Alpha Power Survival Transformation Exponential distribution

Hashim Madlool[1], Kareema Abdel AL-Kadim[2]

[1,2]Department of Mathematics, Faculty of Education for Pure Sciences, Babylon University Hilla, Iraq
Email: hashimalgalby538@gmail.com[1], Kareema.kadim@yahoo.com[2]

Received: 15.04.2024 Revised : 17.05.2024 Accepted: 21.05.2024

ABSTRACT
In this research, the alpha power transformation method was used. This method was modified into the new alpha power transformation method by using the survival function to build the new probabilistic model called alpha power survival transformation exponential distribution. The proposed model is an expansion and transformation of the exponential distribution. With is more flexibility of the original distribution exponential distribution. The basic mathematical properties of this proposed distribution were extracted. The parameters were also estimated and some survival functions were extracted, after which the proposed probabilistic model APSTE was applied to the medical data to compare the new distribution with some probability distributions using the criteria, as the new model gave flexibility and efficiency in representing life data.

Keywords: uniform distribution; exponential distribution; survival function.

INTRODUCTION
The need arose for the desire to find composite, expanded, or transformed distributions that would be more flexible in representing data using different statistical methods, such as composition, expansion, and transformation of probability distributions. Here we will use the new alpha power transformation technique resulting from modifying the alpha power transformationmethod to transform a specific probability distribution into a new distribution. The power (alpha) to transform the exponential probability distribution to obtain a new probability distribution containing two parameters, which we called alpha power survival transformation exponential distribution APSTE. To modify the distribution of the data so that its basic statistical properties can be improved, and then we work to estimate its parameters using the maximum likelihood function.The main purpose of this study is to propose a new probability distribution using the alpha power coefficient technique, to transform the probability exponential distribution into a new distribution to improve the statistical properties of the data.
There are many studies that presented new distributions that rely on different techniques and methods to expand probability distributions and improve their statistical properties, including studies to expand the exponential distribution and these studies.
Mudholkar et al. [1] the exponentiated Weibull (EW) distribution is a bathtub-shaped hazard rate function that is obtained by adding an additional parameter to the Weibull distribution.Nadarajah, & Haghighi.[2] He generalized the exponential distribution. As well as an ancient comprehensive account of the mathematical properties of this generalization.Barreto-Souzaet al. [3] A new distribution called the generalized exponential beta distribution was studied.Ristić, & Balakrishnan. Elgarhy et al. [4] The exponential Weibull distribution, a novel four-parameter continuous model, is introduced. This model is constructed upon the Weibull-G exponential family.
Oguntunde,& Adejumo.[5] A transformed quadratic rank map-based two-parameter probability model that represents an additional generalization of the inverse exponential distribution is presented.
Consider the cumulative density function F_1 (x). A random variable X is said to have a transmuted distribution if the following describes its CDF:

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \quad (1)$$

The PDF of A random variable X is.

$$f_2(x) = f_1(x)[1+\lambda - 2\lambda F_1(x)] \quad (2)$$

Mahdavi and Kundu. [6] presented an alpha power transfer (APT) method. Let F(x) be cumulative distribution function (CDF) of a random variable x Where APT is determined for F(x) for $x \in R$ is.

$$F_{APT}(x) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1}, & \alpha \in R^+, \ \alpha \neq 1 \\ F(x), & \alpha = 1 \end{cases} \tag{3}$$

and the probability density function (PDF) is

$$f_{APT}(x) = \begin{cases} \dfrac{\text{L og}\,\alpha}{\alpha - 1} f(x)\alpha^{F(x)}, & \alpha \in R^+, \ \alpha \neq 1 \\ f(x), & \alpha = 1 \end{cases} \tag{4}$$

1- Alpha Power Transformation Method[19]

Let F(x) be the CDF of a continuous random variable x, then the α-power transformationof $F_{APT}(x)$ for $x \in R$, is defined as follows

$$F_{APT}(x) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1} & \text{iff} \quad \alpha > 0, \ \alpha \neq 1 \\ F(x) & \text{iff} \quad \alpha > 0, \ \alpha = 1 \end{cases} \tag{5}$$

