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The Mohand Transform Approach to Fractional Integro-Differential Equations

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This research investigates specific classes of fractional integro-differential equations using a straightforward fractional calculus technique. The employed methodology yields a variety of compelling outcomes, including a generalized version of the well-established classical Frobenius method. The approach presented in this study primarily relies on fundamental theorems concerning the specific solutions of fractional integro-differential equations, utilizing the Mohand transform and binomial series extension coefficients. Additionally, advanced techniques for solving fractional integro-differential equations effectively are showcased.

Keywords: Riemann-Liouville (RL) fractional integrals; fractional-order differential equation; gamma function; Mittag-Leffler function; Wright function; Mohand transform of the fractional derivative

2010 Subject Classification: 26A33; 31A10; 33C10; 34A05; 35K37.

1 Introduction

Fractional calculus, an exploration of non-integer order integrals and derivatives, has garnered significant attention in mathematics owing to its diverse applications in scientific and engineering domains [4]. Its profound impact arises from robust mathematical foundations and practical implementations. More and more people are interested in making transforms that can solve fractional integro-differential equations. These transforms are often linked to basic ideas like the gamma function, beta function, error function, Mittag-Leffler function, and Mellin-Ross function [8].

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Integral transformations stand as fundamental mathematical tools crucial in addressing various differential equations, including partial differential equations, partial integro-differential equations, delay differential equations, and models describing population dynamics. Out of these, the Mohand transform, which comes from the classical Fourier integral, stands out as a simple and mathematically sound way to solve ordinary differential equations in the time domain. Alongside the Mohand transform, the Fourier, Laplace, Aboodh, and Elzaki transforms [2, 5, 6, 7] constitute the principal mathematical arsenal for solving differential equations. Notably, the Mohand transform shares a close relationship with the Laplace transform.

In recent research, Dubey et al. [12, 13, 14, 15, 16] have extensively explored various aspects of fractional calculus, employing computational techniques to forecast behavior, analyze integral transforms, investigate generalized invexity and duality in optimization problems, and delve into fractal dynamics within the physical sciences. Alongside these contributions, Singh, Purohit, and Kumar [17] compiled a comprehensive book discussing advanced numerical methods for differential equations, while Kumar et al. [18] conducted a computational analysis using fractal calculus to study local fractional partial differential equations.

The Mohand transform [1], like other integral transformations, exhibits certain limitations in its applicability. Its effectiveness often hinges on specific conditions and assumptions, potentially restricting its scope when solving differential equations. Some things about the Mohand approach are the same as the Laplace transform, but it might be hard to get closed-form solutions, especially when there are complicated boundary conditions or nonlinear equations. Recognizing and addressing these limitations is crucial when evaluating its usefulness in solving fractional integro-differential equations.

Aruldoss and Anusuya Devi expanded the use of binomial series extension coefficients and the Aboodh transform of fractional derivatives in 2020 to find exact solutions for fractional differential equations that are not homogeneous [3]. Moreover, Sumudu-based algorithms for differential equations have been extensively explored [9, 10].

We use the Mohand transform of fractional derivatives and binomial series extension coefficients in our research to come up with new ways to solve a number of fractional integro-differential equations. Furthermore, we elucidate properties relevant to our focal investigation.

2 Preliminaries

In this section, we are listing some preliminaries that are useful throughout the paper.

1. For the function $f(t)$, the RL fractional integral [3] of order $\varpi > 0$ is defined as,

$$I^{\varpi}{}_{t} f\left(t\right) = \frac{1}{\Gamma(\varpi)} \int_{a}^{t} \left(t-\zeta\right)^{\varpi-1} \, f\left(\zeta\right) d\zeta.$$

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2. Caputo fractional derivative [2] of the function $f(t)$ is defined by

$$D^{\varpi}{}_{t} f(t) = \begin{cases} f^{m}(t) \quad ; \quad if \ \varpi = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\varpi)} \int_0^{\zeta} \frac{f^{m}(t)}{(t-x)^{\varpi-m+1}} dt \quad ; if \ m-1 < \varpi < m, \end{cases}$$

where the Euler gamma function $\Gamma(\cdot)$ is defined by

$$\Gamma(\phi) = \int_0^{\infty} t^{\phi-1} e^{-t} \, dt \qquad (\mathbb{R} > 0).$$

3. The Mohand transform [1] of a function $f(t)$, $t \in (0, \infty)$ is defined by

$$M[f(t)](s) = F(s) = s^2 \int_0^{\infty} e^{-st} \, f(t) \, dt (\mathrm{s} \in \mathbb{C}).$$

4. The Mittag-Leffler function [11] is defined by

$$E_{\gamma,\delta}(\phi) = \sum_{\wp=0}^{\infty} \frac{\phi^{\wp}}{\Gamma(\gamma\wp + \delta)} \qquad (\gamma, \delta, \phi \in \mathbb{C}, \ \mathbb{R}(\gamma) > 0).$$

