Bivariate Optimal Replacement Policies Under Partial Product Process for Multistate Degenerative Systems with Varying Cost Structures

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ABSTRACT
In this study examines a multistate degenerative system with k-working states having k-distinct rewards and l-failure states with l-different repair costs and its maintenance difficulty. The long-run average cost of a multistate degenerative system within the bivariate replacement polices (T, N), (T^+, N), (U, N), (U^-, N) with varying cost is assessed. The optimality exists under the bivariate replacement policies with partial product process demonstrated.

KEYWORDS: Partial Product Process, Replacement Policy, Renewal Reward Process, Varying Cost and Virtual Repair Time.

AMS subject classification : 60K10, 90B25

1. INTRODUCTION

Because of the aging process and cumulative wear, most real-world scenario systems are degenerative in the sense that they will continue to operate for shorter periods of time between failures, while requiring longer and longer repair durations following failures. Stated otherwise, the repair times are stochastically increasing and eventually go to infinity, but the operational times are stochastically decreasing and eventually dying out. To use this type of characteristic to represent a deteriorating system, Lam (1988) has introduced a Geometric processes and studied replacement problems. Stadje and Zuckerman (1990) have introduced a general monotone process repair model that generalised Lam's work other research works on the geometric process model include Stadje and Zuckerman (1992), Stanley (1993) for repair replacement models, Thangaraj,V., and Sundararajan, R. (1997) for optimal replacement policies for stochastic system.

This study derives the long-run average costs for multistate degenerative systems with varying cost under four bivariate replacement policies with partial product process:

- (T, N) policy : Replace the system after a fixed cumulative working age T or upon N-th failure.
- (U, N) policy : Replace the system after cumulative repair time U or upon N-th failure.
- (T^+, N) policy : Replace the system at the first failure point after cumulative operating time exceeds T or upon N-th failure.
- (U^-, N) policy : Replace the system at the failure point just before total repair time exceeds U or upon N-th failure.

Reliability theory makes the assumption that every system component either functions successfully or fails completely. The fact that a system can have more than two states makes this binary thinking unreliable. For instance, a microwave transmitter may be seen to be fully functional, functioning with a reduced transmission range, or malfunctioning entirely. Multiple distinct failures in a unique kind of multistate system would be an example of failure contribution. Another illustration would be a home security system that is susceptible to electrical or mechanical tampering and could activate on false alarms when cats are found inside. Lesanovsky (1993) has provided a review of research on systems with dual failure modes. Zhang (1994), has introduced a bivariate optimal replacement policy for a two-state repairable system. Govindaraju, Rizwan and Thangaraj (2009) have studied bivariate optimal replacement policy for varying cost structures. Babu, Govindaraju and Rizwan (2018) introduced and studied replacement models where the consecutive repair time follow an increasing partial product process. Raajpandiyan, Syed Tahir Hussainy and Rizwan (2022) have studied optimal replacement models under partial product process.

Generally speaking, a system can have two different failure states in addition to one working state. In a broader sense, the system could have l distinct failure states and k unique working states. This study focuses on a monotone process model for a multi-component system with $(k + l)$ states, namely k-working states and l-failure states. There are several techniques that can generate such a model to match the definition of a multistate degenerative system.

The rest of the paper is organized as follows. In Section 2, we give a general preliminaries. In Section 3, given model assumptions. We also present the monotone process model of a multi-component multistate system and the relevant results regrading their probability structure. In Section 4, we derive explicit expressions for the long-run average cost per unit time for this model under different bivariate replacement policies. Finally, a conclusion in given section 5.

2. PRELIMINARIES

In this part, we first give some definitions. The multi-component multistate system model is then described. We also estimate the conditional probabilities of the operating and failure times based on the current state of the system.

Definition 2.1 ***(Barlow and Proschan, 1965)*** *"A random variable X is said to be* Stochastically Smaller *than anther random variable Y, if $P(X > \alpha) \leq P(Y > \alpha)$, for all real α. It is denoted by $X \leq_{st} Y$."*

Definition 2.2 *"A stochastic process $X_n,\ n = 1,2,\cdots$ is said to be* Stochastically Increasing, *if $X_n \leq_{st} X_{n+1}$, for $n = 1,2,\cdots$."*

Definition 2.3 ***(Shaked and Shanthikumar, 1994)*** *"A Markov process $X_n,\ n = 1,2,\cdots$ with state space* $0,1,2,\cdots$ *is said to be* Stochastically Monotone, *if*

$$(X_{n+1}|X_n = i_1) \leq_{st} (X_{n+1}|X_n = i_2),$$

for any $0 \leq i_1 \leq i_2$."

Clearly, the stochastically monotone concept for a Markov process is defined for a Markov process and is based on the transition probabilities from one state to another state, conditioning on the former state. However, the stochastically monotone concept for a stochastic process defined here is for a general process and is based on the conditional distribution of the successive random variable in the process.

Definition 2.4 *"An integer valued random variable N is said to be a* stopping time *for the* *sequence of independent random variables* $X_1, X_2, \cdots$, *if the event* $\{N = n\}$ *is independent of* $X_{n+1}, X_{n+2}, \cdots$, *for all* $n = 1,2, \cdots$."

Definition 2.5 ***(Renewal process)*** *"If the sequence of nonnegative random variable* $\{X_1, X_2, X_3, \cdots\}$ *is independent and identically distributed, then the counting process* $\{N(t), t \geq 0\}$ *is said to be a* renewal process."

Definition 2.6 *"A life distribution F is said to be* New Better than Used in Expectation (NBUE), *if*

$$\int_0^\infty \overline{F}(t+x)dx \leq \overline{F}(t) \int_0^\infty \overline{F}(x)dx$$

for all $t \geq 0$. To say that the life distribution of an item is new better then used in expectation is equivalent to saying that the mean life length of a new item is greater than the mean residual life length of a non-failed item of age $t > 0$."

Definition 2.7 ***(Revathy, 1997)*** *"At every failure point, a decision is taken whether it can be sent for repair. If the cumulative repair time after this repair is expected to exceed a threshold value* δ, *the repair need not be initiated at that failure time. Such a fictitious repair time is called a* Virtual Repair Time."

Definition 2.8 ***(Babu, Govindaraju and Rizwan, 2018)*** *"Let* $\{X_n, n = 1,2,3, \cdots\}$ *be a sequence of independent and non-negative random variables and let* $F(X)$ *be the distribution function of* X_1. *Then* $\{X_n, n = 1,2,3, \cdots\}$ *is called* Partial Product Process, *if the distribution function of* X_{i+1} *is* $F(\alpha_i X)$ $(i = 1,2,3, \cdots)$, *where* $\alpha_i > 0$ *are real constants and* $\alpha_i = \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_{i-1}$."

Definition 2.9 *"A partial product process is called a* Decreasing Partial Product Process, *if* $\alpha_0 > 1$ *and is called an* Increasing Partial Product Process, *if* $0 < \alpha_0 < 1$."

