A New Family of Tplm Distribution

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ABSTRACT
We have introduced a new family distribution, byusing the beta distribution family throughthe [0,1] intervaltruncated.A new probability distribution known as a Truncated Pareto Distribution- Lomax Distribution (Tplm Distribution)isdiscussed, Probability density function,the function of cumulative distribution,the function of reliabilityand Hazard risk function are presented. Some of these properties will be derived for this distribution. Such as the rth moment, the function of characteristic,Expected, Variance, Kurtosis, Skewness, the function of Shannon entropy. The MSE·s is calculated in parameters estimations, a less MSE can be seen when the default parameter values are small. (β=0.5,θ=0.5,δ =0.5, λ=0.5), for all sample sizes and in all cases MSE decreased with increasing sample size. These results can be reconciled with statistical theory.

Keywords: Pareto distribution, Lomax distribution, reliability function, Hazard risk function, Shannon entropy modules.

INTERDICTION
The Pareto distribution is called after the Italian engineer economist and sociologist Vilfredo Pareto [3]. The Lomax distribution also known as the Type II Pareto distribution is a weighted probability distribution used in the business world of economics queuing theory actuarial science and Internet traffic modeling. [6] Named after KS Lomax. It is a modified Pareto distribution so that support starts from zero [12].
In recent years many researchers have proposed a new distribution." Boshi(2019) defined two methods for generating probability distributions by combining generalized distributions, namely the generalized gamma distribution, the exponentialWeibull distribution, the generalized Gompertz distribution, andthe generalized inverse Weibull distribution "[4]." (2002) Eugene et al,determine the betanormal distribution (BN) through taking G(x) as the function of cumulative distribution of the normal distribution and extracting its first moments. "[8]. "In (2017) Abid et al introduce.A new type of continuous distribution based on the truncated distribution [0,1] by Frechet [1]. (2022) Abid et al [0,1] proposed a truncated Half logistic-Half logistic distribution and derived several important statistical properties of this distribution [2].(2024) Muhi et al introduced a new family of continuous distributions called the power function-LindleyDistribution and production of some important properties of PF-LD [9]. (2022) Hasanain et al Discuss maximum likelihood and the base estimate of two Parameters The shape parameter β and the scale parameter Θ of the Lomax-based distribution on three types of loss functions [11].
The pdf of the Pareto distribution (P-D) is given by[13],

$f(x) = \frac{\beta\theta^{\beta}}{x^{\beta+1}}$, $0 < x < \infty$, $\theta, \beta > 0$ (1)

Thecdf of the distribution is given by,

$F(x) = 1 - \left(\frac{\theta}{x}\right)^{\beta}$, $0 < x < \infty$, $\theta, \beta > 0$ (2)

Truncated$[0,1]$M-QD
Let $Q(x)$ and $q(x)$ be the pdf and cdffor any continuous distribution over the random variable x and let M (.) and m(.) denote the pdf and cdffor any continuous distribution between [0, ∞). General formula of the CDF proposed for this family is based on the writing of M. [6]

$F(x)_{TM-Q} = \frac{M[Q(x)]-M[0]}{M[1]-M[0]}$ (3)

Now, let $M[0] = 0$ Then cdf in (3) we get as,

$F(x)_{TM-Q} = \frac{M[Q(x)]}{M[1]}$ (4)

And pdf, $f(x)\frac{d}{dx}[F(x)]$ will be,

$f(x)_{TM-Q} = \frac{m\ [Q(x)]q(x)}{M[1]}$ (5)

Truncated Pareto distribution - QD

Let $M(.)\ and\ m(.)$ that mentioned in (4) and (5), (TP-QD) call back (1) and (2) to be cdf and pdf for positive parameters (β>0,θ>0). we have M(0)=0 So;

$M[Q(x)] = 1 - \left(\frac{\theta}{Q(x)}\right)^{\beta}$, $L[1] = 1 - (\theta)^{\beta}$