5. The Simplest wright function [11] is defined by

$$\rho(\omega, \phi; \varphi) = \sum_{\wp=0}^{\infty} \frac{1}{\Gamma(\omega\wp + \phi)} \cdot \frac{\varphi^{\wp}}{\wp!} \qquad (\varphi.\phi, \omega \in \mathbb{C}).$$

6. The general Wright function [11] ${}_{i}\lambda_{j}(\varphi)$ is classified as $\varphi \in \mathbb{C}$, $\nu_{1p}, \nu_{2p} \in \mathbb{C}$, and real $\omega_p, \phi_q \in \mathbb{R}$ $(p = 1, \ldots, \ i, \ q = 1, \ldots, \ j)$ by the series

$${}_{i}\lambda_{j}(\nu) = {}_{i}\lambda_{j} \left(\begin{matrix} (\nu_{1p}, \omega_p)_{1,i} \\ (\nu_{2q}, \phi_q)_{1,j} \end{matrix} \mid \varphi \right) = \sum_{r=0}^{\infty} \frac{\prod_{p=1}^{i} \Gamma(\nu_{1p} + \omega_p r)}{\prod_{q=1}^{j} \Gamma(\nu_{2q} + \phi_q r)} \cdot \frac{\varphi^r}{r!}.$$

7. The convolution integral of Mohand transform is

$$M[(f * g)(t)] = \frac{1}{s^2} M[f(t)] M[g(t)].$$

8. The inverse Mohand transform is defined by

$$M^{-1}\Big[\frac{\Gamma(n+1)}{s^{n-1}}\Big] = t^{n}.$$

9. The derivatives of the Mohand transform are

$$M[f^{'}(t)] = sF(s) - s^2 f(0),$$

$$M[f^{''}(t)] = s^2 F(s) - s^3 f(0) - s^2 f^{'}(0).$$

Remark 2.1

$$M[D^{\varpi} f(t)](s) = s^2 \int_0^{\infty} e^{-st} \ [D^{\varpi} \ f(t)] \, dt$$

$$
\begin{aligned}
&= s^2 \int_0^\infty e^{-st} \frac{1}{\Gamma(n-\varpi)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\varpi-n+1}} \, d\zeta \, dt \\
&= \frac{s^2}{\Gamma(n-\varpi)} \int_0^\infty \int_\zeta^\infty e^{-st} \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\varpi-n+1}} \, dt \, d\zeta \\
&= \frac{s^2}{\Gamma(n-\varpi)} \int_0^\infty f^{(n)}(\zeta) \int_0^\infty e^{-s(u+t)} \, u^{n-\varpi-1} \, du \, d\zeta \\
&= \frac{s^2}{\Gamma(n-\varpi)} \int_0^\infty e^{-s\zeta} f^{(n)}(\zeta) \int_0^\infty e^{-su} \, u^{n-\varpi-1} \, du \, d\zeta \\
&= \frac{s^2}{\Gamma(n-\varpi)} \int_0^\infty e^{-s\zeta} f^{(n)}(\zeta) \frac{\Gamma(\mathrm{n}-\varpi)}{s^{n-\varpi}} d\zeta \\
&= s^{\varpi-n+2} \int_0^\infty e^{-s\zeta} \, f^{(n)}(\zeta) \, d\zeta = s^{\varpi-n+2} \, M\left[f^{(n)}(\zeta)\right](s) \\
&= s^{\varpi-n+2}.s^n \left[F(s) - \left(sf(0) + f'(0) + \cdots + s^{2-n} f^{(n-1)}(0)\right)\right] \\
&= s^{\varpi+2} \left[F(s) - sf(0) - f'(0) - \cdots - s^{2-n} f^{(n-1)}(0)\right] \\
&= s^{\varpi+2} \left[M[f(t)] - \sum_{\mathfrak{K}=0}^{n} s^{1-\mathfrak{K}} f^{(\mathfrak{K}-1)}(0)\right].
\end{aligned}
$$

Note: To change the order of integration in the preceding derivative we use Fubini's theorem.

3 Solutions of fractional integro-differential equations

We can strongly suspect thus far in this section that $y(t)$ is enough to ensure that the Mohand transform $M[y(t)]$ proceeds for some value of the parameter s.

