Equivalence Relation with respect to Linear Maps

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Abstract

Semi vector spaces are algebraic structures analogous to vector spaces with the basefields replaced by semifields. Corresponding to each linear map between two semi vector spaces, we may associate an equivalence relation. The equivalence classes corresponding to this equivalence relation have close connection with the nature of the associated linear map.

In this paper we discuss the basic properties of these equivalence classes.

1.Introduction

A non-empty set F with two binary operations $+$ and $\cdot$ defined on it is called a **semifield** if the following conditions are satisfied:

$(F, +)$ is a commutative semigroup

$(F - \{0\}, \cdot)$ is a commutative group, where 0 is the identity element with respect to +, if it exists.

A **semi vector space** over a semifield F is defined to be a non-empty set X equipped with the operations $+ : X \times X \rightarrow X$, called addition, and $\cdot : F \times X \rightarrow X$, called scalar multiplication, satisfying the following conditions:

For each $\alpha, \beta \in F$, $x, y, z \in X$,

$$x + (y + z) = (x + y) + z; \quad x + y = y + x$$
$$(\alpha\beta)\, x = \alpha\, (\beta x)$$
$$1x = x, \text{ where 1 is the multiplicative identity of } F, \text{ if exists}$$
$$\alpha\,(x + y) = \alpha x + \alpha y; \quad (\alpha + \beta)x = \alpha x + \beta x.$$

We shall write $\alpha\, x$ instead of $\alpha \cdot x$, for $x \in X$ and $\alpha \in F$.

Let X,Y be two vector spaces over the same field F. A map $T: X \to Y$ is a linear map if the following two conditions are satisfied:

$$T(x + y) = T(x) + T(y) \text{ for any } x, y \in X,$$
$$T(\alpha x) = \alpha T(x) \text{ for any } x \in X \text{ and } \alpha \in F.$$

An equivalence relation on a set X is a relation that is reflexive, symmetric and transitive, which partitions the elements into equivalence classes, where elements within the same class are considered 'equivalent' under the relation.

2. Linear maps and Equivalence Relation

2.1 Definition

Let $T: X \to Y$ be a linear map, where X and Y are semi vector spaces over $\mathbb{R}_+$. Define the relation ρ_T on X by

$$x\, \rho_T\, z \text{ if } T(x) = T(z), \; x, z \in X. \tag{1}$$

2.2 Proposition

ρ_T is an equivalence relation.

Proof:

Since $T(x) = T(x)$ for all $x \in X$, $x\, \rho_T\, x$ for all $x \in X$ and ρ_T is reflexive.

Let $x, z \in X$. Suppose $x\, \rho_T\, z$.

Then $T(x_1) = T(x_2)$. So, $T(x_2) = T(x_1)$.

Hence $x_2\, \rho_T\, x_1$ and ρ_T is symmetric.

Let $x, y, z \in X$. Let $x\, \rho_T\, y$ and $y\, \rho_T$ z.

Then $T(x) = T(y)$ and $T(y) = T(z)$. Hence , $T(x) = T(z)$.

That is, $x\, \rho_T\, z$. Thus ρ_T is transitive.

Hence ρ_T is an equivalence relation.

□

Note

Since ρ_T is an equivalence relation, ρ_T partitions X into equivalence classes $[a]_T$ given by
$[a]_T = \{x \in X / x\ \rho_T\ a\} = \{x \in X\ /\ T(x) = T(a)\}, a \in X.$ (2)

2.3 Remark

$[0]_T = \{x \in X/\ T(x) = T(0) = 0\} = N(T)$,the null space of T.

$[a]_0 = \{x \in X/0(x) = 0(a)\} = X$,where 0 is the zero map.

$[a]_I = \{x \in X/I(x) = I(a)\} = \{x \in X/x = a\} = \{a\}$,where I is the identity map.

Notation

Let us denote the set of all distinct equivalence classes $[a]_T$ by $[X]_T$.
That is, $[X]_T = \{[a]_T/a \in X\}$.
For any set A let us denote the cardinality of A by $|A|$.

2.4 Definition

A semi vector space X is regular if $x + z = y + z \Rightarrow x = y$ for all $x, y, z \in X$.

2.5 Theorem

Let X and Y be regular semi vector spaces with 0 over $\mathbb{R}_+$. Let $T: X \to Y$ be a linear map.

Then, $|[0]_T| \leq |[a]_T|$ for all $a \in X$.

