M-fractional integral inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present M-fractional integral inequalities of Ostrowski and Polya types.

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1 Introduction

We are inspired by the following results:

Theorem 1 *([2], p. 498, [1], [5]) (Ostrowski inequality)*
Let $f \in C^1([a,b])$, $x \in [a,b]$. Then

$$\left| \frac{1}{b-a} \int_a^b f(z)\,dz - f(x) \right| \leq \left(\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right) \|f'\|_\infty . \tag{1}$$

Inequality (1) is sharp. In particular the optimal function is

$$f^*(z) := |z-x|^\alpha (b-a), \quad \alpha > 1. \tag{2}$$

Theorem 2 *([6], [7, p. 62], [8], [9, p. 83]) (Polya integral inequality)*
Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a,b]$ such that

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\,dx. \tag{3}$$

In this short work we present inequalities of types (1) and (3) involving the left and right fractional local general M-derivatives, see [3], [4].

2 Background

We need

Definition 3 *([4]) Let $f : [a, \infty) \to \mathbb{R}$ and $t > a$, $a \in \mathbb{R}$. For $0 < \alpha \leq 1$ we define the left local general M-derivative of order α of function f, denoted by $D_{M,a}^{\alpha,\beta} f(t)$, by*

$$D_{M,a}^{\alpha,\beta} f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t \mathbb{E}_{\beta}\left(\varepsilon (t-a)^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{4}$$

$\forall\ t > a$, where $\mathbb{E}_{\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + 1)}$, $\beta > 0$, is the Mittag-Leffler function with one parameter.

If $D_{M,a}^{\alpha,\beta} f(t)$ exists over (a, γ), $\gamma \in \mathbb{R}$ and $\lim_{t \to a+} D_{M,a}^{\alpha,\beta} f(t)$ exists, then

$$D_{M,a}^{\alpha,\beta} f(a) = \lim_{t \to a+} D_{M,a}^{\alpha,\beta} f(t). \tag{5}$$

Theorem 4 *([4]) If a function $f : [a, \infty) \to \mathbb{R}$ has the left local general M-derivative of order $\alpha \in (0, 1]$, $\beta > 0$, at $t_0 > a$, then f is continuous at t_0.*

We need

Theorem 5 *([4]) (Mean value theorem) Let $f : [\gamma, \delta] \to \mathbb{R}$ with $\gamma > a$, $0 \notin [\gamma, \delta]$, such that*

(1) f is continuous on $[\gamma, \delta]$,

(2) there exists $D_{M,a}^{\alpha,\beta} f$ on (γ, δ) for some $\alpha \in (0, 1]$.

Then, there exists $c \in (\gamma, \delta)$ such that

$$f(\delta) - f(\gamma) = \left(D_{M,a}^{\alpha,\beta} f(c)\right) \frac{\Gamma(\beta + 1)(c-a)^{\alpha}}{c} (\delta - \gamma). \tag{6}$$

We need

Definition 6 *([3]) Let $f : (-\infty, b] \to \mathbb{R}$ and $t < b$, $b \in \mathbb{R}$. For $0 < \alpha \leq 1$ we define the right local general M-derivative of order α of function f, denoted as ${}_{M,b}^{\alpha,\beta}D f(t)$, by*

$${}_{M,b}^{\alpha,\beta}D f(t) := -\lim_{\varepsilon \to 0} \frac{f\left(t \mathbb{E}_{\beta}\left(\varepsilon (b-t)^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{7}$$

$\forall\ t < b$.

If ${}_{M,b}^{\alpha,\beta}D f(t)$ exists over (γ, b), $\gamma \in \mathbb{R}$ and $\lim_{t \to b-} {}_{M,b}^{\alpha,\beta}D f(t)$ exists, then

$${}_{M,b}^{\alpha,\beta}D f(b) = \lim_{t \to b-} {}_{M,b}^{\alpha,\beta}D f(t). \tag{8}$$

Theorem 7 *([3]) If a function $f : (-\infty, b] \to \mathbb{R}$ has the right local general M-derivative of order $\alpha \in (0,1]$, $\beta > 0$, at $t_0 < b$, then f is continuous at t_0.*

We also need

Theorem 8 *([3]) (Mean value theorem) Let $f : [\gamma, \delta] \to \mathbb{R}$ with $\delta < b$, $0 \notin [\gamma, \delta]$, such that*

(1) f is continuous on $[\gamma, \delta]$,

(2) there exists ${}_{M,b}^{\alpha,\beta}Df$ on (γ, δ) for some $\alpha \in (0,1]$.

Then, there exists $c \in (\gamma, \delta)$ such that

$$f(\delta) - f(\gamma) = \left(-{}_{M,b}^{\alpha,\beta}Df(c)\right)\left(\frac{\Gamma(\beta+1)(b-c)^{\alpha}}{c}\right)(\delta-\gamma). \tag{9}$$

Fractional derivatives $D_{M,a}^{\alpha,\beta}$ and ${}_{M,b}^{\alpha,\beta}D$ possess all basic properties of the ordinary derivatives and beyond, see [3], [4].