Theorem 3.1 Let $1 < \varpi < 2$ and a and $b \in \mathbb{R}$. Then the fractional integro-differential equation

$$
y^{''}(t) + a\, y^{\varpi}(t) + by(t) = \int_0^s \frac{g(t)}{(s-t)^{\varrho}} \, dt \quad ; \quad 0 < \varrho < 1 \tag{1}
$$

With $y(0) = \aleph_0$ and $y'(0) = \aleph_1$ its proposal is provided by

$$
\begin{aligned}
y(t) = \; & \aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} \, t^{2\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp + 2\mathfrak{K} + 1] \, \wp!} \\
& + \aleph_1 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} \, t^{2\mathfrak{K}+1}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp + 2\mathfrak{K} + 2] \, \wp!}
\end{aligned}
$$

$$+\ a\aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}\ t^{2\mathfrak{K}-\varpi+2}}{\mathfrak{K}!} \sum_{r=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}-\varpi+3]\ \ \wp!}$$

$$+\ a\aleph_1 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}\ t^{2\mathfrak{K}-\varpi+3}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}-\varpi+4]\ \ \wp!}$$

$$+\frac{\sin\varrho\pi}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\varrho-1} f(t)\,dt \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}\ t^{2\mathfrak{K}+1}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}+2]\ \ \wp!}\ . \tag{2}$$

Proof:
Utilizing the Mohand transform in (1) and taking into consideration, we have

$$s^2F(s)-s^3f(0)-s^2f'(0)+a\left[s^{\varpi}F(s)-s^{\varpi+1}f(0)-s^{\varpi}f'(0)\right]+bF(s)=M[f(t)]$$

where $f(t)=\int_0^s \frac{g(t)}{(s-t)^{\varrho}}\ dt$,

$$s^2M[y(t)]-s^3y(0)-s^2y'(0)+as^{\varpi}M[y(t)]-as^{\varpi+1}y(0)-as^{\varpi}y'(0)+b\,M[y(t)]=M[f(t)]$$

$$\left(s^2+as^{\varpi}+b\right)\ M[y(t)]=s^3\aleph_0+s^2\aleph_1+as^{\varpi+1}\aleph_0+as^{\varpi}\aleph_1+M[f(t)]$$

$$M[y(t)]=\frac{s^3\aleph_0+s^2\aleph_1+as^{\varpi+1}\aleph_0+as^{\varpi}\aleph_1+M[f(t)]}{(s^2+as^{\varpi}+b)}. \tag{3}$$

Since

$$\begin{aligned}
\frac{1}{(s^2+as^{\varpi}+b)} &= \frac{s^{-\varpi}}{s^{2-\varpi}+a+bs^{-\varpi}} \\
&= \frac{s^{-\varpi}}{(s^{2-\varpi}+a)\left(1+\frac{bs^{-\varpi}}{s^{2-\varpi}+a}\right)} \\
&= \frac{s^{-\varpi}}{s^{2-\varpi}+a}\sum_{\mathfrak{K}=0}^{\infty}\left(\frac{-bs^{-\varpi}}{s^{2-\varpi}+a}\right)^{\mathfrak{K}} \\
&= \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{k}s^{-\varpi\mathfrak{K}-\varpi}}{(s^{2-\varpi}+a)^{\mathfrak{K}+1}} \\
&= \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{\mathfrak{K}}s^{-2\mathfrak{K}-2}}{\left(1+a\ s^{\varpi-2}\right)^{\mathfrak{K}+1}} \\
&= \sum_{\mathfrak{K}=0}^{\infty}(-b)^{\mathfrak{K}}s^{-2\mathfrak{K}-2}\sum_{\wp=0}^{\infty}\left(-as^{\varpi-2}\right)^{\wp}\binom{\mathfrak{K}+\wp}{\wp} \\
&= \sum_{\mathfrak{K}=0}^{\infty}(-b)^{\mathfrak{K}}\sum_{\wp=0}^{\infty}\binom{\mathfrak{K}+\wp}{\wp}\ (-a)^{\wp}\ s^{(\varpi-2)\wp-2\mathfrak{K}-2}
\end{aligned} \tag{4}$$

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and

$$M\left[f\left(t\right)\right]=M\left[\int_0^s \frac{g\left(t\right)}{\left(s-t\right)^{\varrho}}\ dt\right].$$

This is Convolution integral,

$$F\left(P\right)=\frac{1}{s^2}\ K\left(P\right)\ G(P)$$

Where $K(P)$ is the Mohand transform of $K(s)=s^{-\varrho}$

$$M\left[K\left(s\right)\right]=s^{-\varrho}$$

$$K\left(P\right)=\frac{\Gamma(-\varrho+1)}{s^{-\varrho-1}}=s^{\varrho+1}\ \Gamma\left(-\varrho+1\right)$$

$$G\left(P\right)=\frac{p^2F(P)}{p^{\varrho+1}\ \Gamma(1-\varrho)}$$

$$G\left(P\right)=\frac{p^{1-\varrho}F\left(P\right)}{\Gamma\left(1-\varrho\right)}$$

$$G\left(P\right)=\frac{\sin\pi\ \varrho}{\pi}p\ .\ p^{-\varrho}\Gamma\left(\varrho\right)F(P)$$

$$G\left(P\right)=\frac{\sin\pi\ \varrho}{\pi}p\ .M\left[\int_0^s\left(s-t\right)^{\varrho}\ f'(t)\ dt\right] \quad (5)$$

