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On multi-valued nonexpansive mappings in UCBS

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Abstract

In this manuscript, we focus on the approximation of fixed points for multi-valued nonexpansive type mappings within uniformly convex Banach spaces. To achieve this goal, we utilize a three-step iteration scheme that was originally introduced by Ullah et al. Furthermore, we establish the rapid convergence properties of the Ullah et al. iteration scheme through the implementation of numerical examples using Matlab software.

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1 Introduction

In nonlinear analysis, fixed point theory has a great importance over last 90 years. In fact the technique of fixed point also have been used in different fields such as biology, physics, engineering, chemistry, game theory, economics, computer science etc.

Fixed point theorems are developed for both single-valued and multi-valuedd functions over different spaces. Banach contraction principle [2] is one of the pioneering work in the field of fixed point theory and widely used to find out solution of different problems in the field of analysis.

There are lots of fixed point results available concerning single-valued nonexpansive mappings in the literature, while the study of the fixed points of multi-valued nonexpansive mappings are difficult. The multi-valued version of Banach contraction principle was given by Nadler [9] in 1969. Sastry and Babu [11] introduced multi-valued version of Mann [7] and Ishikawa [5] iteration and proved convergence theorems for nonexpansive mappings in Hilbert space. In 2016 Kim et al. [6] introduced multi-valued version of Thakur iteration [14] proved convergence results in uniformly convex Banach space and many more application are discussed on convergence ([17–19]).

The following three-step iteration scheme was introduced by Ullah et al. [15]-
Let $\mathcal{M}$ be a convex subset of a normed space B and $S : \mathcal{M} \to \mathcal{M}$ be a nonlinear mapping. For $w_1 \in \mathcal{M}$, the sequence $\{w_j\}$ in $\mathcal{M}$ is defined by

$$\begin{cases} t_j = (1-\alpha_j)w_j + \alpha_j S w_j, \\ z_j = S t_j, \\ w_{j+1} = S y_j, \ j \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_j\}$ is a sequence in $(0,1)$. Ullah proved that their iterative process converges faster than the iterative process given by Thakur [14].

The concept of Hausdorff metric, to approximate fixed points of multi-valued nonexpansive mapping was introduced by Markin [8] as follows:
Let $CB(\mathcal{M})$=collection of all non-empty closed bounded subset of $\mathcal{M}$ According to [6], a multi-valued mapping $S : \mathcal{M} \to CB(\mathcal{M})$ is said to be nonexpansive if

$$H(Sw, Sy) \leq ||w - y||,$$

for all $w, z \in \mathcal{M}$.

Following is the multi-valued version of Ullah iteration [16]. Let $\mathcal{M}$ be a non-empty subset which is closed convex of a UCBS B and $S : \mathcal{M} \to P(\mathcal{M})$ be a multi-valuedd function. For $w_1 \in \mathcal{M}$, the sequence $\{w_j\}$ in $\mathcal{M}$ is given by

$$\begin{cases} t_j = (1-\alpha_j)w_j + \alpha_j u_j, \\ z_j = w_j, \\ w_{j+1} = v_j, \ j \geq 1, \end{cases} \tag{1.2}$$

where $P_S : \mathcal{M} \to 2^{\mathcal{M}}$ be a multi-valued mapping, $u_j \in P_S(w_j)$, $v_j \in P_S(y_j)$, $w_j \in P_S(t_j)$, and $\{\alpha_j\} \in (0,1)$.

In this study, we prove strong convergence of the iteration scheme given by (1.2), to approximate fixed points for the multi-valuedd nonexpansive functions in uniformly convex Banach space. For convinient,we denote uniformly convex Banach space by

UCBS. We also compare iteration scheme (1.2) with multi-valued version of some well known iteration schemes (refer [20]).

2 Preliminaries

Definition 2.1. Suppose non-empty subset $\mathcal{M}$ of a UCBS B and $S : \mathcal{M} \to 2^{\mathcal{M}}$ is a multi-valued functions. An element $w \in \mathcal{M}$ is known as fixed point of multi-valued functions S, if $w \in Sw$. Trough-out the literature, we represent the set of fixed points of S by $F(S)$.