Then the α-power transformation of f(x) for $x \in R$, is defined as follows:

$$f_{APT}(x) = \begin{cases} \dfrac{\text{L og}\,\alpha}{\alpha - 1} f(x)\alpha^{F(x)} & \text{iff} \quad \alpha > 0, \ \alpha \neq 1 \\ f(x) & \text{iff} \quad \alpha > 0, \ \alpha = 1 \end{cases} \tag{6}$$

2- New Alpha Power Transformation Method

Let S(x) be the survival function of a continuous random variable x, then the new alpha power transformation of S(x) for $x \in R$, is defined as follows:

$$S_A(x) = \begin{cases} \dfrac{1 - \alpha^{S(x)}}{1 - \alpha} & \text{iff} \quad \alpha > 0, \ \alpha \neq 1, \alpha \in R^+ \\ S(x) & \text{iff} \quad \alpha > 0, \ \alpha = 1 \end{cases} \tag{7}$$

Since $\frac{dS(x)}{dx}$= - f(x)

then

$$f_A(x) = \begin{cases} \dfrac{-\ln(\alpha)\alpha^{S(x)}}{1 - \alpha} f(x), & \text{iff} \ \alpha > 0, \ \alpha \neq 1, \ \alpha \in R^+ \\ f(x) & \text{iff} \ \alpha > 0, \ \alpha = 1 \end{cases} \tag{8}$$

Note: $f_A(x)$ is a PDF since that;

1) $f_A(x) \geq 0 \forall$ $x \in R$

Either $\alpha > 1$ or $\alpha < 1$

Where $\alpha > 1 (1 - \alpha) < 0, \ \ln(\alpha) > 0$ so $f_A(x) > 0$

Therefor if $\alpha < 1$

$(1 - \alpha) > 0, \ \ln(\alpha) < 0$ so $f_A(x) > 0$

2) $\int_{-\infty}^{+\infty} f_A \, dx = \int_{-\infty}^{+\infty} \frac{-[\ln(\alpha)]\alpha^{S(x)} \ f(x)}{1-\alpha} dx$

$= \frac{\alpha^{R(\infty)} - \alpha^{R(-\infty)}}{1-\alpha} = 1$

3- New Alpha Power Transformation Exponential Distribution APSTED

Let X~E(λ), the survival function it is.

$S(x) = e^{-\lambda x} I_{(0,\infty)}(x), \ \lambda > 0$

Thus the survival function of the distribution APSTE is as follows:

$$S_A(x) = \begin{cases} \dfrac{1 - \alpha^{e^{-\lambda x}}}{1 - \alpha} & \text{iff} \quad \alpha \neq 1, \alpha \in R^+ \\ e^{-\lambda x} & \text{iff} \quad \alpha = 1 \end{cases} \tag{9}$$

$\lambda > 0$

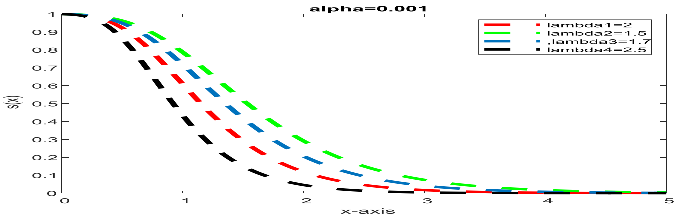

Figure 1. The survival function of APSTED at various values of α, λ,

From the Figure above, it is noted that the survival function as along as the value of x increases.