Remark 2.1 *It is clear that if* $\alpha_0 = 1$, *then the partial product process is a renewal process.*

Remark 2.2 *Let* $E(Y_1) = \mu$, $var(Y_1) = \sigma^2$. *Then for* $j = 1,2,3, \cdots$,

$$E(Y_{j+1}) = \frac{\mu}{\beta_0^{2^{j-1}}}$$

and

$$Var(Y_{j+1}) = \frac{\sigma^2}{\beta_0^{2^j}}$$

where $\beta_0 > 0$.

Theorem 2.1 ***(Wald's equation)*** *"If* $X_1, X_2, X_3, \cdots$ *are independent and identically distribution random variables having finite expectations and if* N *is the stopping time for* $X_1, X_2, \cdots$ *such that* $E[N] < \infty$, *then*

$$E\left[\sum_{n=1}^{N} X_n\right] = E[N]E[X_1]."$$

Theorem 2.2 ***(Wald's equation for partial product process)*** *"Suppose that* $\{Y_n,\ n = 1,2,3, \cdots\}$

forms a partial product process with ratio β_0 *and* $E[Y_1] = \mu < \infty$, *then for* $t > 0$, *we have*

$$E\left[V_{\omega(t)+1}\right] = \mu E\left[1 + \sum_{j=2}^{\omega(t)+1} \frac{1}{\beta_0^{2^{j-2}}}\right],$$

where $\omega(t)$ is the counting process which represents the number of occurrences of an event up to time t."

3 Model Assumptions

We shall now describe the system's states. A $(k+l)$-state multistate system with k operating states and l failure states is considered. When t happens, the state of the system is given by

$$S(t) = \begin{cases} i & if\ the\ system\ is\ in\ the\ i-th\ working\ state\ at\ time\ t\ (i = 1,2,\cdots,k) \\ k+j & if\ the\ system\ is\ in\ the\ j-th\ working\ state\ at\ time\ t\ (j = 1,2,\cdots,l) \end{cases}$$

The state space of a new system is $\Omega = \Omega_1 \cup \Omega_2$ and the working state are $\Omega_1 = \{1,2,\cdots,k\}$ and the failure states are $\Omega_2 = \{k+1, k+2, \cdots, k+l\}$. Let's start with the installation of a brand new, operational system at state 1. If the system fails, it will be repaired. Let s_n be the time of the n-th failure, $n = 1,2,\cdots$ and let t_n be the completion time of the n-th repair, $n = 0,1,\ldots$ with $t_0 = 0$. Next

$$t_0 < s_1 < t_1 < \cdots < s_n < t_n < \cdots < s_{n+1} < \cdots$$

we next describe the probability structure of the model.

Assume that the transition probability from working state $i,\ i = 1,2,\cdots,k$, to failure state $k + j,\ j = 1,2,\cdots,l$, is given by

$$P(S(s_{n+1}) = k+j | S(t_n) = i) = q_j$$

with $\sum_{j=1}^{l} q_j = 1$. Moreover, the transition probability from failure state $k+j,\ j = 1,2,\cdots,l$, to working state i, $i = 1,2,\cdots,k$ is given by

$$P(S(t_n) = i | S(s_n) = k+j) = p_i$$

with $\sum_{i=1}^{k} p_i = 1$.

Let X_1 be the system's running time following installation. Let $Y_n,\ n = 1,2,\ldots$ be the repair time after n-th failure and $X_n,\ n = 2,3,\ldots$ be the system's operating time following $(n-1)$-st repair. Assume that $a_i > 0,\ i = 1,2,\cdots,k$, and that there is a life distribution $U_1(t)$ such that

$$P(X_1 \le t) = U_1(t) \quad (1)$$

and

$$P(X_2 \le t | S(t_1) = i) = U_1(a_i t), \quad (2)$$

$i = 1,2,\cdots,k$, where $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.

In gerenal, $i_j \epsilon \{1,2,\cdots,k\}$, we have

$$P(X_n \leq t|S(t_1) = i_1,\cdots,S(t_{n-1}) = i_{n-1}) = U_1\left(a_i,\cdots,a_{i_{n-1}}t\right), \quad (3)$$

$$j = 1,2,\dots,n-1.$$

Similarly, assume that there exist a life-time distribution $V_i(t)$ and $b_i > 0,\ i = 1,2,\cdots,l$ such that

$$P(Y_1 \leq t|S(s_1) = k+i) = V_1(b_1 t) \quad (4)$$

where $1 \geq b_1 \geq b_2 \geq \cdots \geq b_l > 0$ and in general, for $i_j \epsilon \{1,2,\dots,l\}$

$$P(Y_n \leq t|S(s_1) = k+i_1,\cdots,S(s_n) = k+i_n) = V_1\left(b_{i_1}\cdots b_{i_n}t\right) \quad (5)$$

In particular, if $a_1 = b_1 = 1,\ a_2 = \cdots = a_k = a'$ and $b_2 = \cdots = b_l = b'$ then the $(k+l)$-state system reduces to a two state system. In this case, the equations (3) and (5) become

$$P(X_n \leq t) = U_1((a')^{n-1}t)$$

$$P(Y_n \leq t) = V_1((b')^n t),$$

respectively. Thus the sequence $X_n,\ n = 1,2,\dots$ from a partial product process with ratio $a' > 1$, while the sequence $Y_n,\ n = 1,2,\dots$ from with ratio $0 < b' < 1$. In this case, our model reduces to the model for the multi component two state system introduced by Babu, Govindaraju and Rizwan (2018).

Remarks

For two working states $1 \leq i_1 < i_2 \leq k$, we have

$$(X_2|S(t_1) = i_2) \leq_{st} (X_2|S(t_1) = i_1).$$

Since the system in state i_1 has a stochastically larger operating time than it does in state i_2, working state i_1 is therefore superior to working state i_2. Thus, state 1 is the best working state and state k is the worst working state, with the k-working states defined in decreasing order. An identical exists for two failure states $k+i_1,\ k+i_2$ such that $k+1 \leq k+i_1 < k+i_2 \leq k+l$, we have

$$(Y_1|S(s_1) = k+i_1) \leq_{st} (Y_1|S(s_1) + k + i_2).$$

The failure state $k+i_1$ is superior to the failure state $k+i_2$ because the system in state $k+i_1$ has a stochastically smaller repair time than it does in states $k+i_2$. Consequently, the l failure states are likewise ordered in decreasing order, where $k+1$ is the best failure state and $k+l$ is the worst.

Model Assumptions

Consider the following list of assumptions, **A1 - A8** for a monotone process model of the multistate system covered in this section.

A1 At the beginnig,a new easily repairable system is set up. There are $(k+l)$ potential states for the system, with states $1,2,\cdots,k$ denoting the first working state, the second working state,⋯, k-th working state respectively, and states $(k+1),(k+2),\cdots,(k+l)$ denoting the first failure state, the system's second failure state,⋯ and the l-th failure state in that order. They are mutually exclusive and stochastically occurring failures.

A2 Whenever the system fails, it will be either repaired or replaced. The system will be replaced by an identical new one some times later.

A3 Let the system's operating time after installation will be X_1.The operating system $X_n, n = 2,3,\ldots$ times of the system after the $(n-1)$-st repair in a cycle. It is indicated by the distribution of X_n by $F_n(.)$. Assume that $E(X_1)=\lambda>0$.

