Multivariate Ostrowski-Sugeno Fuzzy inequalities

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Abstract

Here we present multivariate Ostrowski-Sugeno Fuzzy type inequalities. These are multivariate Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties. They give tight upper bounds to the deviation of a multivariate function from its Sugeno-fuzzy multivariate averages.

2010 Mathematics Subject Classification: Primary: 26D07, 26D10, 26D15, 41A44, Secondary: 26A24, 26D20, 28A25.

Keywords and phrases: Sugeno fuzzy integral, multivariate function fuzzy average, deviation from fuzzy multivariate mean, Fuzzy multivariate Ostrowski inequality.

1 Introduction

The famous Ostrowski ([4]) inequality motivates this work and has as follows:

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\left\|f'\right\|_\infty ,$$

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Another motivation comes from author's [2], pp. 507-508, see also [1]:

Let $f \in C^1\left(\prod_{i=1}^{k}[a_i,b_i]\right)$, where $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1,...,k$, and let $x_0 := (x_{01},...,x_{0k}) \in \prod_{i=1}^{k}[a_i,b_i]$ be fixed. Then

$$\left|\frac{1}{\prod_{i=1}^{k}(b_i-a_i)}\int_{a_1}^{b_1}...\int_{a_i}^{b_i}...\int_{a_k}^{b_k} f(z_1,...,z_k)\,dz_1...dz_k - f(x_0)\right| \leq$$

$$\sum_{i=1}^{k}\left(\frac{(x_{0i}-a_i)^2+(b_i-x_{0i})^2}{2(b_i-a_i)}\right)\left\|\frac{\partial f}{\partial z_i}\right\|_{\infty}.$$

The last inequality is sharp, the optimal function is

$$f^*(z_1,...,z_k):=\sum_{i=1}^{k}|z_i-x_{0i}|^{\alpha_i},\ \ \alpha_i>1.$$

Here first we give a survey about Sugeno fuzzy integral and its basic special properties. Then we derive a set of multivariate Ostrowski-like inequalities to all directions in the context of Sugeno integral whitin its basic important properties. We finish with an application to a special multivariate case.

2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented. Also a preparation for the main results Section 3 is given.

Definition 1 *(Fuzzy measure [6, 8]) Let Σ be a σ-algebra of subsets of X, and let $\mu:\Sigma\to[0,+\infty]$ be a non-negative extended real-valued set function. We say that μ is a fuzzy measure iff:*
(1) $\mu(\varnothing)=0$,
(2) $E,F\in\Sigma: E\subseteq F$ imply $\mu(E)\le\mu(F)$ (monotonicity),
(3) $E_n\in\Sigma$ ($n\in\mathbb{N}$), $E_1\subset E_2\subset ...$, imply $\lim_{n\to\infty}\mu(E_n)=\mu(\cup_{n=1}^{\infty}E_n)$ (continuity from below);
(4) $E_n\in\Sigma$ ($n\in\mathbb{N}$), $E_1\supset E_2\supset ...$, $\mu(E_1)<\infty$, imply $\lim_{n\to\infty}\mu(E_n)=\mu(\cap_{n=1}^{\infty}E_n)$ (continuity from above).

Let (X,Σ,μ) be a fuzzy measure space and f be a non-negative real-valued function on X. We denote by $\mathcal{F}_+$ the set of all non-negative real valued measurable functions, and by $L_\alpha f$ the set: $L_\alpha f:=\{x\in X: f(x)\ge\alpha\}$, the α-level of f for $\alpha\ge0$.

Definition 2 *Let (X,Σ,μ) be a fuzzy measure space. If $f\in\mathcal{F}_+$ and $A\in\Sigma$, then the Sugeno integral (fuzzy integral) [7] of f on A with respect to the fuzzy measure μ is defined by*

$$(S)\int_A f d\mu:=\vee_{\alpha\ge0}(\alpha\wedge\mu(A\cap L_\alpha f)),\tag{1}$$

where $\vee$ and $\wedge$ denote the sup and inf on $[0,\infty]$, respectively.