Substituting the above two equations (4) and (5) in (3), we get

$$\begin{aligned} M[y\left(t\right)]= & \ \aleph_0\sum_{\mathfrak{K}=0}^{\infty}\left(-b\right)^{\mathfrak{K}}\sum_{\wp=0}^{\infty}\begin{pmatrix}\mathfrak{K}+\wp\\ \wp\end{pmatrix}\left(-a\right)^{\wp}\ s^{(\varpi-2)\wp-2\mathfrak{K}-1}\\ & +\aleph_1\sum_{\mathfrak{K}=0}^{\infty}\left(-b\right)^{\mathfrak{K}}\sum_{\wp=0}^{\infty}\begin{pmatrix}\mathfrak{K}+\wp\\ \wp\end{pmatrix}\left(-a\right)^{\wp}\ s^{(\varpi-2)\wp-2\mathfrak{K}-2}\\ & +a\aleph_0\sum_{\mathfrak{K}=0}^{\infty}\left(-b\right)^{\mathfrak{K}}\sum_{\wp=0}^{\infty}\begin{pmatrix}\mathfrak{K}+\wp\\ \wp\end{pmatrix}\left(-a\right)^{\wp}\ s^{(\varpi-2)\wp-2\mathfrak{K}+\varpi-3}\\ & +a\aleph_1\sum_{\mathfrak{K}=0}^{\infty}\left(-b\right)^{\mathfrak{K}}\sum_{\wp=0}^{\infty}\begin{pmatrix}\mathfrak{K}+\wp\\ \wp\end{pmatrix}\left(-a\right)^{\wp}\ s^{(\varpi-2)\wp-2\mathfrak{K}+\varpi-4}\\ & +\frac{\sin\pi\ \varrho}{\pi}p\ .M\left[\int_0^s\frac{g\left(t\right)}{\left(s-t\right)^{\varrho}}\ dt\right]\sum_{\mathfrak{K}=0}^{\infty}\left(-b\right)^{\mathfrak{K}}\sum_{\wp=0}^{\infty}\begin{pmatrix}\mathfrak{K}+\wp\\ \wp\end{pmatrix}\left(-a\right)^{\wp}\ s^{(\varpi-2)\wp-2\mathfrak{K}-2}. \end{aligned} \quad (6)$$

Thus, providing inverse Mohand transform on both sides in equation (6), we get

$$y\left(t\right)=\ \aleph_0\sum_{\mathfrak{K}=0}^{\infty}\frac{\left(-b\right)^{\mathfrak{K}}\ t^{2\mathfrak{K}}}{\mathfrak{K}!}\sum_{\wp=0}^{\infty}\frac{\Gamma\left(\mathfrak{K}+\wp+1\right)\ \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma\left[\left(2-\varpi\right)\wp+2\mathfrak{K}+1\right]\ \wp!}$$

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$$+ \aleph_1 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} \, t^{2\mathfrak{K}+1}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}+2] \, \wp!}$$

$$+ a\aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} \, t^{2\mathfrak{K}-\varpi+2}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}-\varpi+3] \, \wp!}$$

$$+ a\aleph_1 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} \, t^{2\mathfrak{K}-\varpi+3}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}-\varpi+4] \, \wp!}$$

$$+ \frac{\sin \varrho\pi}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\varrho-1} f(t) \, dt \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} \, t^{2\mathfrak{K}+1}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{(2-\varpi)}\right)^{\wp}}{\Gamma[(2-\varpi)\wp+2\mathfrak{K}+2] \, \wp!} .$$

Example 3.1 The fractional integro-differential equation is

$$y''(t) + \sqrt{6} \, y^{\left(\frac{3}{2}\right)}(t) + 11y(t) = \int_0^s \frac{g(t)}{(s-t)^{\left(\frac{1}{2}\right)}} \, dt$$

With $y(0) = 1$ and $y'(0) = 1$ its proposal is provided by

$$y(t) = \sum_{\mathfrak{K}=0}^{\infty} \frac{(-11)^{\mathfrak{K}} \, t^{2\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-\sqrt{6} \, t^{\left(\frac{1}{2}\right)}\right)^{\wp}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+2\mathfrak{K}+1\right] \, \wp!}$$

$$+ \sum_{\mathfrak{K}=0}^{\infty} \frac{(-11)^{\mathfrak{K}} \, t^{2\mathfrak{K}+1}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-\sqrt{6}t^{\left(\frac{1}{2}\right)}\right)^{\wp}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+2\mathfrak{K}+2\right] \, \wp!}$$

$$+ \sqrt{6} \sum_{\mathfrak{K}=0}^{\infty} \frac{(-11)^{\mathfrak{K}} \, t^{2\mathfrak{K}+\frac{1}{2}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-\sqrt{6}t^{\left(\frac{1}{2}\right)}\right)^{\wp}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+2\mathfrak{K}+\frac{3}{2}\right] \, \wp!}$$

$$+ \sqrt{6} \sum_{\mathfrak{K}=0}^{\infty} \frac{(-11)^{\mathfrak{K}} \, t^{2\mathfrak{K}+\frac{3}{2}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-\sqrt{6}t^{\left(\frac{1}{2}\right)}\right)^{\wp}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+2\mathfrak{K}+\frac{5}{2}\right] \, \wp!}$$

$$+ \frac{1}{\pi} \frac{d}{ds} \int_0^s (s-t)^{-\frac{1}{2}} f(t) \, dt \sum_{\mathfrak{K}=0}^{\infty} \frac{(-11)^{\mathfrak{K}} \, t^{2\mathfrak{K}+1}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1) \, \left(-at^{\left(\frac{1}{2}\right)}\right)^{\wp}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+2\mathfrak{K}+2\right] \, \wp!} ,$$