3.1. The CDF and PDF of theAPSTE Distribution

The PDF of NATED is.

$$f_A(x) = \begin{cases} \frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha}, & \text{if } \alpha > 0,\ \ \alpha \neq 1, \alpha \in R^+ \\ \lambda\, e^{-\lambda x} & \text{if } \quad \alpha = 1 \end{cases} \quad (10)$$

Note: $f_A(x)$ is a PDF since that

1) $f_A(x) \geq 0 \forall$ x$\in$ R if $\alpha > 1, \ln\alpha > 1,\ (1-\alpha) < 0, \lambda > 1,\ e^{-\lambda x} > 1,\ \forall x \in (0, \infty)$

2) $\int_0^{\infty} f_A\, dx = \int_0^{\infty} \frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha} dx$

$$= \frac{\alpha^{e^{-\infty}} - \alpha^{e^{0}}}{1-\alpha} = \frac{1-\alpha}{1-\alpha} = 1$$

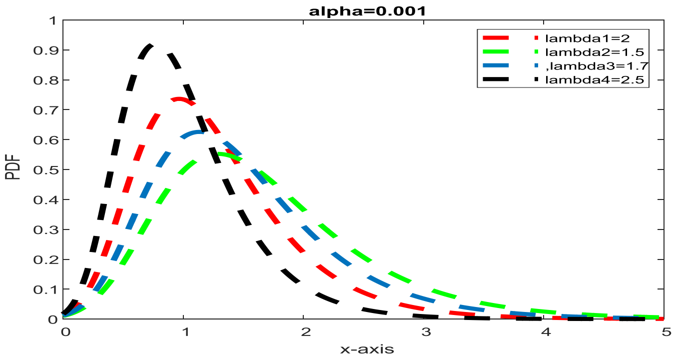

Figure 2. The PDF of APSTED at various values ofα, λ.

From the Figure above, it is noted that the PDF increases up to peak, and then it decreases as along asx value increases.

The CDF of APSTE distribution shown in eq (8)

$F_A(x) = 1 - S_A(x)$

$$F_A(x) = \begin{cases} \frac{\alpha^{e^{-\lambda x}} - \alpha}{1-\alpha}, & \text{if } \quad \alpha > 0,\ \ \alpha \neq 1, \alpha \in R^+ \\ 1 - e^{-\lambda x} & \text{if } \qquad \alpha = 1 \end{cases} \quad (11)$$

$\forall x \in (0, \infty),\ \ \lambda > 0$

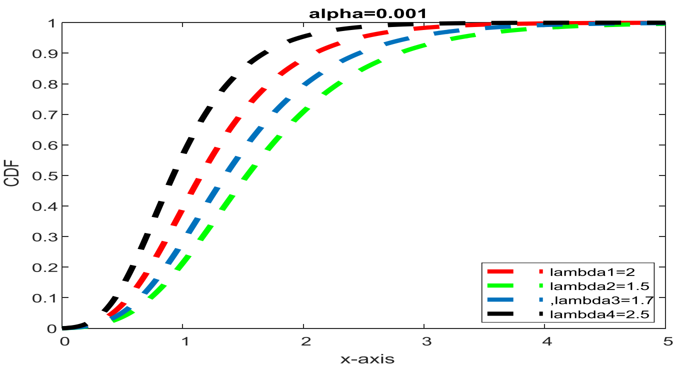

Figure 3. The CDF of APSTED at various values of α, λ.

From the Figure above, it is noted that the CDF increases as along as value of x increases.

3.2. The Hazard Function

The hazard functionof APSTE distributionis.

$$h_A(x) = \frac{f_A(x, \alpha, \lambda)}{S_A(x)}$$

$$h_A(x) = \begin{cases} \frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha^{e^{-\lambda x}}}, & \text{iff} \quad \alpha \neq 1, \alpha \in R^+ \\ \lambda & \text{iff} \quad \alpha = 1 \end{cases} \quad (12)$$

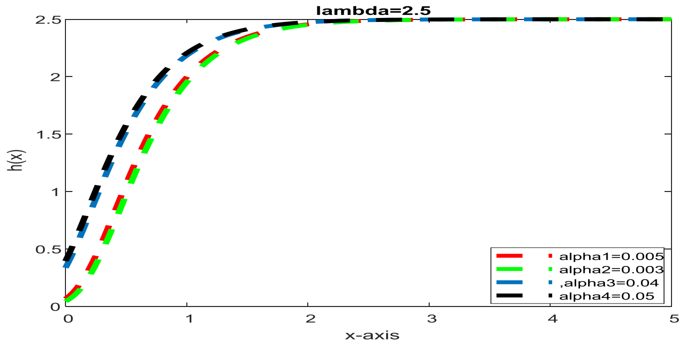

Figure 4. The h(x) of APSTED at various values of α, λ.