And $m[Q(x)] = \frac{\beta\theta^{\beta}}{Q(x)^{\beta+1}}$

Giving to (4) and (5).The cdf and corresponding pdf for the new distribution arenamed [0,1], TP-QD will be,

$$F(x)_{TP-Q} = \frac{1-\left(\frac{\theta}{Q(x)}\right)^{\beta}}{1-(\theta)^{\beta}}, \quad 0 < x < \infty, \quad \beta,\theta > 0 \quad (6)$$

and, $f(x)_{TP-Q} = \frac{\frac{\beta\theta^{\beta}}{Q(x)^{\beta+1}}q(x)}{1-(\theta)^{\beta}}$

$$= \frac{\beta\theta^{\beta}q(x)}{(1-(\theta)^{\beta})Q(x)^{\beta+1}}, \quad 0 < x < \infty, \quad \beta,\theta > 0 \quad (7)$$

T PLM Distribution

suppose that $Q(x)\ and\ q(x)$, the Lomax Distribution (LM) with tow positive parameter $\delta\ and\ \lambda$, cdf and pdf as[5],

$$q(x) = \frac{\delta}{\lambda}\left[1 + \frac{x}{\lambda}\right]^{-(\delta+1)}, \quad 0 \leq x < \infty, \quad \delta,\lambda > 0 \quad (8)$$

$$Q(x) = 1 - \left[1 + \frac{x}{\lambda}\right]^{-\delta}, \quad 0 \leq x < \infty, \quad \delta,\lambda > 0 \quad (9)$$

According to (6), the cdf of new distribution called Tplmdistribution will be,

$$F(x)_{Tplm} = \frac{1-\left(\frac{\theta}{1-\left[1+\frac{x}{\lambda}\right]^{-\delta}}\right)^{\beta}}{1-(\theta)^{\beta}}, 0 \leq x < \infty, \quad \beta,\theta,\delta,\lambda > 0 \quad (10)$$

The pdf Tplm distributioncan be obtained, according to (7) as,

$$f(x)_{Tplm} = \frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{(1-(\theta)^{\beta})\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}, \ ,0 \leq x < \infty, \quad \beta,\theta,\delta,\lambda > 0 \quad (11)$$

The reliability function Tplm distribution, itcan be obtained,

$$R(x)_{Tplm} = 1 - F(x)_{Tplm}$$

$$= 1 - \frac{1-\left(\frac{\theta}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)}\right)^{\beta}}{1-(\theta)^{\beta}}$$

$$= 1 - \frac{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta} - \theta^{\beta}}{(1-(\theta)^{\beta})\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta}}$$

The hazard function of Tplm distribution as,

$$H(x)_{Tplm} = \frac{f(x)_{Tplm}}{R(x)_{Tplm}}$$

$$= \frac{\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{(1-(\theta)^{\beta})\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}}{1-\frac{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta}-\theta^{\beta}}{(1-(\theta)^{\beta})\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta}}}$$

$$=\frac{\left[\left(1-(\theta)^{\beta}\right)\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta}\right]-\left[\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta}-\theta^{\beta}\right]\left[\beta\theta^{\beta}\frac{\delta}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}\right]}{\left[\left(1-(\theta)^{\beta}\right)\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta}\right]\left[1-(\theta)^{\beta}\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}\right]}$$

The characteristics of the Tplm distribution are listed as follows.
r-th moment: It is possible to obtain r-th moment of Tplm distribution
$\int_0^\infty x^r f(x)_{Tplm}\, dx$. According to (7). We get,

$$E(X^r)_{Tplm}=\int_0^\infty x^r\left[\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{\left(1-(\theta)^{\beta}\right)\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right]dx$$

By using the following Formula [10],

$$(a+u)^{-n}=\sum_{i=0}^{\infty}C_i^{-n}\,a^{-n-i}u^i \qquad (12)$$

where $C_i^{-n}=(-1)^i\binom{n+i-1}{i}$,

$$E(X^r)_{Tplm}=\int_0^\infty x^r\left[\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\sum_{i=0}^{\infty}C_i^{-(\delta+1)}\left(\frac{x}{\lambda}\right)^i}{\left(1-(\theta)^{\beta}\right)\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right]dx$$