Figure 1 illustrates the solution behavior of the fractional integro-differential equation of example 1 at various values of ϖ.

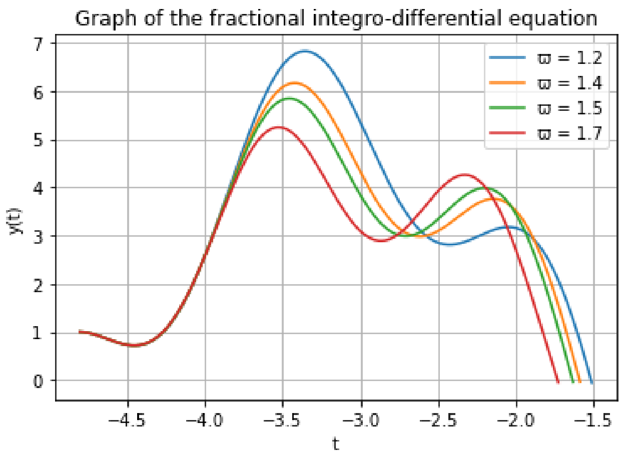

Figure 1: The solution behavior of Example 1.

Theorem 3.2 Let $1 < \varpi < 2$ and a and $b \in \mathbb{R}$. Then the fractional integro-differential equation is

$$y^{\varpi}(t) + a\, y'(t) + by(t) = \int_0^s \frac{g(t)}{(s-t)^{\varrho}}\, dt \quad ; \quad 0 < \varrho < 1 \tag{7}$$

with $y(0) = \aleph_0$ and $y'(0) = \aleph_1$ its proposal is provided by

$$\begin{aligned} y(t) = \; & \aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+1]\ \ \wp!} \\ & + \aleph_1 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}+1}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+2]\ \ \wp!} \\ & + a\aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi-1}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi]\ \ \wp!} \\ & + \frac{\sin\varrho\pi}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\varrho-1} f(t)\, dt \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi-1}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi]\ \ \wp!}. \end{aligned} \tag{8}$$

Proof: Utilizing the Mohand transform in (7) and taking into consideration, we have

$$s^{\varpi} F(s) - s^{\varpi+1} f(0) - s^{\varpi} f'(0) + a\left[s\ F(s) - s^2 f(0)\right] + b\ F(s) = M[f(t)]$$

where $f(t) = \int_0^s \frac{g(t)}{(s-t)^{\varrho}}\, dt$,

$$s^{\varpi} M[y(t)] - s^{\varpi+1} y(0) - s^{\varpi} y'(0) + a\ s\ M[y(t)] - as^2 y(0) + b\ M[y(t)] = M[f(t)]$$

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$$s^{\varpi} M[y(t)] - s^{\varpi+1}\aleph_0 - s^{\varpi}\aleph_1 + a\ s\ M[y(t)] - as^2\aleph_0 + b\ M[y(t)] = M[f(t)]$$

$$M[y(t)] = \frac{s^{\varpi+1}\aleph_0 + s^{\varpi}\aleph_1 + as^2\aleph_0 + M[f(t)]}{(s^{\varpi} + a\ s + b)}. \tag{9}$$

Since

$$\begin{aligned}
\frac{1}{(s^{\varpi} + a\ s + b)} &= \frac{s^{-1}}{s^{\varpi-1} + a + bs^{-1}} \\
&= \frac{s^{-1}}{(s^{\varpi-1} + a)\left(1 + \frac{b\ s^{-1}}{s^{\varpi-1}+a}\right)} \\
&= \frac{s^{-1}}{s^{\varpi-1} + a} \sum_{\mathfrak{K}=0}^{\infty} \left(\frac{-bs^{-1}}{s^{\varpi-1} + a}\right)^{\mathfrak{K}} \\
&= \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} s^{-\mathfrak{K}-1}}{(s^{\varpi-1} + a)^{\mathfrak{K}+1}} \\
&= \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} s^{-\varpi\mathfrak{K}-\varpi}}{\left(1 + a\ s^{1-\varpi}\right)^{\mathfrak{K}+1}} \\
&= \sum_{\mathfrak{K}=0}^{\infty} (-b)^{\mathfrak{K}} s^{-\varpi\mathfrak{K}-\varpi} \sum_{\wp=0}^{\infty} \left(-as^{1-\varpi}\right)^{\wp} \begin{pmatrix} \mathfrak{K}+\wp \\ \wp \end{pmatrix} \\
&= \sum_{\mathfrak{K}=0}^{\infty} (-b)^{\mathfrak{K}} \sum_{\wp=0}^{\infty} \begin{pmatrix} \mathfrak{K}+\wp \\ \wp \end{pmatrix} (-a)^{\wp}\ s^{(1-\varpi)\wp-\varpi\mathfrak{K}-\varpi}
\end{aligned} \tag{10}$$