From the Figure above, it is noted that the hazard function increases as along as value of x increases.

3.3. The Cumulative Hazard Function

The cumulative hazard functionof APSTE distributionis.

$$H_A(x) = -\ln S_A(x)$$

$$H_A(x) = \begin{cases} -\ln \frac{1-\alpha^{e^{-\lambda x}}}{1-\alpha} & \text{iff} \quad \alpha \neq 1, \alpha \in R^+ \\ \lambda x & \text{iff} \quad \alpha = 1 \end{cases} \quad (13)$$

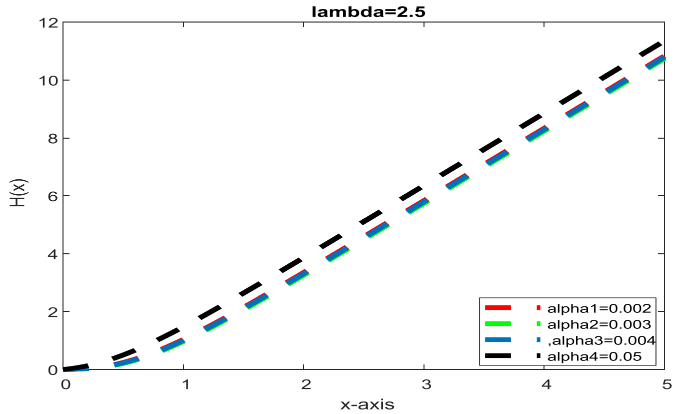

Figure 4. The H(x) of APSTED at various values of α, λ.

From the Figure above, it is noted that thecumulative hazard function increases as along as value of x.

3. 4. Reverse hazard function

Reverse hazard function of APSTE distribution shown is.

$$r(x) = \frac{f_A(x, \alpha, \lambda)}{F_A(x, \alpha, \lambda)}$$

$$r_A(x) = \begin{cases} \frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{\alpha^{e^{-\lambda x}} - \alpha} & \text{iff} \quad \alpha \neq 1, \alpha \in R^+ \\ \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda x}} & \text{iff} \quad \alpha = 1 \end{cases} \quad (14)$$

4. Statistical Properties of the APSTE Distribution

4.1. Quantile

The quantile x_q of the APSTE distribution is as shown.

$$F_A(x_Q) = P(x_q \leq q) = Q, \qquad 0 < q < 1 \quad (13)$$

From eq (7)

$$\frac{\alpha^{e^{-\lambda x_q}} - \alpha}{1 - \alpha} = q$$

$\alpha^{e^{-\lambda x_q}} - \alpha = q(1-\alpha), \qquad \alpha^{e^{-\lambda x_q}} = q(1-\alpha) + \alpha$

Taking the logarithm.

$\ln\alpha^{e^{-\lambda x_q}} = \ln(q(1-\alpha)+\alpha), \qquad e^{-\lambda x_q}\ln\alpha = \ln(q(1-\alpha)+\alpha)$

$$e^{-\lambda x_q} = \frac{\ln(q(1-\alpha)+\alpha)}{\ln\alpha}$$

Taking the logarithm.

$$-\lambda x_q = \ln\left[\frac{\ln(q(1-\alpha)+\alpha)}{\ln\alpha}\right]$$

$$x_q = -\frac{\ln\left[\frac{\ln(q(1-\alpha)+\alpha)}{\ln\alpha}\right]}{\lambda} \tag{15}$$

The median ofAPSTED is defined at q= 1/2 in eq (12)

$$x_{medain} = -\frac{\ln\left[\frac{\ln\left(\frac{1}{2}(1-\alpha)+\alpha\right)}{\ln\alpha}\right]}{\lambda} \tag{16}$$

4.1. Mode

The mode of the APSTE distribution is given as.