By using the following Formula [10],

$(1-u)^{-b}=\sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(b+j)}{r(b)}u^j;\quad |u|<1, b>0$, we get, (13)

$$E(X^r)_{Tplm}=\int_0^\infty x^r\left[\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\sum_{i=0}^{\infty}C_i^{-(\delta+1)}\left(\frac{x}{\lambda}\right)^i\sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}\left[1+\frac{x}{\lambda}\right]^{-j\delta}}{\left(1-(\theta)^{\beta}\right)}\right]dx$$

$$=\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\sum_{i=0}^{\infty}C_i^{-(\delta+1)}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{\left(1-(\theta)^{\beta}\right)}\int_0^\infty x^r\left[\left(\frac{x}{\lambda}\right)^i\left[1+\frac{x}{\lambda}\right]^{-j\delta}\right]dx(14)$$

Let $I=\int_0^\infty x^r\left[\left(\frac{x}{\lambda}\right)^i\left[1+\frac{x}{\lambda}\right]^{-j\delta}\right]dx$

Let $y=\frac{x}{\lambda}$, we find $x=y\lambda$ With $dx=\lambda dy$

This means that ifthelimits of $x=0$ then the limits of $y=0$ and for $x=\infty$ then $y=\infty$. We get:

$$I=\int_0^\infty (y\lambda)^r\left[(y)^i[1+y]^{-j\delta}\right]\lambda dy$$

$$=\lambda^{r+1}\int_0^\infty (y)^{r+i}(1+y)^{-j\delta}dy$$

$$=\lambda^{r+1}\int_0^\infty (y)^{r+i}(1+y)^{-j\delta}dy$$

$$I=\lambda^{r+1}\int_0^\infty (y)^{(r+i+1)-1}(1+y)^{-(r+i+1)-(j\delta-r-i-1)}dy$$

$$=\lambda^{r+1}\int_0^\infty (y)^{(r+i+1)-1}(1+y)^{-[(r+i+1)+(j\delta-r-i-1)]}dy$$

Then, we obtain

$I=\lambda^{r+1}B(r+i+1,j\delta-r-i-1)$

Were

$B(n,m)=\int_0^\infty (y)^{n-1}(1+y)^{-(n+m)}dy$, is beta function of second type [7]

Now, substitute I in (14) we get

$$E(X^r)_{Tplm}=\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\sum_{i=0}^{\infty}C_i^{-(\delta+1)}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{\left(1-(\theta)^{\beta}\right)}\lambda^{r+1}B(r+i+1,j\delta-r-i-1)\quad(15)$$

By setting $r=1$ in (15), we get the mean of X,

$$E(X)_{Tplm}=\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\sum_{i=0}^{\infty}C_i^{-(\delta+1)}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{\left(1-(\theta)^{\beta}\right)}\lambda^2B(i+2,\quad j\delta-i-2)$$

The Second expectation,

$$E(X^2)_{Tplm} = \frac{\beta\theta^\beta \frac{\delta}{\lambda}\sum_{i=0}^{\infty} C_i^{-(\delta+1)} \sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{(1-(\theta)^\beta)}\lambda^3 B(i+3, \quad j\delta - i - 3)$$

the Third expectation,

$$E(X^3)_{Tplm} = \frac{\beta\theta^\beta \frac{\delta}{\lambda}\sum_{i=0}^{\infty} C_i^{-(\delta+1)} \sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{(1-(\theta)^\beta)}\lambda^4 B(i+4, j\delta - i - 4)$$

The Fourth expectation,

$$E(X^4)_{Tplm} = \frac{\beta\theta^\beta \frac{\delta}{\lambda}\sum_{i=0}^{\infty} C_i^{-(\delta+1)} \sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{(1-(\theta)^\beta)}\lambda^5 B(i+5, j\delta - i - 5)$$

So, the mean, variance, the kurtosis and skewness, related measures depending on (15) can be obtained easily.