and we know that,

$$M[f(t)] = M\left[\int_0^s \frac{g(t)}{(s-t)^{\varrho}}\ dt\right].$$

This gives that,

$$G(P) = \frac{\sin\pi\ \varrho}{\pi} p\ .M\left[\int_0^s (s-t)^{\varrho}\ f'(t)\ dt\right]. \tag{11}$$

Substituting the above two equations (10) and (11) in (9), we get

$$\begin{aligned}
y(t) = {} & \aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+1]\ \wp!} \\
& + \aleph_1 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}+1}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+2]\ \wp!} \\
& + a\aleph_0 \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!} \sum_{\wp=0}^{\infty} \frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi-1}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi]\ \wp!}
\end{aligned}$$

$$+\frac{\sin\varrho\pi}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\varrho-1} f(t)\,dt \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{\mathfrak{K}}}{\mathfrak{K}!}\sum_{\wp=0}^{\infty}\frac{\Gamma(\mathfrak{K}+\wp+1)\ (-a)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi-1}}{\Gamma[(\varpi-1)\wp+\varpi\mathfrak{K}+\varpi]\ \wp!}.$$

The Wright function can express this solution as

$$\begin{aligned} y(t) = {} & \aleph_0 \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{\mathfrak{K}}\ t^{\varpi\mathfrak{K}}}{\mathfrak{K}!} 1\lambda 1\left(\begin{matrix}(\mathfrak{K}+1,\ 1\\ (\varpi\mathfrak{K}+1,\ \varpi-1)\end{matrix}\ |-a\ t^{\varpi-1}\right) \\ & + \aleph_1 \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{\mathfrak{K}} t^{\varpi\mathfrak{K}+1}}{\mathfrak{K}!} 1\lambda 1\left(\begin{matrix}(\mathfrak{K}+1,\ 1\\ (\varpi\mathfrak{K}+2,\ \varpi-1)\end{matrix}\ |-a\ t^{\varpi-1}\right) \\ & + a\aleph_0 \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{\mathfrak{K}} t^{\varpi\mathfrak{K}+\varpi-1}}{\mathfrak{K}!} 1\lambda 1\left(\begin{matrix}(\mathfrak{K}+1,\ 1\\ (\varpi\mathfrak{K}+\varpi,\ \varpi-1)\end{matrix}\ |-a\ t^{\varpi-1}\right) \\ & +\frac{\sin\varrho\pi}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\varrho-1} f(t)\,dt \sum_{\mathfrak{K}=0}^{\infty}\frac{(-b)^{\mathfrak{K}} t^{\varpi\mathfrak{K}+\varpi-1}}{\mathfrak{K}!}\ 1\lambda 1\left(\begin{matrix}(\mathfrak{K}+1,\ 1\\ (\varpi\mathfrak{K}+\varpi,\ \varpi-1)\end{matrix}\ |-a\ t^{\varpi-1}\right). \end{aligned} \tag{12}$$

Example 3.2 The fractional integro-differential equation is

$$y^{\frac{3}{2}}(t) - 4y'(t) - 5\ y(t) = \int_0^s \frac{g(t)}{(s-t)^{\frac{1}{2}}}\ dt$$

with $y(0) = 1$ and $y'(0) = 1$ its proposal is provided by

$$\begin{aligned} y(t) = {} & \sum_{\mathfrak{K}=0}^{\infty}\frac{(5)^{\mathfrak{K}}}{\mathfrak{K}!}\sum_{\wp=0}^{\infty}\frac{\Gamma(\mathfrak{K}+\wp+1)\ (4)^{\wp} t^{(\varpi-1)\wp+\varpi\mathfrak{K}}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+1\right]\ \wp!} \\ & + \sum_{\mathfrak{K}=0}^{\infty}\frac{(5)^{\mathfrak{K}}}{\mathfrak{K}!}\sum_{\wp=0}^{\infty}\frac{\Gamma(\mathfrak{K}+\wp+1)\ (4)^{\wp} t^{\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+1}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+2\right]\ \wp!} \\ & -4\sum_{\mathfrak{K}=0}^{\infty}\frac{(5)^{\mathfrak{K}}}{\mathfrak{K}!}\sum_{\wp=0}^{\infty}\frac{\Gamma(\mathfrak{K}+\wp+1)\ (4)^{\wp} t^{\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+\frac{1}{2}}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+\frac{3}{2}\right]\ \wp!} \\ & +\frac{1}{\pi}\frac{d}{ds}\int_0^s (s-t)^{-\frac{1}{2}} f(t)\,dt \sum_{\mathfrak{K}=0}^{\infty}\frac{(5)^{\mathfrak{K}}}{\mathfrak{K}!}\sum_{\wp=0}^{\infty}\frac{\Gamma(\mathfrak{K}+\wp+1)\ (4)^{\wp} t^{\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+\frac{1}{2}}}{\Gamma\left[\left(\frac{1}{2}\right)\wp+\frac{3}{2}\mathfrak{K}+\frac{3}{2}\right]\ \wp!} \end{aligned}$$

