neq 1\\ a\dfrac{\left(\frac{-ln(1-u)}{\lambda b}\right)^{a-1}}{\lambda b(1-u)} & ,if\alpha=1\end{cases} \quad (9)$$

2.1. The APWQED is Special Cases

1. At, the APWQED becomes the WQED. $\alpha = 1$.
2. At, the APWQED decreases to the WD. $\alpha = \lambda = 1$.
3. At, the Quantile Exponential distribution is what the APWQED reduces to.

$\alpha = b = a = 1$.

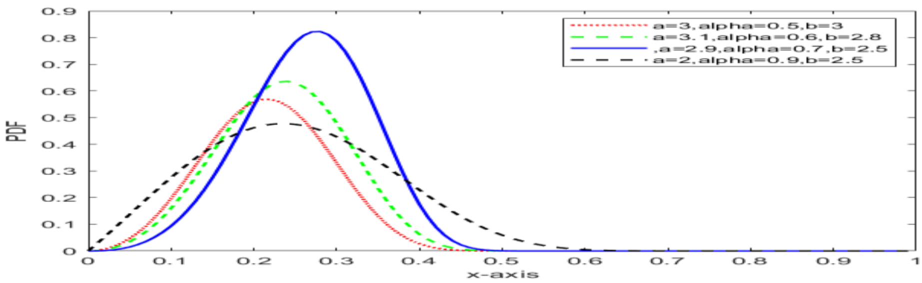

Figure 1. shows APWQED PDF plots for a few chosen values. α,λ,b,a.

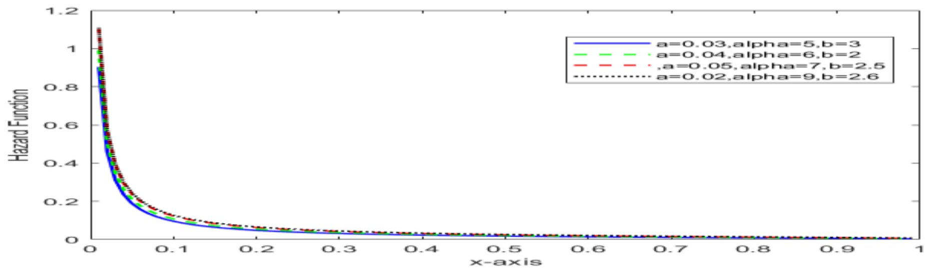

Figure 2. shows APWQED PDF plots for a few chosen values. α,λ,b,a.

Figure 1 illustrates several alternative APWQED density configurations. It includes symmetric, almost symmetric, inverted J-shaped morphologies and is skewed left and right. Figure 2 demonstrates how various uneven shapes are included in the APWQED hazard rate. These results demonstrate how highly adaptive the APWQED is.

2.2. Expansion for the PDF

Using the series representation, the APWQED PDF has a straightforward extension that is given as follows.

$$\alpha^{-z} = \sum_{n=0}^{\infty} \frac{(-log\alpha)^n}{n!} (z)^n \qquad (10)$$

Therefore, expanding $\alpha^{-e^{-\left(\frac{-\ln(1-u)}{\lambda b}\right)^a}}$ In (7) using (10), we have

$$f(u) = \frac{\alpha}{\alpha-1} \frac{a\left(\frac{-ln(1-u)}{\lambda b}\right)^{a-1}}{\lambda b(1-u)} \sum_{n=0}^{\infty} \frac{(-log)^{n+1}}{n!} \left(e^{-(n+1)\left(\frac{-\ln(1-u)}{\lambda b}\right)^a}\right) \qquad (11)$$

3. The APWQED's properties

This section provides the following basic statistical features of the APWQED

3.1. Quantile Function

One may derive the path quantile function $(0 < p < 1)$ of the APWQED as

$$u_p = 1 - e^{-\lambda b\left(-ln\left(1-\frac{ln((\alpha-1)p+1)}{ln\alpha}\right)\right)^{\frac{1}{a}}} \qquad (12)$$

Consequently, the median of the APWQED may be derived as follows when $p = 0.5$ is set:

$$u_{0.5} = 1 - e^{-\lambda b\left(-\ln\left(1-\frac{ln((\alpha-1)0.5+1)}{ln\alpha}\right)\right)^{\frac{1}{a}}} (13)$$