The function of the Characteristic

characteristic functionof theTplm distribution,can be found from,

$$\phi_X(b)_{Tplm} = E(e^{ibx}) = \sum_{r=0}^{\infty}\frac{(ib)^r}{r!}E(X^r)_{TP-LM}$$

$$= \sum_{r=0}^{\infty}\frac{(ib)^r}{r!}\frac{\beta\theta^\beta \frac{\delta}{\lambda}\sum_{i=0}^{\infty} C_i^{-(\delta+1)} \sum_{j=0}^{\infty}\frac{1}{j!}\frac{r(\beta+1+j)}{r(\beta+1)}}{(1-(\theta)^\beta)}\lambda^{r+1} B(r+i+1, j\delta - r - i - 1)$$

Shannon Entropy

To get Shannon entropy of Tplm distribution, we firstly define,

$$-\int_0^\infty \ln\big(f(x)_{Tplm}\big) f(x)_{Tplm}\, dx$$

Through using the natural logarithm, of the pdf in (11), we get.

$$ln\big(f(x)_{Tplm}\big) = ln\left\{\frac{\beta\theta^\beta \frac{\delta}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{(1-(\theta)^\beta)\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right\}$$

$$= ln\left(\frac{\beta\theta^\beta \frac{\delta}{\lambda}}{(1-(\theta)^\beta)}\right) - (1+\delta)ln\left(\frac{x}{\lambda}\right) + (1+\beta)ln\left[1+\frac{x}{\lambda}\right]^{-\delta}$$

$$= ln\left(\frac{\beta\theta^\beta \frac{\delta}{\lambda}}{(1-(\theta)^\beta)}\right) - (1+\delta)ln(x) + (\delta+1)ln(\lambda) - \delta(1+\beta)\ln\left[\frac{x}{\lambda}\right]$$

$$= \left\{\begin{array}{c}\ln\left(\frac{\beta\theta^\beta \frac{\delta}{\lambda}}{(1-(\theta)^\beta)}\right) - (1+\delta)ln(x) + (\delta+1)ln(\lambda) \\ -\delta(\beta+1)ln(x) + \delta(1+\beta)ln(\lambda)\end{array}\right\}$$

$$= \left\{\begin{array}{c}\ln\left(\frac{\beta\theta^\beta \frac{\delta}{\lambda}}{(1-(\theta)^\beta)}\right) - \big[\big((1+\delta)\big) + \delta(1+\beta)\big]\ln(x) \\ +\big((1+\delta)\big)\ln(\lambda) + \delta(1+\beta)\ln(\lambda)\end{array}\right\}$$

By using the following Formula [10],

$$ln(x) = 2\sum_{k=0}^{\infty}\frac{(x-1)^{2k+1}}{(x+1)^{2k+1}}, \qquad x > 0 \qquad (16)$$

And, $(a+b)^n = \sum_{s=0}^{n} C_s^n a^{n-s} b^s$ (17)

$$ln\big(f(x)_{Tplm}\big) = \left\{\begin{array}{c} ln\left(\frac{\beta\theta^\beta \frac{\delta}{\lambda}}{(1-(\theta)^\beta)}\right) - 2[(\delta+1) + \delta(\beta+1)]\sum_{k=0}^{\infty}\frac{(x-1)^{2k+1}}{(x+1)^{2k+1}} \\ +(\delta+1)ln(\lambda) + \delta(\beta+1)ln(\lambda)\end{array}\right\}$$

According to (17) and (12), we get

$$ln\left(f(x)_{Tplm}\right) = \left\{\begin{matrix} ln\left(\dfrac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right) - 2[(\delta+1)+\delta(\beta+1)]\displaystyle\sum_{k=0}^{\infty}\sum_{s=0}^{n} C_s^n(-1)^{(2k+1)-s} \\ \displaystyle\sum_{i=0}^{\infty} C_i^{-(2k+1)}\, x^{i+s} + (\delta+1)\ln(\lambda) + \delta\,(\beta+1)\ln(\lambda) \end{matrix}\right\}$$

The Shannon entropy, Tplm distribution can be obtained as:

$$SH_{Tplm} = -\int_0^{\infty}\left\{\begin{matrix} ln\left(\dfrac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right) - 2[(\delta+1)+\delta(\beta+1)]\displaystyle\sum_{k=0}^{\infty}\sum_{s=0}^{n} C_s^n(-1)^{(2k+1)-s} \\ \displaystyle\sum_{i=0}^{\infty} C_i^{-(2k+1)}\, x^{i+s} + (\delta+1)\ln(\lambda) + \delta\,(\beta+1)\ln(\lambda) \end{matrix}\right\} f(x)_{Tplm}\, dx$$