Figure 2 illustrates the solution behavior of the fractional integro-differential equation of example 1 at various values of ϖ.

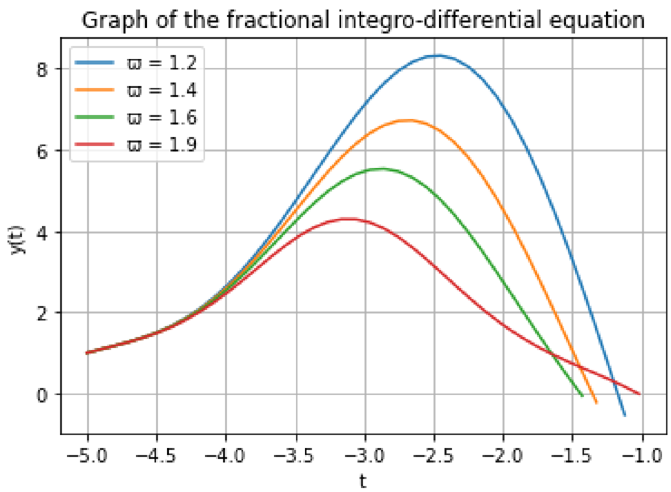

Figure 2: The solution behavior of Example 2.

Proposition 3.1 Let $1 < \varpi, \varrho < 2$ and $b \in \mathbb{R}$. Then the fractional integro-differential equation is

$$y^{\varpi}(t) - by(t) = \int_0^s \frac{g(t)}{(s-t)^{\varrho}}\, dt \quad ; \quad 0 < \varrho < 1 \tag{13}$$

With $y(0) = \aleph_0$ its proposal is provided by

$$y(t) = \aleph_0 \sum_{\mathfrak{K}=0}^{\infty} b^{\mathfrak{K}} \frac{t^{\varpi\mathfrak{K}}}{\Gamma(\varpi\mathfrak{K}+1)} + \frac{\sin\varrho\pi}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\varrho-1} f(t)\, dt \sum_{\mathfrak{K}=0}^{\infty} \frac{(-b)^{\mathfrak{K}} t^{\varpi+\varpi\mathfrak{K}-1}}{\Gamma(\varpi+\varpi\mathfrak{K})}$$

$$= \aleph_0 E_{\alpha}(bt^{\varpi}) + \frac{\sin\varrho\pi}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\varrho-1} f(t)\, dt\; t^{\varpi-1} E_{\varpi,\varpi}(bt^{\varpi}). \tag{14}$$

Proof: The proof of this proposition as like as previous theorem.

Remark 3.1 Accordingly, $a = 0$ in (7), then the derivative is

$$y^{\varpi}(t) + by(t) = \int_0^s \frac{g(t)}{(s-t)^{\varrho}}\, dt \;\; ; \; 1 < \varpi \le 2, \; 0 < \varrho < 1 \tag{15}$$

With $y(0) = \aleph_0$ and $y'(0) = \aleph_1$ its proposal is provided by

$$y(t) = \aleph_0 E_{\varpi,1}(-bt^{\varpi}) + \aleph_1 E_{\varpi,2}(-bt^{\varpi}) + \frac{\sin\varrho\pi}{\pi} \frac{d}{ds} \int_0^s (s-t)^{\varrho-1} f(t)\, dt\; t^{\varpi-1} E_{\varpi,\varpi}(-bt^{\varpi}). \tag{16}$$

Proposition 3.2 A nearly simple harmonic vibration integro-differential equation

$$y^{\varpi}(t) + z^2 y(t) = \int_0^s \frac{g(t)}{(s-t)^{\varrho}}\, dt \;\; ; \; 1 < \varpi \le 2, \; 0 < \varrho < 1 \tag{17}$$

With $y(0) = \aleph_0$ and $y'(0) = \aleph_1$ its proposal is provided by

$$y(t) = \aleph_0 E_{\varpi,1}\left(-z^2 t^{\varpi}\right) + \aleph_1 E_{\varpi,2}\left(-z^2 t^{\varpi}\right) + \frac{\sin\varrho\pi}{\pi}\frac{d}{ds}\int_0^s (s-t)^{\varrho-1} f(t)\, dt\, t^{\varpi-1} E_{\varpi,\varpi}\left(-z^2 t^{\varpi}\right).$$

Proof: The above proof is accomplished by implanting $b = z^2$ in equation (16).