3.2. Moments

If $U\sim$ APWQED(α, λ, u, a), Subsequently, the rth instant of u may be acquired.

$$\mu_r = E(U^r) = \int_0^{\infty} u^r f(u)\, du = \sum_{n=0}^{\infty} \frac{(-log)^{n+1}}{n!} \frac{a\alpha}{\alpha-1} \int_0^{\infty} u^r \frac{\left(\frac{-ln(1-u)}{\lambda b}\right)^{a-1}}{\lambda b(1-u)} \left(e^{-(n+1)\left(\frac{-ln(1-u)}{\lambda b}\right)^a}\right) du$$

Substituting $y = \frac{-\ln(1-u)}{\lambda b} \Longrightarrow u = 1 - e^{-\lambda by}$, $du = \lambda b e^{-\lambda by}\, dy$, the rt moment of the APWQED can be expressed as

$$\mu_r = E(U^r) = \sum_{n=0}^{\infty}\sum_{i=0}^{r}\sum_{k=0}^{\infty} \frac{(-log)^{n+1}\binom{r}{i}(-1)^{i+k}(n+1)^k a\alpha}{n!\,(\alpha-1)k!} \frac{\Gamma(ak+a)}{(\lambda bi)^{ak+a}} \qquad (14)$$

Consequently, it is simple to represent the mean of the APWQED as

$$\mu = E(U) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-log)^{n+1}(-1)^k(n+1)^k a\alpha}{n!\,(\alpha-1)k!} \frac{\Gamma(ak+a)}{(\lambda bi)^{ak+a}} \qquad (15)$$

Additionally, (18) *and* (19), the variance for the APWQED can be given by

$$\sigma^2 = E(U^2) - \mu^2$$

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{2}\sum_{k=0}^{\infty} \frac{2(-log\alpha)^{n+1}(-1)^{i+k}(n+1)^k a\alpha}{n!\,(\alpha-1)k!\,i!\,(2-i)!} \frac{\Gamma(ak+a)}{(\lambda bi)^{ak+a}} - \mu^2 \qquad (16)$$

The rth moment about the meanμ_rA random variable U with pdf is called the Central moment, as stated by

$$E(U-\mu)^r = \sum_{n=0}^{\infty}\sum_{i=0}^{r}\sum_{j=0}^{i}\sum_{k=0}^{\infty} \frac{(log)^{n+1}\binom{r}{i}\binom{i}{j}(-1)^{n+1+r-i+j+k}(n+1)^k a\alpha}{n!\,(\alpha-1)k!} \frac{\Gamma(ak+a)}{(\lambda bj)^{ak+a}} \qquad (17)$$

3.3. Characteristic functions and moment-generating functions: an easy way to express the (MGF) of APWQED is as

$$M_u(t) = E(e^{tu}) = \int_0^{\infty} e^{tu} f(u)\, du$$

Using the same steps for finding the moment function and using the binomial series, then the moment generating function and characteristic function of APWQED

$$M_u(t)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{r}\sum_{k=0}^{\infty}\frac{(-log)^{n+1}\binom{r}{i}(-1)^{i+k}t^r(n+1)^k a\alpha}{n!\,(\alpha-1)k!\,r!}\frac{\Gamma(ak+a)}{(\lambda bi)^{ak+a}} \quad (18)$$

Similarly,

$$\phi_u(t)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{r}\sum_{k=0}^{\infty}\frac{(-log)^{n+1}\binom{r}{i}(-1)^{i+k}(ti)^r(n+1)^k a\alpha}{n!\,(\alpha-1)k!\,r!}\frac{\Gamma(ak+a)}{(\lambda bi)^{ak+a}} \quad (19)$$

3.4. Order Statistics

Suppose a random sample $U_1, U_2, \ldots\ldots, U_n$ from the APWQED and $u_{i:n}$Give its order statistics. The order statistic's density is specified in

$$f_{k:n}(u_i)=\frac{n!}{(k-1)!\,(n-1)!}[F(u)]^{k-1}[1-F(u)]^{n-k}f(u) \quad (20)$$