Proof

Let $a \in X$.
$[a]_T = \{x \in X/T(x) = T(a)\}$ and $[0]_T = \{x \in X/T(x) = 0\}$.

Claim : $x \in [0]_T$ *if and only if* $x + a \in [a]_T$. (3)

$x \in [0]_T$ if and only if $x\ \rho_T\ 0$

if and only if $T(x) = T(0) = 0$

if and only if $T(x) + T(a) = T(a)$, since Y is regular

if and only if $T(x + a) = T(a)$

if and only if $(x + a)\ \rho_T\ a$

if and only if $x + a \in [a]_T$.

Define $F: [0]_T \to [a]_T$ by

$$F(x) = x + a. \quad (4)$$

Then F is well defined because of (3).

Claim : F is 1-1.

Suppose $F(x_1) = F(x_2)$, where $x_1, x_2 \in [0]_T$.

That is, $x_1 + a = x_2 + a$. Then, $x_1 = x_2$, since X is regular.

So, F is 1-1.

Thus there is an 1-1 map $F: [0]_T \to [a]_T$ and hence $|[0]_T| \leq |[a]_T|$.

□

2.6 Proposition

T is 1-1 if and only if $[a]_T$ is a singleton set for all $a \in X$.

Proof:

Assume that T is 1-1.
Let $a \in X$. Then $a \in [a]_T$.
Suppose $b \in [a]_T$. Then $b \; \rho_T \; a$.
So, $T(b) = T(a)$. Since, T is 1-1, $b = a$.
Thus T contains no element other than a.
Now assume that $[a]$ is a singleton set for all $a \in X$.
To prove that T is 1-1 ,suppose $T(a) = T(b)$, where $a, b \in X$.
Then $a \; \rho_T \; b$. So $b \in [a]_T$. Also $a \in [a]_T$. But $[a]_T$ is a singleton set. Hence $b = a$.
Thus, $T(a) = T(b)$ implies a$=$ b and T is 1-1.

□

2.7 Definition

A subset S of X is said to be convex if for all $a, b \in S$ and $0 \leq r \leq 1$, $ra + (1 - r)b \in S$.

2.8 Proposition

$[a]_T$ is a convex set for all $a \in X$.

Proof:

Let $x_1, x_2 \in [a]_T$ and $0 \leq r \leq 1$.
Then $x_1 \; \rho_T \; a$ and $x_2 \; \rho_T a$. So $T(x_1) = T(a) \; and \; T(x_2) = T(a)$.
Now $T(rx_1 + (1 - r)\, x_2) = r\, T(x_1) + (1 - r)\, T\,(x_2)$
$= r\, T(a) + (1 - r)\, T\,(a) \; = T(a)$.
So, $(rx_1 + (1 - r)x_2) \; \rho_T \; a$ and so $rx_1 + (1 - r)x_2 \in [a]_T$.

□

2.9 Theorem

Let $T: X \to Y$ be a linear map, where X and Y are regular seemi vector spaces. Let $x, z \in X$. Then for any $a \in X$,

(i) $x\ \rho_T\ z\ if\ and\ only\ if\ x + a\ \rho_T\ z + a$

(ii) $x\ \rho_T\ z\ if\ and\ only\ if\ \alpha x\ \rho_T\ \alpha z$, where $\alpha \neq 0$.

The proof is direct. □

2.10 Proposition

Let $a, b \in X$. Then, $[a + b]_T \supset [a]_T + [b]_T$. (5)

If T is $1 - 1$, $[a + b]_T = [a]_T + [b]_T$. (6)

Proof:

Let $x_1 + x_2 \in [a]_T + [b]_T$ so that $x_1 \in [a]_T\ and\ x_2 \in [b]_T$.

Then $x_1\ \rho_T\ a$ and $x_2\ \rho_T\ b$.

That is, $T(x_1) = T(a)\ and\ T(x_2) = T(b)$. (7)

So, $T(x_1 + x_2) = T(x_1) + T(x_2)$

$= T(a) + T(b) = T(a + b)$.

Thus, $x_1 + x_2 \in [a + b]_T$.

Hence $[a]_T + [b]_T \subset [a + b]_T$. (8)

Now assume that T is 1-1.

Let $x \in [a + b]_T$. Then, $x\ \rho_T\ (a + b)$.

Hence, $T(x) = T(a + b)$.

But T is 1-1.