$$x_{mode} = \frac{d\ln f(x,\alpha,\lambda)}{dx} = 0$$

$$\frac{d}{dx}\ln\left[\frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha}\right] = 0$$

$$\frac{d}{dx}[\ln(-\ln(\alpha)) + \ln\lambda - \lambda x + \ln\alpha^{e^{-\lambda x}} - \ln(1-\alpha)] = 0$$

$$\frac{d}{dx}[\ln(-\ln(\alpha) + \ln\lambda - \lambda x + e^{-\lambda x}\ln\alpha - \ln(1-\alpha)] = 0$$

$$x_{mode} = -\frac{\ln\frac{1}{\ln\alpha^{-1}}}{\lambda} \tag{17}$$

4.2. Moments

The rth moment of APSTE distribution about the origin and mean is.

$$E(X^r) = -\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k(\ln\alpha)^{n+1}\Gamma(r+k+1)}{k!\,n!\,(1-\alpha)(n)^{r+k+1}\lambda^r} \tag{18}$$

$$E(X-\mu)^r = -\sum_{i=0}^{r}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{r}{i}(-\mu)^{r-n}\frac{(-1)^k(\ln\alpha)^{n+1}}{k!\,n!\,(1-\alpha)}\times\frac{\Gamma(i+k+1)}{\lambda^i(n)^{i+k+1}} \tag{19}$$

4.3.Moment Generating Function

Proposition1: The moment generating function of APSTED is.

$$M_X(t) = -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^j(-1)^k(\ln\alpha)^{n+1}\Gamma(j+k+1)}{j!\,k!\,n!\,(1-\alpha)(n)^{j+k+1}\lambda^i} \tag{20}$$

4.4. Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from with the $F_A(x,\alpha,\lambda)$ and $f_A(x,\alpha,\lambda)$ given in the equation (6) and (7) respectively.

Let $Y_{1:n} \le Y_{2:n} \le \cdots \le Y_{n:n}$ denote the corresponding order statistics, then the PDF of $Y_{k:n}$ is defined by.

$$f_{h,n}(x,\Phi) = \frac{n!}{(h-1)!\,(n-h)!}f_A(x,\Phi)(F_A(x,\Phi))^{h-1}[1-F_A(x,\Phi)]^{n-h} \tag{21}$$

The PDF of the statistics with the smallest order, denoted $f_{1:n}(x,\Phi)$, and those with the largest order, denoted $f_{n:n}(x,\Phi)$, can then be calculated as follows:

$$f_{1,n}(x,\Phi) = n\,\frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha}\left[1-\frac{\alpha^{e^{-\lambda x}}-\alpha}{1-\alpha}\right]^{n-1}$$

$$f_{1,n}(x,\Phi) = n\,\frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha}\left[\frac{1-\alpha^{e^{-\lambda x}}}{1-\alpha}\right]^{n-1} \tag{22}$$

$$f_{n,n}(x,\Phi) = n\,\frac{-\ln(\alpha)\lambda e^{-\lambda x}\alpha^{e^{-\lambda x}}}{1-\alpha} \times [\frac{\alpha^{e^{-\lambda x}}-\alpha}{1-\alpha}]^{n-1} \qquad (23)$$

4.5 Maximum Likelihood Estimation

It is considered one of the important methods of estimation and its estimators have excellent properties. Let a random sample $X_1, X_2, \ldots, X_k$ from the probability distribution defined (6), is taken then the likelihood function will be:

$$L(x_1, x_2, \ldots, x_k, \lambda, \alpha) = \prod_{i=1}^{k} f_A(x_i, \lambda, \alpha) = [\frac{-\ln(\alpha)\lambda}{1-\alpha}]^k \prod_{i=1}^{k} e^{-\lambda x_i}\alpha^{e^{-\lambda x_i}} \qquad (24)$$

We take the natural logarithm of both sides of the equation (35)

$$\ln L = k\ln(-\ln\alpha) + k\ln\lambda - k\ln(1-\alpha) - \sum_{i=1}^{k}\lambda x_i + \ln\alpha\sum_{i=1}^{k} e^{-\lambda x_i} \qquad (25)$$