The basic properties of Sugeno integral follow:

Theorem 3 *([5, 8]) Let (X, Σ, μ) be a fuzzy measure space with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. Then*

1) $(S)\int_A f d\mu \leq \mu(A)$;
2) $(S)\int_A k d\mu = k \wedge \mu(A)$ for a non-negative constant k;
3) if $f \leq g$ on A, then $(S)\int_A f d\mu \leq (S)\int_A g d\mu$;
4) if $A \subset B$, then $(S)\int_A f d\mu \leq (S)\int_B f d\mu$;
5) $\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S)\int_A f d\mu \leq \alpha$;
6) if $\mu(A) < \infty$, then $\mu(A \cap L_\alpha f) \geq \alpha \Leftrightarrow (S)\int_A f d\mu \geq \alpha$;
7) when $A = X$, then $(S)\int_A f d\mu = \vee_{\alpha \geq 0}(\alpha \wedge \mu(L_\alpha f))$;
8) if $\alpha \leq \beta$, then $L_\beta f \subseteq L_\alpha f$;
9) $(S)\int_A f d\mu \geq 0$.

Theorem 4 *([8], p. 135) Here $f \in \mathcal{F}_+$, the class of all finite nonnegative measurable functions on (X, Σ, μ). Then*

1) if $\mu(A) = 0$, then $(S)\int_A f d\mu = 0$, for any $f \in \mathcal{F}_+$;
2) if $(S)\int_A f d\mu = 0$, then $\mu(A \cap \{x | f(x) > 0\}) = 0$;
3) $(S)\int_A f d\mu = (S)\int_X f \cdot \chi_A d\mu$, where χ_A is the characteristic function of A;
4) $(S)\int_A (f + a) d\mu \leq (S)\int_A f d\mu + (S)\int_A a d\mu$, for any constant $a \in [0, \infty)$.

Corollary 5 *([8], p. 136) Here $f, f_1, f_2 \in \mathcal{F}_+$. Then*

1) $(S)\int_A (f_1 \vee f_2) d\mu \geq (S)\int_A f_1 d\mu \vee (S)\int_A f_2 d\mu$;
2) $(S)\int_A (f_1 \wedge f_2) d\mu \leq (S)\int_A f_1 d\mu \wedge (S)\int_A f_2 d\mu$;
3) $(S)\int_{A \cup B} f d\mu \geq (S)\int_A f d\mu \vee (S)\int_B f d\mu$;
4) $(S)\int_{A \cap B} f d\mu \leq (S)\int_A f d\mu \wedge (S)\int_B f d\mu$.

In general we have

$$(S)\int_A (f_1 + f_2) d\mu \neq (S)\int_A f_1 d\mu + (S)\int_A f_2 d\mu,$$

and

$$(S)\int_A a f d\mu \neq a (S)\int_A f d\mu, \text{ where } a \in \mathbb{R},$$

see [8], p. 137.

Lemma 6 *([8], p. 138) $(S)\int_A f d\mu = \infty$ iff $\mu(A \cap L_\alpha f) = \infty$ for any $\alpha \in [0, \infty)$.*

We need

Definition 7 *([3]) A fuzzy measure μ is subadditive iff $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for all $A, B \in \Sigma$.*

We mention

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Theorem 8 *([3]) If μ is subadditive, then*

$$(S)\int_X (f+g)\, d\mu \leq (S)\int_X f d\mu + (S)\int_X g d\mu, \tag{2}$$

for all measurable functions $f, g : X \to [0,\infty)$.

Moreover, if (2) holds for all measurable functions $f, g : X \to [0,\infty)$ and $\mu(X) < \infty$, then μ is subadditive.

Notice here in (1) we have that $\alpha \in [0,\infty)$.