We let X_{i+1} be the operating time after the i-th repair, for $i=1,2,3,\cdots$. Then the distribution function of X_{i+1} is $F\left(\alpha_0^{2^{i-1}}x\right)$, where $\alpha_0(>1)$ is a constant. Now

$$E(X_{i+1})=\frac{\lambda}{\alpha_0^{2^{i-1}}}$$

for $i=1,2,3,\cdots$. Let $X_n, n=1,2,3,\cdots$ be the successive operating time after repair, constitute a decreasing partial product process.

A4 let Y_1 denote the repair time and $G(y)$ be its distribution after the first failure function of Y_1. Assume that $E(Y_1)=\mu\geq 0$. It means that if $\mu=0$, it indicates that the anticipated repair time is negligibly small. Let Y_{j+1} be the repair time after $(j+1)$-st failure for $j=1,2,3,\cdots$ and $G\left(\beta_0^{j-1}y\right)$ the distribution function of Y_{j+1}, where $0<\beta_0\leq 1$ is a constant and $E\left(Y_{j+1}\right)=\frac{\mu}{\beta_0^{2^{j-1}}}$ for $j=1,2,3,\cdots$.

The Y_j represent the sequential repair durations from an increasing partial product process are $\left\{Y_j, j=1,2,3,\cdots\right\}$

A5 Let r be the reward rate when the system in working state i is operating, let c be the repair cost when the system in failure state $(k+1)$ is repaired, and the replacement cost is composed of two parts, one is the basic replacement cost R and the other proportional to the replacement time z at rate c_p. That is, the replacement cost by $R+c_pZ$.

A6 Assume that $1\geq b_1\geq b_2\geq\cdots\geq b_l>0$ and $1\leq a_1\leq a_2\leq\cdots\leq a_k$.

A7 Assume that $G_n(t)$ be the cumulative distribution of $M_n=\sum_{i=1}^{n} Y_i$ and $F_n(t)$ be the cumulative distribution of $L_n=\sum_{i=1}^{n} X_i$.

A8 The replacement time Z, $(n=1,2,\cdots)$ are independent random variables and the repair time Y_n, the working time X_n.

4. BIVARIATE REPLACEMENT POLICIES

4.1 The Bivariate Policy (T, N)

This section presents and examines a bivariate replacement policy (T, N) under partial product process with varying cost for the multistate degenerative system, where in the system is replaced at working age T or at the time of N-th failure, whichever happens first. The problem is to determine an optimal replacement policy $(T, N)^*$ so that the long-run average cost per unit time is minimized.

The working age T of the system at time t is the cumulative life-time given by

$$T(t) = \begin{cases} t - M_n, & : \quad L_n + M_n \leqslant t < L_{n+1} + M_n \\ L_{n+1}, & : \quad L_{n+1}, + M_n \leqslant t < L_{n+1} + M_{n+1}. \end{cases}$$

Initially let $L_n = \sum_{i=1}^{n} X_i$ and $M_n = \sum_{j=1}^{n} Y_j$ and $L_0 = M_0 = 0$.

Following Lam (2005), the distribution of the survival time X_n in **A3** and the distribution of the repair time Y_n in **A4** are given by

$$P(X_n \leq t) = \sum_{\sum_{j=1}^{k} i_j = n-1} \frac{(n-1)!}{i_1!\, i_2! \cdots i_k!} p_1^{i_1} \cdots p_k^{i_k} U\left(a_1^{i_1} \cdots a_k^{i_k}\right) t \qquad (6)$$

where $i_1, i_2, \cdots, i_k \epsilon z^+$ and

$$P(Y_n \leq t) = \sum_{\sum_{i=1}^{l} j_i = n} \frac{(n)!}{j_1!\, j_2! \cdots j_l!} q_1^{j_1} \cdots q_l^{j_l} V\left(b_1^{j_1} \cdots b_l^{j_l}\right) t \qquad (7)$$

where $j_1, j_2, \cdots, j_l \in Z^+$ and if $E(X_1) = \lambda$, then the mean survival time is

$$E(X_n) = \frac{\lambda}{\alpha_0^{2^{n-1}}} \qquad (8)$$

for $n > 1$, where

$$a = \left(\sum_{i=1}^{k} \frac{p_i}{a_i}\right)^{-1} \qquad (9)$$

and if $E(Y_1) = \mu$, then the mean repair time is

$$E(Y_n) = \frac{\mu}{\beta_0^{2^{n-1}}} \qquad (10)$$

for $n > 1$, where

$$b_{-1} = \left(\sum_{j=1}^{l} \frac{q_j}{b_j}\right)^{-1}. \quad (11)$$

Similarly, if $R_n = r_i$, where $S(s_{n-1}) = i$, $i = 1,2,\cdots,k$, represents the reward received after the n-th repair, the mean reward received after the (n-1)-st repair is $E(R,X) = r\lambda$, and for $n \geq 2$, the expected reward after installation is given by

$$E(R_n X_n) = \frac{r\lambda}{\alpha_0^{2^{n-1}}} \quad (12)$$

where $r = \sum_{i=1}^{k} \frac{r_i p_i}{a_i}$, and if $C_n = c_i$ where $S(s_n) = k + i, i = 1,2,\cdots,l$ shows the cost of repairs after the n-th failure, therefore the average cost of repairs after the n-th failure is

$$E(C_n Y_n) = \frac{c\mu}{\beta_0^{2^{n-1}}} \quad (13)$$

where $c = \sum_{l}^{i=1} \frac{r_i q_i}{b_i}$.

The length of a cycle and its mean

Then length of a cycle under the bivariate replacement policy (T, N) with partial product process is

$$w = \left(T + \sum_{i=1}^{\eta} Y_i\right)\chi_{(L_N > T)} + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(L_N \leq T)} + Z,$$

where $\eta = 1,2,\cdots,N-1$ represents the number of failures prior to the system's working age surpassing T.

$$\chi_{(A)} = \begin{cases} 1 & : if\ the\ event\ A\ occurs \\ 0 & : if\ the\ event\ A\ does\ not\ occur, \end{cases}$$

denote the indicator function and $E[\chi_{(A)}] = P(A)$.
From Leung (2006), we have

$$E[\chi_{(L_i \leq T < L_N)}] = P(L_i \leq T < L_N)$$

$$= P(L_i \leq T) - P(L_N \leq T)$$

$$= F_i(T) - F_N(T).$$

Lemma 4.1 *The mean length of a cycle under the policy (T, N) is*

$$E(W) = \int_0^T \overline{F}_N(u)du + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} F_i(T) + \tau. \tag{14}$$