$$= \left\{\begin{matrix} -ln\left(\dfrac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right) + 2[(\delta+1)-\delta(\beta+1)]\displaystyle\sum_{k=0}^{\infty}\sum_{s=0}^{n} C_s^n(-1)^{(2k+1)-s} \\ \displaystyle\sum_{i=0}^{\infty} C_i^{-(2k+1)}\int_0^{\infty} x^{i+s}\, f(x)_{Tplm}\, dx - (\delta+1)ln(\lambda) - \delta(\beta+1)ln(\lambda) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} -\ln\left(\dfrac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right) + 2[(\delta+1)-\delta(\beta+1)]\displaystyle\sum_{k=0}^{\infty}\sum_{s=0}^{n} C_s^n(-1)^{(2k+1)-s} \\ \displaystyle\sum_{i=0}^{\infty} C_i^{-(2k+1)}\, E(X^{i+t}) - (\delta+1)\ln(\lambda) - \delta(\beta+1)\ln(\lambda) \end{matrix}\right\}$$

Were,
(X^{i+t}) as in (15) with $(r = i + t)$.

Estimation of the Tplm distribution Parameters

we obtain the maximum likelihood estimate (MLE) of the Tplm distribution parameters. Let $x_1, x_2, \dots, x_n$ be a random sample size from the probability function $X \sim Tplm(\beta, \theta, \delta, \lambda)$, likelihood function is,

$$L(\beta,\theta,\delta,\lambda \backslash X) = \prod_{i=1}^{n}[f(x_i \backslash \beta,\theta,\delta,\lambda)]$$

$$= \prod_{i=1}^{n}\left(\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{(1-(\theta)^{\beta})\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right)$$

$$= \left(\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right)^{n}\prod_{i=1}^{n}\left(\frac{\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right)$$

So, log-likelihood function is,

$$\ln L(\beta,\theta,\delta,\lambda \backslash X) = n\ln\left(\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right) + \sum_{i=0}^{n}\ln\left(\frac{\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right) \quad (18)$$

Let $I = ln\left(\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right)$ and $II = ln\left(\frac{\left[1+\frac{x}{\lambda}\right]^{-(\delta+1)}}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right)$

Now

$$I = ln\left(\frac{\beta\theta^{\beta}\frac{\delta}{\lambda}}{(1-(\theta)^{\beta})}\right)$$

$$= \beta\, ln(\theta) + ln(\delta) \ln\ (\beta) + - \ln(\lambda) - \ln\left(1-(\theta)^{\beta}\right)$$

$$II = ln\left(\frac{\left[1+\frac{x}{\lambda}\right]^{-(1+\delta)}}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)^{\beta+1}}\right)$$

$$= -(1+\delta) ln\ \left(1+\frac{x}{\lambda}\right) - (1+\beta) ln\ \left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)$$

Substitute I and II in (18) we get,

$$\ln L(\beta,\theta,\delta,\lambda \setminus X) = \left\{\begin{matrix} n\left[ln(\beta) + \beta\, ln(\theta) + ln(\delta) - ln(\lambda) - ln\left(1-(\theta)^{\beta}\right)\right] \\ + \left[\sum_{i=0}^{n}\left(-(\delta+1) ln\ \left(1+\frac{x}{\lambda}\right) - (\beta+1) ln\ \left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)\right)\right] \end{matrix}\right\}$$

The MLEs $\hat{\beta}, \hat{\theta}, \hat{\delta}$ and $\hat{\lambda}$ are obtained respectively by solving the four nonlinear equations,

$$\frac{\partial \ln L(\beta,\theta,\delta,\lambda \setminus X)}{\beta} = \left\{\begin{matrix} \frac{n}{\beta} + n\, ln(\theta) + \frac{n}{(1-(\theta)^{\beta})}(\theta)^{\beta}\, ln(\theta) \\ + \left[\sum_{i=0}^{n}\left(-ln\ \left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)\right)\right] \end{matrix}\right\} = 0$$

$$\frac{\partial \ln L(\beta,\theta,\delta,\lambda \setminus X)}{\theta} = \left\{\frac{n\beta}{\theta} - \frac{n\beta\theta^{\beta-1}}{(1-(\theta)^{\beta})}\right\} = 0$$