Definition 2.2. [6] Suppose a non-empty $\mathcal{M}$ subset of a UCBS B. Then $\mathcal{M}$ is known as proximal if for each $w \in B$, there exists an element $y \in \mathcal{M}$, we have

$$||w - z|| = d(w, \mathcal{M}) = \inf\{||w - t|| : t \in \mathcal{M}\}.$$

Definition 2.3. [6] Suppose a non-empty $\mathcal{M}$ be subset of a UCBS B and $\{w_j\}$ in B is known as Fejer monotone subset $\mathcal{M}$, if

$$||w_{j+1} - p|| \leq ||w_j - q||,$$

for all $q \in \mathcal{M}$, $j \geq 1$.

Proposition 2.1. *[6] Suppose a non-empty M be subset of a UCBS B and $\{w_j\}$ is Fejer monotone sequence with respect to $\mathcal{M}$. Then, the followings are true:*

(a) $\{w_j\}$ is bounded.

(b) For each $w \in M$, $\{||w_j - w||\}$ converges.

Note that the concept of Condition (I) in Banach space was given by Dotson and Senter [13]. Given below are multi-valued version of Condition (I).

Definition 2.4. suppose $\mathcal{M}$be a non-empty subset of a UCBS B. A multi-valuedd nonexpansive function $S : j \to CB(\mathcal{M})$ holds Condition (I), if non-decreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(r) > 0$ for each $r \in (0, \infty)$ such that $||w - Sw|| \geq g(d(w, F(S)))$ for all $w \in M$.

Lemma 2.2. *[12] Suppose B be a UCBS and $\{\alpha_j\}$ is a sequence in $[\gamma, 1-\gamma]$ for some $\gamma \in (0, 1)$. Let $\{w_j\}$ and $\{z_j\} \in X$ then,* $\limsup_{j\to\infty} ||w_j|| \leq q$, $\limsup_{j\to\infty} ||z_j|| \leq q$, *and* $\limsup_{j\to\infty} ||\alpha_j w_j + (1-\alpha_j)y_j|| = q$ *for some* $q \geq 0$. *Then* $\lim_{j\to\infty} ||w_j - z_j|| = 0$.

Lemma 2.3. *[3] suppose $S : \mathcal{M} \to P(\mathcal{M})$ is a multi-valuedd function with $F(S) \neq \emptyset$ and let $P_S : \mathcal{M} \to 2^{\mathcal{M}}$ be a multi-valued function given by*

$$P_S(w) = \{z \in Sw : ||w - z|| = d(w, Sw)\}, w \in \mathcal{M}.$$

Then the following conclusion holds:

(a) P_S is multi-valuedd function from $\mathcal{M} \to P(\mathcal{M})$.

(b) $F(S) = F(P_S)$.

(c) $P_S(q) = \{q\}$, *for every* $q \in F(S)$.

(d) For each $w \in M$, $P_S(w)$ *and* Sw *is a compact because its a closed.*

(e) $d(w, Sw) = d(w, P_S(w))$ *for each* $w \in \mathcal{M}$.

3 Primary result

Lemma 3.1. *Considering* $\mathcal{M}$ *be a non-empty closed convex subset of a UCBS* B*. Let* $S : \mathcal{M} \to P(\mathcal{M})$ *be a multi-valuedd function such that* $F(S) \neq \emptyset$*, and* $P_S : \mathcal{M} \to 2^{\mathcal{M}}$ *be a multi-valuedd nonexpansive function. Let* $\{w_j\}$ *be a sequence in* $\mathcal{M}$ *given by (1.2), then* $\lim_{j\to\infty} ||w_j - t||$ *exists for all* $t \in F(S)$.

Proof. By our assumption, that $F(S) \neq \emptyset$, so suppose that $t \in F(S)$. Then, by Lemma 2.3, we have $t \in P_S(t) = \{t\}$. Also by Hausdorff metric, we have

$$H(P_S(w_j), P_S(t)) = \max\{\sup d(u_j, P_S(t)), \sup d(P_S(w_j), t)\},$$

where $d(u_j, P_S(t)) = \inf_{t\in P_S(t)} ||u_j - t||$.
By (1.2) we have,

$$\begin{aligned} ||t_j - t|| &= ||(1-\alpha_j)w_j + \alpha_j u_j - t|| \\ &\leq (1-\alpha_j)||w_j - t|| + \alpha_j ||u_j - t|| \\ &\leq (1-\alpha_j)||w_j - t|| + \alpha_j H(P_S(w_j), P_S(t)) \\ &\leq (1-\alpha_j)||w_j - t|| + \alpha_j ||w_j - t|| = ||w_j - t||, \end{aligned}$$

$$\begin{aligned} ||z_j - t|| &= ||w_j - t|| \\ &\leq H(P_S(t_j), P_S(t)) \\ &\leq ||t_j - t||, \end{aligned}$$

and

$$\begin{aligned} ||w_{j+1} - t|| &= ||v_j - t|| \\ &\leq H(P_S(z_j), P_S(t)) \\ &\leq ||z_j - t|| \\ &\leq ||w_j - t||. \end{aligned}$$