$$=\frac{n!}{(k-1)!\,(n-1)!}\sum_{t=0}^{n-k}\sum_{t=0}^{k-1}\binom{n-k}{t}\binom{k-1}{t}\left(\frac{\alpha}{\alpha-1}\right)^{n-k-t}\left(\frac{-\alpha^{-e^{\left(\frac{-\ln(1-u}{\lambda b}\right)^{a}}}}{\alpha-1}\right)^{t}\left(\frac{\alpha^{1-e^{-\left(\frac{-ln(1-u}{\lambda b}\right)^{a}}}1}{\alpha-1}\right)^{k-1-t}$$

$$\left(\frac{-1}{\alpha-1}\right)^{t}\frac{log\alpha}{\alpha-1}\frac{a\left(\frac{-ln(1-u}{\lambda b}\right)^{a-1}e^{-\left(\frac{-ln(1-u}{\lambda b}\right)^{a}}}{\lambda b(1-u)}\alpha^{1-e^{-\left(\frac{-ln(1-u}{\lambda b}\right)^{a}}} \quad (21)$$

4. Maximum Likelihood Estimates

If$u_1, u, u_3, \ldots, u_n$It is a random sample. The vector of parameters $\theta = (\alpha, \lambda, b, a)$ may be found using the log-likelihood (ℓ) of size n from the APWQED.

$$\ell(\alpha,\lambda,b,a,u)=nln\left(\frac{ln\alpha}{(\alpha-1}\right)+nln\left(\frac{a\alpha}{\lambda b}\right)+\sum_{i=1}^{n}(a-1)ln\left(\frac{-ln(1-u_i)}{\lambda b}\right)-\sum_{i=1}^{n}\ln(1-u_i)$$
$$-\sum_{i=1}^{n}\left(\frac{-ln(1-u_i)}{\lambda b}\right)^{a}-ln\alpha\sum_{i=1}^{n}e^{\left(\frac{-\ln(1-u_i)}{\lambda b}\right)^{a}} \quad (22)$$

The partial derivatives for each parameter in (32)

$$\frac{\partial\ell}{\partial\alpha}=\frac{n\left(\frac{\alpha-1}{\alpha}-ln\alpha\right)}{(\alpha-1)ln\alpha}+\frac{n}{\alpha}-\frac{1}{\alpha}\sum_{i=1}^{n}e^{-\left(\frac{-ln(1-u_i)}{\lambda b}\right)^{a}} \quad (23)$$

$$\frac{\partial\ell}{\partial\lambda}=-\frac{n}{\lambda}-\frac{(a-1)}{\lambda}-a\sum_{i=1}^{n}\frac{\left(\frac{-\ln(1-u_i)}{\lambda b}\right)^{a-1}\ln(1-u_i)}{\lambda^2 b}\left[1+ln\alpha\left(e^{-\left(\frac{-\ln(1-u_i)}{\lambda b}\right)^{a}}\right)\right] \quad (24)$$

$$\frac{\partial\ell}{\partial b}=-\frac{n}{b}-\frac{(a-1)}{b}-\sum_{i=1}^{n}\frac{a\left(\frac{-ln(1-u_i}{\lambda b}\right)^{a-1}ln(1-u_i)}{b^2\lambda}\left[1+ln\alpha\left(e^{-\left(\frac{-ln(1-u_i}{\lambda b}\right)^{a}}\right)\right] \quad (25)$$

$$\frac{\partial\ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}ln\left(\frac{\ln(1-u_i)}{\lambda b}\right)-\sum_{i=1}^{n}ln\left(\frac{-ln(1-u_i)}{\lambda b}\right)\left(\frac{-ln(1-u_i)}{\lambda b}\right)^{a}\left[1+ln\alpha\left(e^{-\left(\frac{-\ln(1-u_i)}{\lambda b}\right)^{a}}\right)\right] \quad (26)$$

5. Applications

Real medical data sets were employed to evaluate APWQED, and the distributions' fitting was compared to a few other well-known ones.

Survival Time Data

The original 85 patients' survival periods were included in the dataset From [4] who had prostate and gallbladder cancer. The results are shown as follows. 0.04,0.3,0.31,0.56,0.94,1.07,1.12,1.25,1.28,1.28,1.3,1.43,1.48,1.51,1.51,1.57,1.62,1.62,1.65,1.65,1.76,1.81,1.87,1.88,1.91,1.91,1.91,1.98,2.01,2.01,2.038,2.09,2.09,2.14,2.15,2.19,2.19,2.22,2.22,2.23,2.3,2.32,2.39,2.48,2.61,2.63,2.63,2.65,2.66,2.69,2.82,2.89,2.9,2.93,2.96,2.96,3,3,3.1,3.11,3.12,3.17,3.34,3.38,3.44,3.47,3.48,3.51,3.58,3.61,3.78,3.92,4.04,4.12,4.17,4.24,4.26,4.28,4.31,4.38,4.45,4.49,4.57,4.6,4.66.