Proof. Examine

$$E(w) = E\left[\left(T + \sum_{i=1}^{\eta} Y_i\right)\chi_{(L_N>T)}\right] + E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(L_N\leq T)}\right] + E(Z)$$

$$= E\left[T_{\chi_{(L_n>T)}}\right] + E\left[\left(\sum_{i=1}^{\eta} Y_i\right)\chi_{(L_N>T)}\right]$$

$$+ E\left\{E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(L_N\leq T)} | L_N = u\right]\right\} + E(Z)$$

$$= T\overline{F}_N(T) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \quad E\left[\chi_{(L_i\leq T<L_N)}\right] + \int_0^T udF_N(u) + \int_0^T \sum_{i=1}^{N-1} (Y_i)dF_N(u) + \tau$$

$$= T\overline{F}_N(T) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \; P(L_i < T < L_N) + \int_o^T udF_N(u) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \; F_N(T) + \tau$$

$$= T\overline{F}_N(T) + \int_o^T udF_N(u) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \; [F_i(T) - F_N(T)] + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \; F_N(T) + \tau$$

$$= \int_0^T \overline{F}_N(u)du + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \; F_i(T) + \tau,$$

as intended and the lemma's proof is now complete. ■

Lemma 4.2 *If $L_N \leq T$ and $n \geq 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N\leq T)}\right] = \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} \int_0^T udF_N(u).$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N\le T)}\right] = E\left\{E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N\le T)}|L_N\right]\right\}$$

$$= \int_0^T E\left(\sum_{n=2}^{N} R_n X_n|L_N = u\right) dF_N(u)$$

$$= \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} \int_0^T u dF_N(u)$$

Lemma 4.3 *If $L_N > T$ and $n \ge 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{\eta} R_n X_n\right)\chi_{(L_N>T)}\right] = \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} [F_n(T) - F_N(T)].$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{\eta} R_n X_n\right)\chi_{(L_N>T)}\right] = E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N<T<L_N)}\right]$$

$$= \sum_{n=2}^{N} E(R_n X_n) E\left[\chi_{(L_N<T<L_N)}\right]$$

$$= \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} [F_n(T) - F_N(T)],$$

Lemma 4.4 *If $L_N \le T$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(L_N\le T)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} F_N(T).$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(L_N\le T)}\right] = E\left[E\left(\sum_{n=1}^{N-1} C_n Y_n|L_N = u\right)\chi_{(L_N\le T)}\right]$$

$$= \int_0^T E\left(\sum_{n=1}^{N-1} C_n Y_n|L_N = u\right) dF_N(u)$$

$$= \int_0^T \sum_{n=1}^{N-1} E(C_n Y_n) dF_N(u),$$

Lemma 4.5 *If $L_N > T$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right)\chi_{(L_N>T)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}}[F_n(T) - F_N(T)].$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right)\chi_{(L_N>T)}\right] = E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(L_n<T<L_N)}\right]$$

$$= \sum_{n=1}^{N-1} E(C_n Y_n) E\left[\chi_{(L_n<T<L_N)}\right]$$

$$= \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} P[L_n < T < L_N]$$

$$= \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}}[F_n(T) - F_N(T)],$$

The long-run average cost under policy (T, N)

Assume that T_1 is the initial replacement time and that the interval between the $(n-1)$-st and n-th replacements is $T_n (n \leq 2)$. A renewal process is then formed by the sequence T_n, $n = 1,2, \ldots$. A renewal cycle is the inter arrival time between two successive replacements. The long-run average cost per unit of time for a multistate degenerative system with varying cost structures under the multistate bivariate replacement policy (T, N) with partial product process is determined by the renewal reward theorem by Ross (1996).

$$C(T,N) = \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}$$

$$= \frac{\left[E\left\{\left(\sum_{n=1}^{\eta} C_n Y_n - T\sum_{n=1}^{\eta} R_n\right)\chi_{(L_N>T)} + c_p E(Z)\right\} + E\left\{\left(\sum_{n=1}^{N-1} C_n Y_n - \sum_{n=1}^{N} R_n X_n\right)\chi_{(L_N\leq T)}\right\} + R\right]}{E(W)}$$

Using Lemmas 4.1 to 4.5, we obtain

$$\mathcal{C}(T,N)=\frac{\begin{bmatrix}\sum_{n=1}^{N-1}\frac{c\mu}{\beta_0^{2^{n-1}}}[F_n(T)-F_N(T)]-\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[F_n(T)-F_N(T)]+r_1T\\+\sum_{n=1}^{N-1}\frac{c\mu}{\beta_0^{2^{n-1}}}F_N(T)-\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^T udF_N(u)+c_p\tau+R-r_1\lambda\end{bmatrix}}{\int_0^T\overline{F}_N(u)du+\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}F_i(T)+\tau}$$

After combining the aforementioned facts, we arrive at the following conclusion.

Theorem 4.1 *The long run average cost per unit time for a multistate degenerative systemwith varying cost structures under the bivariate replacement policy* (T,N) *with partial product process for the model presented in section 3 under the* **A1** *through* **A8** *is provided by.*

$$\mathcal{C}(T,N)=\frac{\begin{bmatrix}\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[F_n(T)-F_N(T)]-r_1(\lambda+T)\\+\sum_{n=1}^{N-1}\frac{c\mu}{\beta_0^{2^{n-1}}}F_N(T)-\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^T udF_N(u)+c_p\tau+R\end{bmatrix}}{\int_0^T\overline{F}_N(u)du+\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}F_i(T)+\tau} \qquad (15)$$

Deductions

The long-run average cost $\mathcal{C}(T,N)$ is a bivariate function in T and N. Obviously, when N is fixed, $\mathcal{C}(T,N)$ is a function of T. For fixed $N=m$, it can be written as

$$\mathcal{C}(T,N)=\mathcal{C}_m(T), \qquad m=1,2,\cdots.$$

Thus, for a fixed m, we can find T_m^* by analytical or numerical methods such that $\mathcal{C}_m(T_m^*)$ is minimised. That is, when $N=1,2,\cdots,m,\cdots$, we can find $T_1^*,T_2^*,\cdots,T_m^*,\cdots$, respectively, such that the corresponding, $C_1(T_1^*),C_2(T_2^*),\cdots,C_m(T_m^*),\cdots$, are minimised. Because the total lifetime of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. So we can determine the minimum of the long-run average cost per unit time based on $C_1(T_1^*),C_2(T_2^*),\cdots,C_m(T_m^*),\cdots$. Then, if the is denoted by $\mathcal{C}_n(T_n^*)$, we obtain the bivariate optimal replacement policy $(T,N)^*$ such that

$$\begin{aligned}\mathcal{C}((T,N)^*)&=\min_n C_n(T_n^*)\\&=\min_n\left[\min_T\mathcal{C}(T,N)\right]\\&\le\mathcal{C}(\infty,N)\\&=C(N^*)\end{aligned}$$

The optimal policy $(T,N)^*$ is better than the optimal policy N^*. Moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy T^*. So under the same conditions, an optimal policy $(T,N)^*$ is better than the optimal replacement policies N^* and T^*.

4.2 The Bivariate Policy (U, N)

The bivariate replacement policy (U, N) with partial product process for the multistate degenerative system with varying cost is defined and examined here. The system is replaced when it reaches its N-th failure or when the overall period of time to repair surpasses U, whichever occurs first. Choosing the optimal replacement policy $(U, N)^*$ to minimize the long-run average cost per unit of time over the problem.