It follows that the sequence $\{w_j\}$ is a Fejer monotone with respect to $F(S)$. Hence from the Proposition 2.1, sequence $\{w_j\}$ is bounded and $\lim_{j\to\infty} ||w_j - t||$ exists for all $t \in F(S)$. □

Lemma 3.2. *Let $\mathcal{M}$ be a non-empty closed convex subset of a UCBS B. Let $S : \mathcal{M} \to P(M)$ be a multi-valued mapping such that $F(S) \neq \emptyset$, and $P_S : \mathcal{M} \to 2^{\mathcal{M}}$ be a multi-valued nonexpansive mapping. Let $\{w_j\}$ be a sequence in $\mathcal{M}$ defined by (1.2), then $\lim_{j\to\infty} d(w_j, Sw_j) = 0$.*

Proof. By Lemma 3.1, we have $\lim_{j\to\infty} ||w_j - t||$ exists for all $t \in F(S)$. Let $\lim_{j\to\infty} ||w_j - t|| = a$. If $a = 0$, then

$$\begin{aligned} d(w_j, Sw_j) &\leq ||w_j - u_j|| \\ &\leq ||w_j - t|| + ||t - u_j|| \\ &\leq ||w_j - t|| + H(P_S(w_j), P_S(t)) \\ &\leq 2||w_j - t|| \ \to \ 0 \ as \ j \to \infty. \end{aligned}$$

Let $a > 0$. Since $\lim_{j\to\infty} ||w_j - t|| = a$, we have $\limsup_{j\to\infty} ||w_j - t|| \leq a$. Also

$$||z_j - t|| \leq ||w_j - t|| \Rightarrow \limsup_{j\to\infty} ||z_j - t|| \leq a.$$

In addition to,

$$\begin{aligned} \limsup_{j\to\infty} ||u_j - t|| &\leq \limsup_{j\to\infty} H(P_S(w_j), P_S(t)) \\ &\leq \limsup_{j\to\infty} ||w_j - t|| \\ &\leq a. \end{aligned}$$

Also

$$\begin{aligned} \limsup_{j\to\infty} ||v_j - t|| &\leq \limsup_{j\to\infty} H(P_S(z_j), P_S(t)) \\ &\leq \limsup_{j\to\infty} ||z_j - t|| \\ &\leq a. \end{aligned}$$

Here, for each $\{\alpha_j\}$ in $[\gamma, 1-\gamma]$ for some $\gamma \in (0, 1)$, one has

$$\begin{aligned} \limsup_{j\to\infty} ||\alpha_j(w_j - t) + (1-\alpha_j)(u_j - t)|| &\leq \alpha_j \limsup_{j\to\infty} ||w_j - t|| \\ &\quad + (1-\alpha_j) \limsup_{j\to\infty} ||u_j - t|| \\ &\leq a. \end{aligned}$$

Hence, from Lemma 2.2, we have $\lim_{j\to\infty} ||(w_j - t) - (u_j - t)|| = 0$, i.e., $\lim_{j\to\infty} ||w_j - u_j|| = 0$.
Since

$$d(w_j, Sw_j) \leq ||w_j - u_j||,$$

we have

$$\lim_{j\to\infty} d(w_j, Sw_j) = 0.$$

□

Theorem 3.3. *Consider $\mathcal{M}$ is a non-empty closed convex subset of a UCBS X. Let $S : \mathcal{M} \to P(\mathcal{M})$ be a multi-valued mapping such that it satisfy Condition (I). Let $F(S) \neq \emptyset$, and $P_S : \mathcal{M} \to 2^{\mathcal{M}}$ be a multi-valued nonexpansive function. Then the sequence $\{w_j\}$ defined by (1.2), strongly converges to a fixed point of S.*