So, $x = a + b \in [a]_T + [b]_T$, since $a \in [a]_T$ and $b \in [b]_T$

Thus, $[a + b]_T \subset [a]_T + [b]_T$. (9)

From (8) and (9) we get,

$[a + b]_T = [a]_T + [b]_T$ if T is 1-1.

□

2.11 Definition

Let X be a semi vector space over $\mathbb{R}_+$. For an element $a \in X$, the semi subspace generated by a is the set $\{ra/r \in \mathbb{R}_+\}$ and is denoted by $\langle a]$.That is

$\langle a] = \{ra/r \in \mathbb{R}_+\}$ (10)

Note

$\langle a]$ is a semi vector space contained in X.

2.12 Proposition

(i) For $a \in X$ and $0 \neq \alpha \in \mathbb{R}_+$, $\langle \alpha a] = \langle a]$

(ii) For $a, b \in X$, $\langle a + b] \subseteq \langle a] + \langle b]$.

Proof:

Let $x \in \langle \alpha a]$. Then, $x = r\,(\alpha a)$ for some $r \in \mathbb{R}_+$

$= (r\alpha)\, a \in \langle a]$, since $r\alpha \in \mathbb{R}_+$.

Thus, $\langle \alpha a] \subset \langle a]$. (11)

Now let $x \in \langle a]$. Then, $x = ra$ for some $r \in \mathbb{R}_+$.

Consider, $x = ra = (\alpha\alpha^{-1})\,(ra)$, since $\alpha \neq 0$

$= \alpha\,(\alpha^{-1}\,ra) = (\alpha^{-1}\,r)\,(\alpha a) \in \langle \alpha a]$.

So $\langle a] \subset \langle \alpha a]$. (12)

From (11) and (12), $\langle a] = \langle \alpha a]$. □

2.13 Definition

Let $T : X \to Y$ be a linear map, where X and Y are semi vector spaces over $\mathbb{R}_+$.

For $a \in X$, define

$$\langle a]_T = \{x \in X/T(x) \in \langle T(a)]\}$$

$$= \{x \in X/T(x) = r\,T(a) \text{ for some } r \in \mathbb{R}_+\}.$$

2.14 Remark

(a) $\langle\, a\,]_0 = X$, where 0 is the zero map, $a \in X$

(b) $\langle\, a\,]_{\mathrm{I}} = \langle\, a\,]$ for $Y = X$, where I is the identity map

(c) $\langle\, 0\,]_T = N\,(T)$, the null space of T.

2.15 Theorem

$\langle\, a\,]_{\mathrm{T}}$ is a semi vector space over $\mathbb{R}_+$.

Proof:

Let $x_1, x_2 \in \langle\, a\,]_{\mathrm{T}}$.

Then $T(x_1) = r\,T(a)$ and $T(x_2) = sT(a)$ for some $r, s \in \mathbb{R}_+$.

Now, $T(x_1 + x_2) = T(x_1) + (x_2) = r\,T(a) + s\,T(a) = (r + s)\,T(a)$.

Hence $x_1 + x_2 \in \langle\, a\,]_{\mathrm{T}}$.

Now, let $x \in \langle\, a\,]_{\mathrm{T}}$ and $\alpha \in \mathbb{R}_+$.

Then, $T(x) = kT(a)$ for some $k \in \mathbb{R}_+$.

So, $T(\alpha x) = \alpha\,T(x) = \alpha\,kT(a)$. Hence $\alpha\,x \in \langle\, a\,]_{\mathrm{T}}$.

Thus $\langle\, a\,]_{\mathrm{T}}$ is a sub semi vector space of X and hence a semi vector space.

□

References

[1] I.N. Herstein- Topics in Algebra. Wiley Eastern Limited,(1975).

[2] Kaplemsky, Irving – Fields and Rings. University of Chicago , (1972).

[3] Walter Rudin - Functional Analysis. Tata McGraw Hill (New Delhi), (2011).

[4] K. Chandrasekara Rao - Functional Analysis. Norosa Publishing House, New Delhi (2009).

[5] E. Kreyszig- Introductory Functional Analysis with Applications. John Wiley and Sons, New York, (1978).

[6] B.V. Limaye - Functional Analysis. New Age International Publishes, New Delhi, (1996).

[7] Vasantha Kandasamy - W.B., Smarandache Semirings, Semifields and Semivector spaces. American Research Press, Rehoboth, (2002).

[8] Josef Janyˇska , Marco Modugno , Raffaele Vitolo – Semi Vector Spaces and Units of Measurement. (2007).