4.2.1 The length of a cycle and its mean

The length of a cycle W under the bivariate replacement policy (U, N) with partial product process is

$$W = \left(\sum_{i=1}^{\eta} X_i + U\right)\chi_{(M_N > U)} + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(M_N \le U)} + Z,$$

where $\eta = 1,2,\cdots, N-1$ represents the number of failures prior to the system's overall repair time surpassing U and $\chi_{(.)}$ denote the indicator function.

$$\chi_{(A)} = \begin{cases} 1 & : if\ the\ event\ A\ occurs \\ 0 & : if\ the\ event\ A\ does\ not\ occur. \end{cases}$$

Lemma 4.6 *The mean length of the cycle under policy the (U, N) is*

$$E(W) = \int_0^U \overline{G}_N(u)du + \frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U) + \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}}\ G_{i-1}(U) + \tau. \qquad (16)$$

Proof. Examine

$$E(w) = \left[\left(\sum_{i=1}^{\eta} X_i + U\right)\chi_{(M_N > U)}\right] + E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(M_N \le U)}\right] + E(Z)$$

$$= E\left[\left(\sum_{i=1}^{\eta} X_i\right)\chi_{(M_N > U)}\right] + E\left[U_{\chi_{(M_N > U)}}\right]$$

$$+E\left\{E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(M_N \le U)} | M_N = u\right]\right\} + E(Z)$$

$$= E\left(\sum_{i=1}^{N} X_i\right)\ E\left[\chi_{(M_N \le U)}\right] + \int_0^U u dG_N(u) + \sum_{i=1}^{N-1} E(X_i)\ E\left[\chi_{(M_{i-1} \le U < M_N)}\right]$$

$$+UE\left[\chi_{(M_N>U)}\right]+\tau$$

$$=\int_0^U udG_N(u)+\sum_{i=1}^{N}\frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U)+\sum_{i=1}^{N-1}E(X_i)\,P(M_{i-1}\le U<M_N)$$
$$+\ U\overline{G}_N(U)+\tau$$

$$=\int_0^U udG_N(u)+\sum_{i=1}^{N}\frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U)+\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}\ [G_{i-1}(U)-G_N(U)]$$
$$+\ U\overline{G}_N(U)+\tau,$$

as intended and the lemma's proof is now complete. ■

Lemma 4.7 *If $M_N \le U$ and $n \ge 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N\le U)}\right]=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^U udG_N(u).$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N\le U)}\right]=E\left\{E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N\le U)}|M_N\right]\right\}$$

$$=\int_0^U E\left(\sum_{n=2}^{N}R_nX_n|U_N=u\right)dG_N(u)$$

$$=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^U udG_N(u)$$

Lemma 4.8 *If $M_N > T$ and $n \ge 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{\eta}R_nX_n\right)\chi_{(M_N>U)}\right]=\sum_{n=2}^{N-1}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[G_n(U)-G_N(U)].$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{\eta}R_nX_n\right)\chi_{(M_N>U)}\right]=E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N<U<M_N)}\right]$$

$$=\sum_{n=2}^{N}E(R_nX_n)E\left[\chi_{(M_N<U<M_N)}\right]$$

$$= \sum_{n=2}^{N-1} \frac{r\lambda}{\alpha_0^{2^{n-2}}} [G_n(U) - G_N(U)],$$

Lemma 4.9 *If $M_N \leq U$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right) \chi_{(M_N \leq U)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} G_N(U).$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right) \chi_{(M_N \leq U)}\right] = E\left[E\left(\sum_{n=1}^{N-1} C_n Y_n | M_N = u\right) \chi_{(M_N \leq U)}\right]$$

$$= \int_0^U E\left(\sum_{n=1}^{N-1} C_n Y_n | M_N = u\right) dG_N(u)$$

$$= \int_0^U \sum_{n=1}^{N-1} E(C_n Y_n) dG_N(u),$$

Lemma 4.10 *If $M_N > U$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right) \chi_{(M_N > U)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} [G_n(U) - G_N(U)].$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right) \chi_{(M_N > U)}\right] = E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right) \chi_{(M_n < U < M_N)}\right]$$

$$= \sum_{n=1}^{N-1} E(C_n Y_n) E\left[\chi_{(M_n < U < M_N)}\right]$$

$$= \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} [G_n(U) - G_N(U)],$$

The long-run average cost under policy (U, N)

Assume that U_1 is the initial replacement time and that the interval between the $(n-1)$-

st and n-th replacements is $U_n (n \leq 2)$. A renewal process is then formed by the sequence U_n, $n = 1,2, \ldots$. A renewal cycle is the inter arrival time between two successive replacements. The long-run average cost per unit of time for a multistate degenerative system with varying cost structures under the multistate bivariate replacement policy (U, N) with partial product process is.

$$\mathcal{C}(U,N) = \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}$$

$$= \frac{\begin{bmatrix} E\left\{\left(U \sum_{n=1}^{\eta} C_n - \sum_{n=1}^{\eta} R_n X_n\right)\chi_{(M_N > U)}\right\} + c_p E(Z) \\ + E\left\{\left(\sum_{n=1}^{N-1} C_n Y_n - \sum_{n=1}^{N} R_n X_n\right)\chi_{(M_N \leq U)}\right\} + R \end{bmatrix}}{E(W)}$$

Using Lemmas 4.6 to 4.10, we obtain

$$\mathcal{C}(U,N) = \frac{\begin{bmatrix} \sum_{n=1}^{N-1} \frac{c\lambda}{\beta_0^{2^{n-1}}}[G_n(U) - G_N(U)] - \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}}[G_n(U) - G_N(U)] + r_1 U \\ + \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} G_N(U) - \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} \int_0^U u dG_N(u) + c_p\tau + R - r_1\lambda \end{bmatrix}}{\int_0^U \overline{G}_N(u)du + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} G_i(U) + \tau}$$

After combining the aforementioned facts, we arrive at the following conclusion.

Theorem 4.2 *The long run average cost per unit time for a multistate degenerative systemwith varying cost structures under the bivariate replacement policy (U, N) with partial product process for the model presented in section 3 under the* **A1** *through* **A8** *is provided by.*

$$\mathcal{C}(T,N) = \frac{\begin{bmatrix} \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}}[G_n(U) - G_N(U)] - r_1(\lambda + U) \\ + \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} G_N(U) - \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} \int_0^U u dG_N(u) + c_p\tau + R \end{bmatrix}}{\int_0^U \overline{G}_N(u)du + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} G_i(U) + \tau} \quad (17)$$

Deductions

The long-run average cost $\mathcal{C}(U, N)$ is a bivariate function in U and N. Obviously, when N is fixed, $\mathcal{C}(U, N)$ is a function of U. For fixed $N = m$, it can be written as

$$\mathcal{C}(U,N) = \mathcal{C}_m(U), \qquad m = 1,2, \cdots.$$

Thus, for a fixed m, we can find U_m^* by analytical or numerical methods such that $\mathcal{C}_m(U_m^*)$ is minimised. That is, when $N = 1,2, \cdots, m, \cdots$, we can find $U_1^*, U_2^*, \cdots, U_m^*, \cdots$, respectively, such that the corresponding, $C_1(U_1^*), C_2(U_2^*), \cdots, C_m(U_m^*), \cdots$, are minimised. Because the total lifetime of a multistate degenerative system is limited, the minimum of the long-run average cost per unit time exists. So we can determine the minimum of the long-run average cost per unit time based on $C_1(U_1^*), C_2(U_2^*), \cdots, C_m(U_m^*), \cdots$. Then, if the is denoted by $\mathcal{C}_n(U_n^*)$, we obtain the bivariate optimal replacement policy $(U,N)^*$ such that

$$\mathcal{C}((U,N)^*) = \min_m C_m(U_m^*)$$

$$= \left[\min_U \mathcal{C}(U,N)\right]$$

$$\leq \mathcal{C}(\infty, N)$$

$$= C(N^*)$$

The optimal policy $(U,N)^*$ is better than the optimal policy N^*. Moreover, under some mild conditions the optimal replacement policy N^* is better than the optimal policy U^*. So under the same conditions, an optimal policy $(U,N)^*$ is better than the optimal replacement policies N^* and U^*.