Proof. By Lemma 3.1, one has

$$||w_{j+1} - t|| \leq ||w_j - t||,$$

it gives that

$$d(w_{j+1}, F(S)) \leq d(w_j, F(S)).$$

This implies that $\lim_{j\to\infty} d(w_j, F(S))$ exists. Since S satisfy Condition (I) and by Lemma 3.2, we have $\lim_{j\to\infty} d(w_j, Sw_j) = 0$, we have $\lim_{j\to\infty} d(w_j, F(S)) = 0$.
Next we prove that $\{w_j\}$ is Cauchy sequence in M. As, we have $\lim_{j\to\infty} d(w_j, F(S)) = 0$ and $\epsilon > 0$, there is a constant j_0 for all $j \geq j_0$, one has

$$d(w_j, F(S)) < \frac{\epsilon}{4}.$$

In particular and must $p \in F(S)$ then we obtain

$$||w_{j_0} - p|| < \frac{\epsilon}{2}.$$

For $j, m \geq j_0$, we obtain

$$\begin{aligned} ||w_{j+m} - w_j|| &\leq ||w_{j+1} - p|| + ||p - w_j|| \\ &< \epsilon. \end{aligned}$$

It follows that $\{w_j\}$ is a Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is closed subset of UCBS B, it must converges in $\mathcal{M}$ and $w \in M$ thus $\lim_{j\to\infty} ||w_j - w|| = 0$. Now

$$\begin{aligned} 0 \leq d(w, P_S(w)) &\leq ||w_j - w|| + d(w_j, P_S(w_j)) + H(P_S(w_j), P_S(w)) \\ &\leq ||w_j - w|| + ||w_j - u_j|| + ||w_j - w|| \to 0 \ as \ j \to \infty. \end{aligned}$$

So $d(w, P_S(w)) = 0$. By Lemma 2.2, $F(P_S)$ is closed, therefore $w \in F(P_S) = F(S)$. □

Theorem 3.4. *Consider $\mathcal{M}$ is a non-empty closed convex subset of a UCBS B and $S : \mathcal{M} \to P(\mathcal{M})$ be a multi-valuedd function with $F(S) \neq \emptyset$, and $P_S : \mathcal{M} \to 2^{\mathcal{M}}$ is a multi-valuedd nonexpansive function. Then, the sequence $\{w_j\}$ given by (1.2), strongly converges to a fixed point of S if and only in $\liminf_{j\to\infty} d(w_j, F(S)) = 0$.*

Proof. If $\liminf_{j\to\infty} d(w_j, F(S)) = 0$, then it is obvious that the sequence $\{w_j\}$ strongly converges to a fixed point of S.
For the conversation, suppose that $\liminf_{j\to\infty} d(w_j, F(S)) = 0$, then $\lim_{j\to\infty} d(w_j, F(S)) = 0$. Using similar argument of the proof as in Theorem 3.3, we obtain that $\{w_j\}$ is a Cauchy sequence in $\mathcal{M}$. Let $\lim_{j\to\infty} w_j = q$. Then

$$\begin{aligned} 0 \leq d(q, P_S(q)) &\leq ||w_j - q|| + d(w_j, P_S(w_j)) + H(P_S(w_j), P_S(q)) \\ &\leq ||w_j - q|| + ||w_j - u_j|| + ||w_j - q|| \to 0 \ as \ j \to \infty. \end{aligned}$$

It follows that $d(q, P_S(w)) = 0$. Since $F(P_S)$ is closed, therefore $q \in F(P_{S)} = F(S)$. □

4 Numerical example

Example 4.1. *Let $B = \mathbb{R}$ be a UCBS, with $||.||$ is produced by a metric d such that $d(w, y) = ||w - z||$, for all $w, y \in B$. Let $\mathcal{M} = [0, 1]$ be a non-empty subset of B. Let $S : \mathcal{M} \to P(\mathcal{M})$ defined by*

$$Sx = \begin{cases} [0, \frac{w+1}{2}],\ w \in [0, \frac{1}{2}), \\ \{0\},\ w \in [\frac{1}{2}, 1). \end{cases}$$

Consider the following cases:

Case I: when $w \in [\frac{1}{2}, 1)$. Then $F(S) = \{0\}$ and from Lemma 2.3, we have $F(S) = \{0\} = F(P_S)$ and $P_S(0) = \{0\}$.