By differentiating both sides of the equation (36), respect each the parameter as

$$\frac{\partial L}{\alpha} = \frac{k}{\alpha\ln(\alpha)} + \frac{k}{(1-\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{k} e^{-\lambda x_i}$$

$$\frac{k}{\hat{\alpha}\ln(\hat{\alpha})} + \frac{k}{(1-\hat{\alpha})} + \frac{1}{\hat{\alpha}}\sum_{i=1}^{k} e^{-\hat{\lambda} x_i} = 0 \qquad (26)$$

$$\frac{\partial L}{\lambda} = \frac{k}{\lambda} - \sum_{i=1}^{k} x_i - \ln\alpha\sum_{i=1}^{k} x_i e^{-\lambda x_i}$$

$$\frac{k}{\hat{\lambda}} - \sum_{i=1}^{k} x_i - \ln\hat{\alpha}\sum_{i=1}^{k} x_i e^{-\hat{\lambda} x_i} = 0 \qquad (27)$$

By solving the two equations (37) and (38) using numerical methods, the maximum potential estimators of the two parameters $\hat{\alpha}, \hat{\lambda}$ are given.

4.6. Application

The statistical criteria Akaike information criterion, Hannan-Quinn information criterion, consistent Akaike information criterion and Bayesian information criterion, were adopted to determine whether the distribution that was proposed APSTED better represents real data from some selected distributions, such as the exponential distribution and the exponential Weibull distribution.

$$AIC = -2\hat{\ell} + 2q \;,\quad BIC = -2\hat{\ell} + q\log(n) \qquad (39)$$

$$CAIC = -2\hat{\ell} + \frac{2qn}{n-q-1} \;, HQIC = -2\,\hat{\ell} + 2q\log(\log(n)) \qquad (40)$$

Data set: The following data set are the remission times (in months) of a random sample of 128 bladder cancer patients [8].

[0.08 , 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34,5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53,12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69].

Table 1: The MLEs of the parameters(α, λ).

Model	Parameters		
APED(α, λ)	$\hat{\alpha} = 0.91$	$\hat{\lambda} = 0.109$	
APSTED(α, λ)	$\hat{\alpha} = 0.107$	$\hat{\lambda} = 0.169$	
ED(λ)	$\hat{\lambda} = 0.035$		

Table 1 shows the parameter values for the proposed distribution and other distributions used.

Table 2: Represents the results of statistical tests (BIC, CAIC, AIC) on the data.

Model	LL	AIC	BIC	HQIC	CAIC
APED(α, β, λ)	-396.0574	796.1148	801.8189	798.4324	796.2108
APSTED(α, λ)	-420.2310	844.460	850.1660	846.7796	844.5580
ED(λ)	-471.0661	944.1322	946.9843	945.2910	944.1640

Table 2 shows the quality criteria values for the proposed distribution and other distributions used.

REFERENCES

[1] Xie, M., & Lai, C. D. (1996). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliability Engineering & System Safety, 52(1), 87-93.

[2] Nadarajah, S., & Haghighi, F. (2011). An extension of the exponential distribution. Statistics, 45(6), 543-558.

[3] Barreto-Souza, W., Santos, A. H., & Cordeiro, G. M. (2010). The beta generalized exponential distribution. Journal of statistical Computation and Simulation, 80(2), 159-172.

[4] Elgarhy, M., Shakil, M., & Kibria, G. (2017). Exponentiated Weibull-exponential distribution with applications. Applications and Applied Mathematics: An International Journal (AAM), 12(2), 5.

[5] Oguntunde, P. E., & Adejumo, A. O. (2015). The transmuted inverse exponential distribution. International Journal of Advanced Statistics and Probability, 3(1), 1-7.

[6] Mahdavi, A., & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. Communications in Statistics-Theory and Methods, 46(13), 6543-6557.

[7] Nasiru, S., Mwita, P. N., & Ngesa, O. (2019). Alpha power transformed Frechet distribution. Applied Mathematics & Information Sciences, 13(1), 129-14

[8] Al-Kadim K. A. and Mahdi A. A. (2018) ' Exponentiated Transmuted Exponential Distribution', Journal of Babylon University/Pure and Applied Sciences/ No.(2)/ Vol.(26).