4.3 The Bivariate Policy (T^+, N)

This policy involves a multistate degenerative system with varying cost that is replaced at the first failure point when the cumulative operating time exceeds T or when the N-th failure occurs, whichever occurs first. Muth (1977) used this method of replacing the system at the first failure point once the total operating time higher than a specified value.

4.3.1 The length of a cycle and its mean

The length of a cycle W under the bivariate replacement policy (T^+, N) with partial product process is

$$W = \left(\sum_{i=1}^{\eta} X_i + Y_{i-1}\right)\chi_{(L_N > T)} + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(L_N \leq T)} + Z,$$

where $\eta = 1,2,\cdots,N-1$ represents the number of failures prior to the system's working age surpassing T.

$$P(\eta = j) = P\big(X_1 \leq T, X_2 \leq T, \cdots, X_{\eta-1} \leq T, X_\eta > T\big);\ \ j = 1,2,\cdots$$

$$= F^{j-1}\overline{F}(T).$$

Since η is a random variable,

$$E(\eta - 1) = \sum_{j=1}^{\infty} (j-1)p(\eta = j)$$

$$= F^{j-1}(T)\sum_{j=1}^{\infty} (j-1)\overline{F}(T)$$

$$= \frac{F(T)}{\overline{F}(T)}.$$

Lemma 4.11 *The mean length of the cycle under policy the (T^+, N) is*

$$E(W) = \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \ [(1-b)F_N(T) + bF_i(T)] + \frac{F(T)}{\overline{F}(T)} \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} \ [(F_i(T) - F_N(T)] + \int_0^T u dF_N(u) + \tau. \quad (18)$$

Proof. Examine

$$E(w) = E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(L_N \le T)}\right] + E\left[\left(\sum_{i=1}^{\eta} X_i + \sum_{i=1}^{\eta} Y_{i-1}\right)\chi_{(L_N > T)}\right] + E(Z)$$

$$= E\left\{E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(L_N \le T)} | L_N = u\right]\right\}$$

$$+E\left[\left(\sum_{i=1}^{\eta} X_i\right)\chi_{(L_N > T)}\right] + E\left[\left(\sum_{i=1}^{\eta} Y_{i-1}\right)\chi_{(L_N > T)}\right] + E(Z)$$

$$= \int_o^T u dF_N(u) + \int_o^T \sum_{i=1}^{N-1} E(Y_i) dF_N(u) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \ P(L_i < T < L_N)$$

$$+ \sum_{i=1}^{N-1} E(X_i | \eta = N-1) \ P(L_i \le T < L_N) + \sum_{i=1}^{N-1} E(Y_{i-1}) \ E\left[\chi_{(L_i \le T < L_N)}\right] + \tau$$

$$= \int_o^T u dF_N(u) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \ F_N(T) + \frac{F(T)}{\overline{F}(T)} \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} \ [F_i(T) - F_N(T)]$$

$$+ \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} \ [F_i(T) - F_N(T)] + \tau,$$

as intended and the lemma's proof is now complete. ■

Lemma 4.12 *If $L_N \le T$ and $n \ge 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N\leq T)}\right]=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^T u dF_N(u).$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N\leq T)}\right]=E\left\{E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N\leq T)}|L_N\right]\right\}$$

$$=\int_0^T E\left(\sum_{n=2}^{N} R_n X_n|L_N=u\right)dF_N(u)$$

$$=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^T u dF_N(u)$$

Lemma 4.13 *If $L_N > T$ and $n \geq 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{\eta} R_n X_n\right)\chi_{(L_N>T)}\right]=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[F_n(T)-F_N(T)].$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{\eta} R_n X_n\right)\chi_{(L_N>T)}\right]=E\left[\left(\sum_{n=2}^{N} R_n X_n\right)\chi_{(L_N<T<L_N)}\right]$$

$$=\sum_{n=2}^{N}E(R_n X_n)E\left[\chi_{(L_N<T<L_N)}\right]$$

$$=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[F_n(T)-F_N(T)],$$

Lemma 4.14 *If $L_N \leq T$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(L_N\leq T)}\right]=\sum_{n=1}^{N-1}\frac{c\mu}{\beta_0^{2^{n-1}}}F_N(T).$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(L_N\leq T)}\right]=E\left[E\left(\sum_{n=1}^{N-1} C_n Y_n|L_N=u\right)\chi_{(L_N\leq T)}\right]$$

$$= \int_0^T E\left(\sum_{n=1}^{N-1} C_n Y_n | L_N = u\right) dF_N(u)$$

$$= \int_0^T \sum_{n=1}^{N-1} E(C_n Y_n) dF_N(u),$$

Lemma 4.15 *If $L_N > T$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right)\chi_{(L_N>T)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}}[F_n(T) - F_N(T)].$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right)\chi_{(L_N>T)}\right] = E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(L_n<T<L_N)}\right]$$

$$= \sum_{n=1}^{N-1} E(C_n Y_n) E\left[\chi_{(L_n<T<L_N)}\right]$$

$$= \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}}[F_n(T) - F_N(T)],$$

The long-run average cost under policy (T^+, N)

The long-run average cost per unit of time for a multistate degenerative system with varying cost structures under the multistate bivariate replacement policy (T^+, N) with partial product process is

$$\mathcal{C}(T^+, N) = \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}$$

$$= \frac{\begin{bmatrix} E\left\{\left(\sum_{n=1}^{\eta} C_n Y_n - \sum_{n=1}^{\eta} R_n X_n\right)\chi_{(L_N>T)} + c_p E(Z)\right\} \\ +E\left\{\left(\sum_{n=1}^{N-1} C_n Y_n - \sum_{n=1}^{N} R_n X_n\right)\chi_{(L_N\leq T)}\right\} + R \end{bmatrix}}{E(W)}$$

Using Lemmas 4.10 to 4.15, we obtain

$$C(T,N)=\frac{\begin{bmatrix}\sum_{n=1}^{N-1}\frac{c\mu}{\beta_0^{2^{n-1}}}[F_n(T)-F_N(T)]-\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[F_n(T)-F_N(T)]+r_1T\\+\sum_{n=1}^{N-1}\frac{c\mu}{\beta_0^{2^{n-1}}}F_N(T)-\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^T udF_N(u)+c_p\tau+R-r_1\lambda\end{bmatrix}}{\begin{bmatrix}\int_o^T udF_N(u)+\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}\ F_N(T)+\frac{F(T)}{\overline{F}(T)}\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}\ [F_i(T)-F_N(T)]\\+\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}\ [F_i(T)-F_N(T)]+\tau\end{bmatrix}}$$

After combining the aforementioned facts, we arrive at the following conclusion.