Case II: when $w \in [0, \frac{1}{2})$. Then $F(S) = [0, \frac{w+1}{2}]$. Now

$$\begin{aligned} P_S(w) &= \{z \in Sw : d(w, Sw) = ||z - w||\} \\ P_S(w) &= \{z \in Sw : ||z - w|| = d(w, [0, \frac{w+1}{2}])\} \\ &= \{z \in Sw : ||z - w|| = ||w - \frac{w+1}{2}||\} \\ &= \{z \in Sw : ||z - w|| = ||\frac{w-1}{2}||\} \\ &= \{z = \frac{w+1}{2}\} \end{aligned}$$

Therefore, we have $F(S) = F(P_S) = [0, \frac{w+1}{2}]$

Next, we show that a sequence $\{w_j\} \in \mathcal{M}$ given by (1.1) converges strongly to a point of $F(S)$.
Start with initial value $w_1 = \frac{1}{2}$ and choose $\alpha_k = \frac{2}{3}$, then we have

$$P_S(w_1) = \{\frac{w_1 + 1}{2}\} = \{\frac{3}{4}\}.$$

Choose $u_1 \in P_S(w_1)$, then $u_1 = \frac{3}{4}$. Now

$$\begin{aligned} z_1 &= (1 - \alpha_1)w_1 + \alpha_1 u_1 \\ &= \frac{1}{6} + \frac{1}{2} \\ &= \frac{2}{3}. \end{aligned}$$

$$P_S(t_1) = \{\frac{z_1 + 1}{2}\} = \frac{5}{6}.$$

Choose $w_1 \in P_S(t_1)$, then $w_1 = \frac{5}{6}$. Hence $z_1 = w_1 = \frac{5}{6}$. Choose $v_1 \in P_S(z_1) = \{\frac{z_1+1}{2}\} = \frac{11}{12}$.

Choose $w_2 = v_1 = \frac{11}{12}$, then doing same procedure, we have $u_2 = \frac{23}{24}$, $t_2 = \frac{17}{18}$, $z_2 = w_2 = \frac{35}{36}$ and $w_3 = v_2 = \frac{71}{72}$. Continuing the process, we get that $w_1 < 1$, $w_2 < 1$,...$w_n < 1, \ldots$. Hence, we conclude that sequence $\{w_k\} \in M$ given by (1.2) converges strongly to a point of $F(S)$.

Here we show the fastness of the iteration scheme (1.2) by comparing some well known iteration schemes with the help of some given below examples.

Example 4.2. *Let $B = \mathbb{R}$ convex Banach space which is uniformly, equipped with $||.||$ is produced by a metric d thus $d(w, z) = ||w - z||$, for all $w, z \in B$. Let $M = [0, \infty)$ be a non-empty subset of B. Let $S : \mathcal{M} \to P(\mathcal{M})$ defined by*

$$Sw = \begin{cases} \{0\},\ w \in [0, \frac{1}{100}) = A', \\ [0, \frac{w}{5}],\ w \in [\frac{1}{100}, \infty) - \{\frac{8}{7}\} = B', \\ [0, \frac{9}{10}],\ w \in \{\frac{8}{7}\} = C'. \end{cases}$$

We prove that S is multi-valued nonexpansive functions. Consider the following cases

Case I: when $w, z \in A'$ and $w, z \in C'$, then it is clear that $H(Sw, Sz) \leq ||w - z||$.

Case II: when $w, z \in B'$, then

$$\begin{aligned} H(Sw, Sz) &= H([0, \frac{w}{5}], [0, \frac{z}{5}]) \\ &= ||\frac{w}{5} - \frac{z}{5}|| \\ &= \frac{1}{5}||w - z|| < ||w - z||. \end{aligned}$$

Case III: when $w \in A'$ and $z = \frac{8}{7}$. Then $H(Sw, Sz) = H(\{0\}, [0, \frac{9}{10}]) = \frac{9}{10} < 1 = |w - z||$.

Case IV: when $w \in B'$ and $z = \frac{8}{7}$. Then

$$\begin{aligned} ||w - z|| = d(w, z) &= d(w, \frac{8}{7}) \\ &= ||w - \frac{8}{7}|| \\ &\leq ||w|| + \frac{8}{7}. \end{aligned}$$

and

$$\begin{aligned} H(Sw, Sy) &= H([0, \frac{w}{5}], [0, \frac{9}{10}]) \\ &= ||\frac{w}{5} - \frac{9}{10}|| \\ &\leq ||\frac{w}{5}|| + \frac{9}{10}. \end{aligned}$$

Clearly $H(Sw, Sy) \leq ||w - z||$.