4 Conclusion

The utilization of the Mohand transform to solve fractional integro-differential equations stands as a pivotal focus of this article. Exploring the intricate relationship between the Mohand transform and the Laplace transform has yielded invaluable insights, enriching our comprehension of these integral transformations. This study uses a unique method that combines the Mohand transform with binomial series extension coefficients to come up with a new way to solve fractional integro-differential equations. Beyond its mere application, this research delves into elucidating various properties and providing illustrative examples, substantiating the efficacy and adaptability of the proposed methodology. Looking ahead, future research endeavors aim to refine the Mohand transform's applicability by addressing its limitations in specific scenarios. Also, looking into how it can be used in different scientific fields and combining different types of methods are both good ways to improve how differential equations are solved. In conclusion, this study not only introduces a novel approach but also sets the stage for broader investigations, seeking to expand the practical utility and deepen the understanding of the Mohand transform in diverse mathematical problem-solving domains.

References

[1] M. M. Abdelrahim, The New Integral Transform Mohand Transform, *Adv Theoret Appl Mathe*, 12, 2, 113-120 (2017).

[2] K. S. Aboodh, The New Integral transform Aboodh transform, *Global Journal of Pure and Applied Mathematics*, 9, 1, 35 - 43 (2013).

[3] R. Aruldoss, and R. Anusuya Devi, Aboodh Transform for Solving Fractional Differential Equations, *Global Journal of Pure and Applied Mathematics*, 16, 2, 145-153 (2020).

[4] M. Caputo, Elasticity dissipazione (Elasticity and anelastic dissipation), *Zanichelli, Bologna,* 4, 98(1969).

[5] A.E. Elsayed Mohamed, Elzaki transformation for Linear Fractional Differential Equations, *Journal of Computational and Theoretical Nanoscience*, 12, 2303 - 2315 (2015).

[6] A. Kashuri, A. Fundo, R. Liko, New Integral transform for solving some fractional differential equations, *International Journal of Pure and Applied Mathematics*, 103, 4, 675-682, (2015).

[7] S. Lin, C. Lu, Laplace transform for solving some families of fractional differential equations and its applications, *Advances in Difference Equations*, 1-9 (2013).

[8] I. Podlubny, Fractional Differential Equation, *Academic Press, San Diego*, (1999).

[9] F. S. Silva, D. M. Moreira, and M. A. Moret, Conformable Laplace transform of fractional differential equations, *Axioms*, 7, 3, 55 (2018).

[10] J. A. Zhang, Sumudu based algorithm for solving differential equations, *Computer Science Journal of Moldova*, 15, 3, 45-53 (2007).

[11] P. Raghavendran, T. Gunasekar, H. Balasundaram, S. S. Santra, D. Majumder, and D. Baleanu, Solving fractional integro-differential equations by Aboodh transform. *J. Math. Computer Sci*, 229-240, 32 (2023).

[12] V.P. Dubey, J. Singh, A.M. Alshehri, S. Dubey, and D. Kumar, Forecasting the behavior of fractional order Bloch equations appearing in NMR flow via a hybrid computational technique. *Chaos, Solitons & Fractals*, 164, 112691, (2022).

[13] V.P. Dubey, J. Singh, A.M. Alshehri, S. Dubey, and D. Kumar, Some integral transform results for Hilfer–Prabhakar fractional derivative and analysis of free-electron laser equation. *Iranian Journal of Science*, 1333-1342, 47, 4 (2023).

[14] V.P. Dubey, J. Singh, A.M. Alshehri, S. Dubey, and D. Kumar, Analysis of Cauchy Problems and Diffusion Equations Associated with the Hilfer–Prabhakar Fractional Derivative via Kharrat–Toma Transform. *Fractal and Fractional*, 413, 7, 5 (2023).

[15] V.P. Dubey, D. Kumar, A.M. Alshehri, J. Singh, and D. Baleanu, Generalized invexity and duality in multiobjective variational problems involving non-singular fractional derivative. *Open Physics*, 939-962, 20, 1 (2022).

[16] V.P. Dubey, J. Singh, A.M. Alshehri, S. Dubey, and D. Kumar, Analysis and Fractal Dynamics of Local Fractional Partial Differential Equations Occurring in Physical Sciences. *Journal of Computational and Nonlinear Dynamics*, 031001, 18, 3 (2023).

[17] H. Singh, J. Singh, S. D. Purohit, & D. Kumar, (Eds.). Advanced Numerical Methods for Differential Equations: Applications in Science and Engineering (1st ed.), (2021) CRC Press. https://doi.org/10.1201/9781003097938.

[18] D. Kumar, V.P. Dubey, S. Dubey, J. Singh, and A.M. Alshehri, Computational analysis of local fractional partial differential equations in realm of fractal calculus. *Chaos, Solitons & Fractals*, 113009, 167 (2023).