Theorem 4.3 *The long run average cost per unit time for a multistate degenerative systemwith varying cost structures under the bivariate replacement policy* (T^{+},N) *with partial product process for the model presented in section 3 under the* **A1** *through* **A8** *is provided by.*

$$C(T^{+},N)=\frac{\begin{bmatrix}c\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}[(1-b)F_N(T)+bF_i(T)]\\+\ r\int_0^T udF_N(u)+\frac{F(T)}{\overline{F}(T)}\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}[F_i(T)-F_N(T)]+c_p\tau+R\end{bmatrix}}{\begin{bmatrix}\int_o^T udF_N(u)+\frac{F(T)}{\overline{F}(T)}\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}[F_i(T)-F_N(T)]\\+\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}[(1-b)F_N(T)+bF_i(T)]+\tau\end{bmatrix}}. \tag{19}$$

The process used to determine the optimal policy $(T,N)^{*}$ is also employed to obtain the bivariate optimal replacement policy $(T^{+},N)^{*}$ with partial product process.

4.4 The Bivariate Policy $(\boldsymbol{U}^{-},\boldsymbol{N})$

The multistate degenerative system will be replaced at the failure point in accordance with policy (U^{-},N), either when the N-th failure occurs, whichever occurs first, or shortly before the overall repair time surpasses U.

4.4.1 Virtual Repair Times

There may be an optimal policy in the policy (U^{-},N), that requires replacing the system in the middle of the repair time. Since we might have been able to save cost on repairs, the question of whether it would have been cheaper to replace the system at the point of failure itself naturally comes up. In fact, Stadje and Zuckerman (1992) demonstrated for their policy U that there is an optimal replacement policy that does not replace during a repair period if Y_s are new better then used in expectation. The strategy of not replacing system components in the middle of the operating cycle is cost-effective because our policies do not impose additional costs for replacement in the event of failure.

4.4.2 The length of a cycle and its mean

The length of a cycle W under the bivariate replacement policy (U^{-},N) with partial product process is

$$W = \left(\sum_{i=1}^{\eta} X_i + \sum_{i=o}^{v} Y_i\right)\chi_{(M_N>U)} + \left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(M_N\leq U)} + Z,$$

where $\eta = 1,2,\cdots,N-1$ represents the number of failures prior to the system's overall repair time surpassing U and $\chi_{(.)}$ And the number of repairs before the overall repair time above U is denoted by $V = 0,1,2,\cdots,N-1$. If $M_i \leq U < M_{i+1}$ for $i = 1,2,\cdots,N-1$, then $U - M_i$ will be the virtual repair time.

Lemma 4.16 *The mean length of the cycle under policy* (U^-,N) *is*

$$E(W) = \int_0^U udG_N(u) + \frac{G(U)}{\overline{G}(U)}\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}\ [G_i(U) - G_N(U)] + \sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}\ G_{i-1}(U) + \frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U) + \tau \qquad (20)$$

Proof. Examine

$$E(W) = \left[\left(\sum_{i=1}^{\eta} X_i + \sum_{i=o}^{v} Y_i\right)\chi_{(M_N>U)}\right] + E\left[\left(\sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i\right)\chi_{(M_N\leq U)}\right] + E(Z)$$

$$= \int_0^U udG_N(u) + \sum_{i=1}^{N}\frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U) + E\left[\sum_{i=1}^{\eta} X_{i\chi_{(M_N>U)}}\right] + E\left[\sum_{i=0}^{v} Y_{i\chi_{(M_N>U)}}\right] + E(Z)$$

$$= \int_0^U udG_N(u) + \sum_{i=1}^{N}\frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U) + \sum_{i=1}^{N} E(X_i)\ P[M_{i-1} \leq U < M_N] + \sum_{i=0}^{N-1} E(Y_i|v)\ P[M_i \leq U < M_N] + E(Z)$$

$$= \int_0^U udG_N(u) + \sum_{i=1}^{N}\frac{\lambda}{\alpha_0^{2^{N-1}}}\ G_N(U) + \sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}\ [G_{i-1}(U) - G_N(U)] + \sum_{i=0}^{N-1} E(Y_i)E(v-1)\ [G_i(U) - G_N(U)] + \tau$$

$$= \int_0^U udG_N(u) + \frac{G(U)}{\overline{G}(U)}\sum_{i=1}^{N-1}\frac{\mu}{\beta_0^{2^{i-1}}}\ [G_i(U) - G_N(U)]$$

$$+\sum_{i=1}^{N}\frac{\lambda}{\alpha_0^{2^{N-1}}}\;G_N(U)+\sum_{i=1}^{N-1}\frac{\lambda}{\alpha_0^{2^{i-1}}}\;[G_{i-1}(U)-G_N(U)]+\tau,$$

as desired and this completes the proof of the lemma. ■

Lemma 4.17 *If $M_N \leq U$ and $n \geq 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N\leq U)}\right]=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^U u dG_N(u).$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N\leq U)}\right]=E\left\{E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N\leq U)}|M_N\right]\right\}$$

$$=\int_0^U E\left(\sum_{n=2}^{N}R_nX_n|U_N=u\right)dG_N(u)$$

$$=\sum_{n=2}^{N}\frac{r\lambda}{\alpha_0^{2^{n-2}}}\int_0^U u dG_N(u)$$

Lemma 4.18 *If $M_N > T$ and $n \geq 2$, then the expected reward earned is*

$$E\left[\left(\sum_{n=2}^{\eta}R_nX_n\right)\chi_{(M_N>U)}\right]=\sum_{n=2}^{N-1}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[G_n(U)-G_N(U)].$$

Proof. Examine

$$E\left[\left(\sum_{n=2}^{\eta}R_nX_n\right)\chi_{(M_N>U)}\right]=E\left[\left(\sum_{n=2}^{N}R_nX_n\right)\chi_{(M_N<U<M_N)}\right]$$

$$=\sum_{n=2}^{N}E(R_nX_n)E\left[\chi_{(M_N<U<M_N)}\right]$$

$$=\sum_{n=2}^{N-1}\frac{r\lambda}{\alpha_0^{2^{n-2}}}[G_n(U)-G_N(U)],$$

Lemma 4.19 *If $M_N \leq U$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(M_N \le U)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} G_N(U).$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(M_N \le U)}\right] = E\left[E\left(\sum_{n=1}^{N-1} C_n Y_n | M_N = u\right)\chi_{(M_N \le U)}\right]$$

$$= \int_0^U E\left(\sum_{n=1}^{N-1} C_n Y_n | M_N = u\right) dG_N(u)$$

$$= \int_0^U \sum_{n=1}^{N-1} E(C_n Y_n) dG_N(u),$$

Lemma 4.20 *If $M_N > U$, then the expected repair cost is*

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right)\chi_{(M_N > U)}\right] = \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} [G_n(U) - G_N(U)].$$

Proof. Examine

$$E\left[\left(\sum_{n=1}^{\eta-1} C_n Y_n\right)\chi_{(M_N > U)}\right] = E\left[\left(\sum_{n=1}^{N-1} C_n Y_n\right)\chi_{(M_n < U < M_N)}\right]$$

$$= \sum_{n=1}^{N-1} E(C_n Y_n) E\left[\chi_{(M_n < U < M_N)}\right]$$

$$= \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} [G_n(U) - G_N(U)],$$

The long-run average cost under policy (U^-, N)