Case V: when $w \in A'$ *and* $z \in B'$. *Then*

$$\begin{aligned} H(Sw, Sy) &= H(\{0\}, [0, \frac{z}{5}]) \\ &= ||\frac{z}{5}|| < ||z|| \\ &< ||w|| + ||z|| \\ &= ||w - z||. \end{aligned}$$

Hence, we conclude that S *is multi-valued nonexpansive mapping.*

Now with the help of Matlab software program, we compare iteration scheme (1.2) with multi-valued version of different iteration schemes given in [20].

Table 1: Strong convergence of multivalued version of Ullah (1.2), Abbas [1], Ishikawa [5], Noor [10], Picard S [4] and Thakur [6] iterations to the fixed point $x = 0$ of S in Example 4.2.

Iteration	Ullah	Thakur	Abbas	Noor	Ishikawa	Picard S
0	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000	0.50000000
1	-0.01200000	-0.04400000	- 0.02800000	-0.58800000	-0.62000000	-0.09999911
2	0.00009600	0.00140800	0.00078400	0.32222400	0.34720000	-0.02000009
3	-0.00000026	-0.00002378	-0.00001417	-0.11466401	-0.12190578	0.00240000
4	0.00000000	0.00000024	0.00000018	0.03009930	0.03047644	-0.00005333
5	0.00000000	-0.00000000	-0.00000000	-0.00623946	-0.00580272	0.00000213
6	0.00000000	0.00000000	0.00000000	0.00106579	0.00087685	-0.00000017
7	0.00000000	0.00000000	-0.00000000	-0.00015444	-0.00010809	0.00000001
8	0.00000000	0.00000000	0.00000000	-0.00001939	0.00001108	-0.00000000

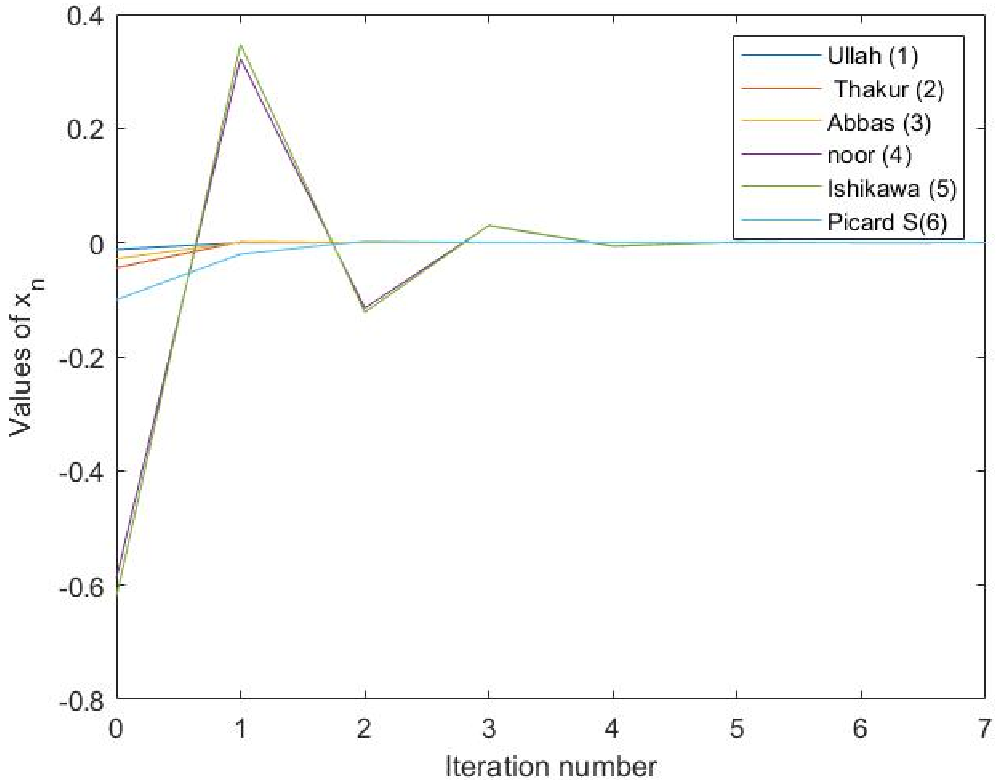

Figure 1. Behavior of Ullah iteration (magenta), Thakur iteration (carrot orange), Abbas iteration (yellow), Noor iteration (purple), Ishikawa iteration (green), Picard S iteration (cyan) to the fixed point $x = 0$ of the mapping S .

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