APWQED and three other distributions were compared.With the use of the density functions listed below

1. Alpha power exponential (APE) distribution by [5]

$$f_{APE}(x)=\begin{cases}\frac{\log\alpha}{\alpha-1}\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}} & if\alpha\neq 1,\alpha>0\\ \lambda e^{-\lambda x} & if\alpha=1\geq 0\end{cases}$$

2. Alpha power inverse Weibull (APIW) distribution by [6]

$$f_{APIW}(x) = \begin{cases} \frac{\log\alpha}{\alpha-1}\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}} & if\alpha \neq 1, \alpha > 0 \\ \lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}} & if\alpha = 1 \geq 0 \end{cases}$$

3. The Alpha Power Weibull Exponential Distribution (APWED) by [7]

$$f_{APWE}(x) = \begin{cases} \frac{log\alpha}{\alpha-1}\frac{\alpha\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{\alpha-1} e^{-\left(\frac{\lambda x}{\gamma}\right)^{\alpha}} \alpha^{1-e^{-\left(\frac{\lambda x}{\gamma}\right)^{\alpha}}} & , \alpha > 0, \alpha \neq 1 \\ \frac{\alpha\lambda}{\gamma}\left(\frac{\lambda x}{\gamma}\right)^{\alpha-1} e^{-\left(\frac{\lambda x}{\gamma}\right)^{\alpha}} & , \alpha = 1 \end{cases}$$

To determine if the proposed model was more valid than the other models, the GOOF-testing standards listed below were taken into account: The negative statistics include Kolmogorov-Smirnov (K–S), Hannan–Quinn (HQIC), Cramer–von Mises (W), the information criterion of Akaike (AIC), and the corrected Akaike (CAIC) information criterion. It fits better when these numbers are lower.

Table 1 Shows how well the APWQED performs compared to alternate distributions for the aforementioned real data sets.

Table 2 Demonstrates that, compared to the other distributions, the APWQED had the lowest ratings (AIC, CAIC, HQIC, K-S, and W), demonstrating its effectiveness in fitting the actual data sets.

Table 1. MLEs (SEs in parentheses) for survival time data.

Distribution	Estimated Parameters			
APWQE $(\hat{\alpha}, \hat{\lambda}, \hat{b}, \hat{a})$	2.0750 (0.0056)	1.0375 (0.0014)	0.9000 (0.0100)	0.0519 (3.5156e-06)
APE $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\lambda}})$	34.1428 (13.3994)	0.7691 (0.0592)		
APWE $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{a}})$	25.6685 (40.1965)	2.6851 (199.6717)	4.9713 (369.7458)	1.5886 (0.3246)
APIW $(\hat{\alpha}, \hat{\lambda}, \hat{\beta}, \hat{a})$	3.9152 (1.2192)	0.1948 (0.0235)	1.239 (0.0695)	

Table 2. GOF criteria for survival time data.

Distribution	AIC	CAIC	HQIC	K-S	W	$-\ell(\hat{\theta})$
APWQE	228.1090	228.6090	232.0235	0.14651	0.21151	110.0545
APE	281.9912	282.1375	283.9562	0.1603	0.6024	138.9956
APWE	267.4350	267.9350	271.3650	0.0652	0.0478	129.7175
APIW	354.8962	355.1925	357.8437	0.2619	1.2682	174.4481

6. CONCLUSIONS

This work introduces A unique methodology for constructing distributions that provide an excellent level of adaptability for representing real-world data in several fields. The methodology integrates widely used Advanced Persistent Threat (APT) methods. The latest distribution, APWQED, has been revealed. Using this technique, the proposed distribution's hazard rate and density functions provide appealing structures for incorporating various data patterns. The APWQE's fundamental statistical characteristics included moments, quantile, median, mean residual life, order statistics, and maximum likelihood estimators (MLEs). The new distribution's practicality was shown by successfully fitting a real dataset. Overall, the APWQED model demonstrated superior performance to other well-established rival models.

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