3 Main Results

We present the following M-fractional Ostrowski type inequality:

Theorem 9 *Let $a < \gamma < \delta < b$, $0 \notin [\gamma, \delta]$, $f : [a,b] \to \mathbb{R}$, which is continuous over $[\gamma, \delta]$. We assume that $D_{M,a}^{\alpha,\beta}$, ${}_{M,b}^{\alpha,\beta}D$ exist and are continuous over $[\gamma, x_0]$ and $[x_0, \delta]$, respectively, where $x_0 \in [\gamma, \delta]$, for some $\alpha \in (0,1]$. Then*

$$\left|\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} f(x)\,dx - f(x_0)\right| \leq \frac{\Gamma(\beta+1)}{2(\delta-\gamma)}$$

$$\left[\left\|\frac{D_{M,a}^{\alpha,\beta}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}(x_0-a)^{\alpha}(x_0-\gamma)^2 + \left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}(b-x_0)^{\alpha}(\delta-x_0)^2\right]. \tag{10}$$

Proof. Let $x \in [\gamma, x_0]$, the by Theorem 5, there exists $c_1 \in (x, x_0)$, such that

$$f(x_0) - f(x) = \left(\frac{D_{M,a}^{\alpha,\beta}f(c_1)}{c_1}\right)\Gamma(\beta+1)(c_1-a)^{\alpha}(x_0-x). \tag{11}$$

Thus

$$|f(x) - f(x_0)| = \left|\frac{D_{M,a}^{\alpha,\beta}f(c_1)}{c_1}\right|\Gamma(\beta+1)(c_1-a)^{\alpha}|x-x_0| \leq$$

$$\left\|\frac{D_{M,a}^{\alpha,\beta}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}\Gamma(\beta+1)(x_0-a)^{\alpha}|x-x_0|, \tag{12}$$

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$\forall\ x \in [\gamma, x_0]$.

Let now $x \in [x_0, \delta]$, then by Theorem 8, there exists $c_2 \in (x_0, x)$, such that

$$f(x) - f(x_0) = -\left(\frac{{}_{M,b}^{\alpha,\beta}Df(c_2)}{c_2}\right)\Gamma(\beta+1)(b-c_2)^{\alpha}(x-x_0). \tag{13}$$

Thus

$$|f(x) - f(x_0)| = \left|\frac{{}_{M,b}^{\alpha,\beta}Df(c_2)}{c_2}\right|\Gamma(\beta+1)(b-x_0)^{\alpha}|x-x_0| \leq$$

$$\left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}\Gamma(\beta+1)(b-x_0)^{\alpha}|x-x_0|, \tag{14}$$

$\forall\ x \in [x_0, \delta]$.

We have that

$$\left|\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} f(x)\,dx - f(x_0)\right| = \frac{1}{\delta-\gamma}\left|\int_{\gamma}^{\delta}(f(x)-f(x_0))\,dx\right| \leq$$

$$\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}|f(x)-f(x_0)|\,dx = \tag{15}$$

$$\frac{1}{\delta-\gamma}\left[\int_{\gamma}^{x_0}|f(x)-f(x_0)|\,dx + \int_{x_0}^{\delta}|f(x)-f(x_0)|\,dx\right] \overset{\text{(by (12), (14))}}{\leq}$$

$$\frac{1}{\delta-\gamma}\left[\left\|\frac{D_{M,a}^{\alpha,\beta}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}\Gamma(\beta+1)(x_0-a)^{\alpha}\int_{\gamma}^{x_0}(x_0-x)\,dx\right.$$

$$\left.+\left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}\Gamma(\beta+1)(b-x_0)^{\alpha}\int_{x_0}^{\delta}(x-x_0)\,dx\right] =$$

$$\frac{\Gamma(\beta+1)}{2(\delta-\gamma)}\left[\left\|\frac{D_{M,a}^{\alpha,\beta}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}(x_0-a)^{\alpha}(x_0-\gamma)^2 + \right. \tag{16}$$

$$\left.\left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}(b-x_0)^{\alpha}(\delta-x_0)^2\right].$$

The theorem is proved. ■

Next we give two M-fractional Polya type inequalities:

Theorem 10 *All as in Theorem 9 and* $f(x_0) = 0$. *Then*

$$\left|\int_{\gamma}^{\delta} f(x)\,dx\right| \leq \int_{\gamma}^{\delta} |f(x)|\,dx \leq \frac{\Gamma(\beta+1)}{2}$$

$$\left[\left\|\frac{D_{M,a}^{\alpha,\beta} f(x)}{x}\right\|_{\infty,[\gamma,x_0]} (x_0 - a)^{\alpha} (x_0 - \gamma)^2 + \left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]} (b - x_0)^{\alpha} (\delta - x_0)^2\right]. \quad (17)$$

Proof. Same as in the proof of Theorem 9, by setting $f(x_0) = 0$. ∎

Corollary 11 *(to Theorem 10, case of* $x_0 = \frac{\gamma+\delta}{2}$*) All as in Theorem 9 and* $f\left(\frac{\gamma+\delta}{2}\right) = 0$. *Then*

$$\int_{\gamma}^{\delta} |f(x)|\,dx \leq \frac{\Gamma(\beta+1)(\delta-\gamma)^2}{8}$$

$$\left[\left\|\frac{D_{M,a}^{\alpha,\beta} f(x)}{x}\right\|_{\infty,\left[\gamma,\frac{\gamma+\delta}{2}\right]} \left(\left(\frac{\gamma+\delta}{2}\right) - a\right)^{\alpha} + \left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,\left[\frac{\gamma+\delta}{2},\delta\right]} \left(b - \left(\frac{\gamma+\delta}{2}\right)\right)^{\alpha}\right]. \quad (18)$$

Proof. Apply (17) for $x_0 = \frac{\gamma+\delta}{2}$. ∎

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