The long-run average cost per unit of time for a multistate degenerative system with varying cost structures under the multistate bivariate replacement policy (U^-, N) with partial product process is.

$$C(U, N) = \frac{\text{the expected cost incurred in a cycle}}{\text{the expected length of a cycle}}$$

$$= \frac{\begin{bmatrix} E\{(\sum_{n=1}^{\eta} C_n Y_n - U\sum_{n=1}^{\eta} R_n X_n)\chi_{(M_N>U)}\} + c_p E(Z) \\ +E\{(\sum_{n=1}^{N-1} C_n Y_n - \sum_{n=1}^{N} R_n X_n)\chi_{(M_N\le U)}\} + R \end{bmatrix}}{E(W)}$$

Using Lemmas 4.16 to 4.20, we obtain

$$\mathcal{C}(U,N) = \frac{\begin{bmatrix} \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}}[G_n(U) - G_N(U)] - \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}}[G_n(U) - G_N(U)] + r_1 U \\ + \sum_{n=1}^{N-1} \frac{c\mu}{\beta_0^{2^{n-1}}} G_N(U) - \sum_{n=2}^{N} \frac{r\lambda}{\alpha_0^{2^{n-2}}} \int_0^U u dG_N(u) + c_p\tau + R - r_1\lambda \end{bmatrix}}{\int_0^U \overline{G}_N(u)du + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} G_i(U) + \tau}$$

After combining the aforementioned facts, we arrive at the following conclusion.

Theorem 4.4 *The long run average cost per unit time for a multistate degenerative systemwith varying cost structures under the bivariate replacement policy* (U^-, N) *with partial product process for the model presented in section 3 under the* **A1** *through* **A8** *is provided by.*

$$\mathcal{C}(U^-,N) = \frac{\begin{bmatrix} c\int_0^U u dG_N(u) + \frac{G(U)}{\overline{G}(U)} \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} [G_i(U) - G_N(U)] \\ - r\sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} G_{i-1}(U) + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + c_p\tau + R \end{bmatrix}}{\begin{bmatrix} \sum_{i=1}^{N-1} \frac{\lambda}{\alpha_0^{2^{i-1}}} G_{i-1}(U) + \frac{\lambda}{\alpha_0^{2^{N-1}}} G_N(U) + \int_0^U u dG_N(u) \\ + \frac{G(U)}{\overline{G}(U)} \sum_{i=1}^{N-1} \frac{\mu}{\beta_0^{2^{i-1}}} [G_i(U) - G_N(U)] + \tau \end{bmatrix}}. \quad (21)$$

5. CONCLUSION

By considering a repairable system for a monotone process model of a multi component multistate degenerative system varying cost structures, explicit expressions for the long-run average cost per unit time under a bivariate replacement policies (T,N), (U,N), (T^+,N) and (U^-,N) with partial product process have been derived. Existence of optimal value of has been deduced.

6. REFERENCES

[1] Babu, D., Govindaraju, P. and Rizwan, U. (2018), Partial Product Processes and Replacement Problem, International Journal of Current Advanced Research, 7 (1), pp. 139 – 142.

[2] Barlow, R.E. and Proschan, F. (1965), Mathematical Theory of Reliability, John Willey and Sons, New York.

[3] Barlow, R.E. and Proschan, F. (1974), Statistical Theory of Reliability and Life Testing Probability Models, John Willey and Sons, New York.

[4] Brown, M. and Proschan, F. (1983), Imperfect Repair, Journal of Applied Probability, 20 (4), pp. 851 – 859.

[5] Govindaraju, P., Rizwan, U. and Thangaraj, V. (2009), Bivariate Optimal Replacement Policies for Multistate Degenerative Systems with varying cost structures, International Journal of computational mathematics and numerical simulation, 2 (1), pp. 1 – 10.
[6] Govindaraju, P., Rizwan, U. and Thangaraj, V. (2011), Bivariate Optimal Replacement Policies for Multistate Degenerative Systems, International Journal of Reliability and Safety, 5 (1), pp. 1 – 20.
[7] Lam Yeh, (1988), Geometric Processes and Replacement Problem, Acta Mathematicae Applicatae Sinica, 4, pp. 366 – 377.
[8] Lam Yeh, (1988), A Note on the Optimal Replacement Problem, Advances in Applied Probability, 20 (2), pp. 479 – 482.
[9] Lam Yeh, (2005), A Monotone Process Maintenance Model for a Multistate System, Journal of Applied Probability, 42, pp. 1 – 14.
[10] Lam Yeh and Zhang, Y.L, (2002), A Geometric Process Equivalent Model for a Multistate Degenerative System, European Journal of Operational Research, 142, pp. 21 – 29.
[11] Lesanovsky, A. (1993), System with Two Dual Failure Models - a Survey, Microelectronics and reliability, 33, pp. 1597 – 1626.Leung, K.N.G (2006), A Note on a Bivariate Optimal Replacement Policy for a Repairable System, Engineering Optimization, 38 , pp. 621 – 625.
[12] Raajpandiyan, T.R., Syed Tahir Hussainy and Rizwan, U. (2022), Optimal Replacement Model under Partial Product Process, Stochastic Modeling and Applications, 26 (3), pp. 170 – 176.
[13] Ross, S. M., (1996), Stochastic Processes 2nd edition, John Wiley and Sons, New York.
[14] Shaked, M. and Shantikumar, J.G. (1994), Stochastic Orders and their Applications, Academic Press, New York.
[15] Thangaraj,V.,and Sundararajan, R. (1997), Multivariate Optimal Replacement Policies for a system subject to shocks, Optimization, 41(2), pp. 173 – 195.
[16] Stadje, W. and Zuckerman, D. (1990), Optimal Strategies for some Repair Replacement Models, Advances in Applied Probability, 22 (1), pp. 641 – 656.
[17] Stadje, W. and Zuckerman, D. (1992), Optimal Repair Policies with General Degree of Repair in Two Maintenance Models, Operations Research Letters, 11, pp. 77 – 80.
[18] Stanley, A.D.J. (1993), On Geometric Processes and Repair Replacement Problems, Microelectronics and Reliability, 33, pp. 489 – 491.
[19] Thangaraj, V. and Rizwan, U. (2001), Optimal Replacement Policies in Burn-in Process for an Alternative Repair Model, International Journal of Information and Management Sciences, 12(3), pp. 43 – 56.
[20] Zhang, Y.L. (1994), A Bivariate Optimal Replacement Policy Replacement Policy for a Repairable System, Journal of Applied Probability, 31, pp.1123 – 1127.
[21] Zhang, Y.L., Richard CM Yam and Ming J. Zuo. (2007), A Bivarite Optimal Replacement Policy for a Multistate Repairable System, Reliability Engineering and System Safety, 92(4), pp. 535 – 542.
[22] Zhang, Y.L. and Wang, G. J. (2010), An Optimal Replacement Policy for a Multistate Degenerative Simple System, Applied Mathematical Modelling, 34 (12), pp. 4138 – 4150.