$$\frac{\partial \ln L(\beta,\theta,\delta,\lambda \setminus X)}{\delta} = \left\{\begin{matrix} \frac{n}{\delta} \\ + \left[\sum_{i=0}^{n}\left(-ln\ \left(1+\frac{x}{\lambda}\right)\right) - (1+\beta)\frac{\left[1+\frac{x}{\lambda}\right]^{-\delta} \ln\left(1+\frac{x}{\lambda}\right)}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)}\right] \end{matrix}\right\} = 0$$

$$\frac{\partial \ln L(\beta,\theta,\delta,\lambda \setminus X)}{\lambda} = \left\{\begin{matrix} \frac{n}{\lambda} \\ + \left[\sum_{i=0}^{n}\left((1+\delta)\frac{x\lambda^{-2}}{\left(1+\frac{x}{\lambda}\right)} + (1+\beta)\frac{\delta x\lambda^{-2}\left[1+\frac{x}{\lambda}\right]^{-\delta-1}}{\left(1-\left[1+\frac{x}{\lambda}\right]^{-\delta}\right)}\right)\right] \end{matrix}\right\} = 0$$

Empirical study

We simulate a data of random variable from Tplm distribution for different sample sizes (30, 60, 150) and different parameter values, it can be simulated by numerical solution the above nonlinear equations and the MSE,s are calculated a for parameters estimations .By using MATLAB codes, we obtained the results. Empirical MSE for the parameter's estimation of Tplm distribution.

Default parameter value				**Sample size**	**Empirical MSE**			
β	θ	δ	λ	n	$\hat{\beta}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\lambda}$
	0.5	0.5	0.5	30	0.001226	0.000858	0.001448	0.0001242
				60	0.001225	0.000557	0.001483	0.000476
				150	0.001031	0.000188	0.001315	0.000182
			1.5	30	0.001355	0.001032	0.01412	0.0001245
				60	0.001204	0.000705	0.01677	0.000806
				150	0.001333	0.000431	0.001621	0.000521
		1.5	0.5	30	0.001354	0.00113	0.02517	0.002160
				60	0.001419	0.000905	0.001962	0.001008
				150	0.001345	0.000575	0.001824	0.000632
			1.5	30	0.001742	0.001045	0.002462	0.001917
				60	0.001475	0.000925	0.001939	0.000913
				150	0.001259	0.000447	0.001813	0.000472
	1.5	0.5	0.5	30	0.001623	0.001315	0.002832	0.001580

0.5				60	0.001536	0.001082	0.002731	0.001624
				150	0.001462	0.000721	0.001852	0.001613
			1.5	30	0.001693	0.001427	0.002722	0.001534
				60	0.001673	0.001275	0.002285	0.001364
				150	0.001563	0.000823	0.001834	0.001124
		1.5	0.5	30	0.001797	0.001452	0.002685	0.001465
				60	0.001771	0.001122	0.002234	0.001214
				150	0.001653	0.001167	0.001856	0.001211
			1.5	30	0.001921	0.001622	0.002721	0.001634
				60	0.001853	0.001357	0.002354	0.001486
				150	0.001755	0.000765	0.001818	0.000653

In this table you can see:

1- In all cases,the MSE decreases as the sample sizeincreasing. This result is consistent with statistical theory.
2- We notice that the lowest MSE when the estimated parameters are small (β=0.5, θ =0.5, δ =0.5, λ=0.5), and for all sample sizes.
3- As the sample size increases, the parameters get closer and closer to the true parameter values.
4- In all cases, the variance decreases as the sample size increases.

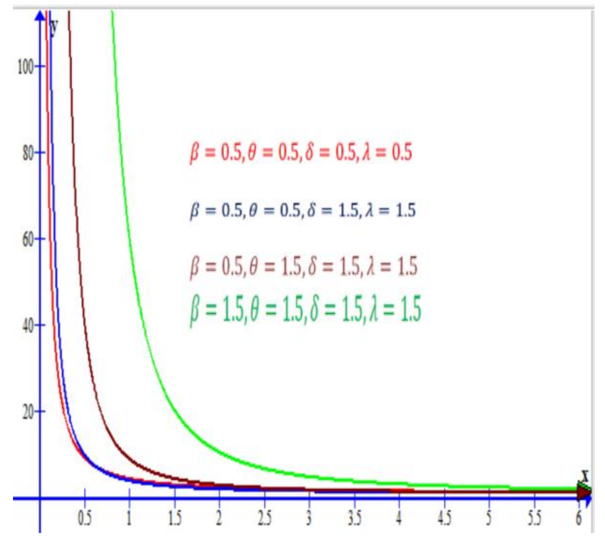

Figure 1. Graph of pdf of the Tplm Distribution

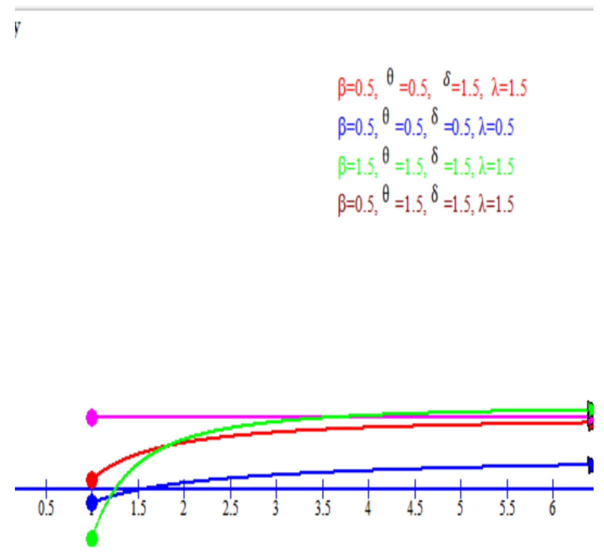

Figure 2. Graph of cdf of the Tplm Distribution

CONCLUSION

In this search, we created a new continuous distribution, named Tplm distribution,the probability pdfand cdf, are indicated. In addition to more important statistical properties such as our moment reliability the Shannon entropy of the hazard rate function. A Simulation study of the proposed model are also explained, in all cases, the variance decreases as the sample size increases.

REFERENCES

[1] Abid S, Abdulrazak R. [0,1] truncated Fréchet-G Generator of Distributions. Applied Mathematics. 2017; 7(3): 51-66.
[2] Abid S, Kneehr A, Muhi, E. [0,1] Truncated half logistic- half logistic distribution J. Nonlinear Anal. Appl.2022; 13 (1), 2959-2969.
[3] Amoroso L. "VILFREDO PARETO". Econometrica (Pre-1986); Jan 1938; 6, 1; ProQuest. 6.
[4] Boshi MA. New classes of probability distribution. 2019; Thesis submitted to the council of the college of science at the Mustansiriya University.
[5] Chakraborty T. A note on the Lomax distribution Journal reference: Soc. Netw. Anal. Min. 2019; 11, 43.

[6] Chen J J, Addie R G, Zukerman M, Neame T D. Performance Evaluation of a Queue Fed by a Poisson Lomax Burst Process. IEEE Communications Letters. 2015; 19, 3, 367-370.
[7] Elbiely M M, Yousof H M. A new extension of the Lomax distribution and its application. Journal of Statistics and Applications. 2019; 2(1), 18-34.
[8] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. Commun. Statist. - Theory and Methods. 2002 ; 31:497-512.
[9] Hasanain, W., AL-Ghezi, N., Soady, A. Bayes estimation of Lomax parameters under different loss functions using Lindley's approximation. Italian Journal Of Pure And Applied Mathematics – N. 48–2022 (630–640).
[10] Muhi E F.Some Issues Related With Logistic Distribution. 2022; A Dissertation submitted to the council of the college of science at the Mustansiriya University.
[11] Muhi, E.,Oleiwi, N.,Al-khairullah, N. A New Family of Power Function- Lindley Distribution.Advances in Nonlinear Variational Inequalities.2024:27(1), 325-337.
[12] Van H M , Vose D. A Compendium of Distributions [ebook]. Vose Software; 2009. Ghent, Belgium. Available at www.vosesoftware.com.
[13] Walck C. statistical distributions for experimentalists. hand book, Internal Report.2007; SUF–PFY/96–01,133.