We have

Corollary 9 *If μ is aubadditive, $n \in \mathbb{N}$, and $f : X \to [0,\infty)$ is a measurable function, then*

$$(S)\int_X nf d\mu \leq n\,(S)\int_X f d\mu, \tag{3}$$

in particular it holds

$$(S)\int_A nf d\mu \leq n\,(S)\int_A f d\mu, \tag{4}$$

for any $A \in \Sigma$.

Proof. By (2). ■

A very important property of Sugeno integral follows.

Theorem 10 *If μ is subadditive measure, and $f : X \to [0,\infty)$ is a measurable function, and $c > 0$, then*

$$(S)\int_A cf d\mu \leq (c+1)\,(S)\int_A f d\mu, \tag{5}$$

for any $A \in \Sigma$.

Proof. Let the ceiling $\lceil c \rceil = m \in \mathbb{N}$, then by Theorem 3 (3) and (4) we get

$$(S)\int_A cf d\mu \leq (S)\int_A mf d\mu \leq m\,(S)\int_A f d\mu \leq (c+1)\,(S)\int_A f d\mu,$$

proving (5). ■

From now on in this article we work on the fuzzy measure space $(Q, \mathcal{B}, \mu)$, where $Q \subset \mathbb{R}^k$, $k \geq 1$ is a convex compact subset, $\mathcal{B}$ is the Borel σ-algebra on Q, and μ is a finite fuzzy measure on $\mathcal{B}$. Typically we take it to be subadditive.

The functions f we deal with here are continuous from Q into $\mathbb{R}_+$.

We make

Remark 11 *Let $f \in C(Q, \mathbb{R}_+)$, and μ is a subadditive fuzzy measure such that $\mu(Q) > 0$, $x \in Q$. We will estimate*

$$E(x) := \left| (S)\int_Q f(t)\, d\mu(t) - \mu(Q) \wedge f(x) \right| \tag{6}$$

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(by Theorem 3 (2))

$$= \left| (S) \int_Q f(t)\, d\mu(t) - (S) \int_Q f(x)\, d\mu(t) \right|.$$

We notice that

$$f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x),$$

then (by Theorem 3 (3) and Theorem 4 (4))

$$(S) \int_Q f(t)\, d\mu(t) \leq (S) \int_Q |f(t) - f(x)|\, d\mu(t) + (S) \int_Q f(x)\, d\mu(t), \quad (7)$$

that is

$$(S) \int_Q f(t)\, d\mu(t) - (S) \int_Q f(x)\, d\mu(t) \leq (S) \int_Q |f(t) - f(x)|\, d\mu(t). \quad (8)$$

Similarly, we have

$$f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t),$$

then (by Theorem 3 (3) and Theorem 8)

$$(S) \int_Q f(x)\, d\mu(t) \leq (S) \int_Q |f(t) - f(x)|\, d\mu(t) + (S) \int_Q f(t)\, d\mu(t),$$

that is

$$(S) \int_Q f(x)\, d\mu(t) - (S) \int_Q f(t)\, d\mu(t) \leq (S) \int_Q |f(t) - f(x)|\, d\mu(t). \quad (9)$$

By (8) and (9) we derive that

$$\left| (S) \int_Q f(t)\, d\mu(t) - (S) \int_Q f(x)\, d\mu(t) \right| \leq (S) \int_Q |f(t) - f(x)|\, d\mu(t). \quad (10)$$

Consequently it holds

$$E(x) \overset{(by\ (6),\ (10))}{\leq} (S) \int_Q |f(t) - f(x)|\, d\mu(t), \quad (11)$$

where $t = (t_1, ..., t_k)$, $x = (x_1, ..., x_k)$.

We will use (11).

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3 Main Results

We make

Remark 12 *Here* $Q := \prod_{i=1}^{k} [a_i, b_i]$, *where* $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1, ..., k$; $x = (x_1, ..., x_k) \in \prod_{i=1}^{k} [a_i, b_i]$ *is fixed, and* $f \in C^1 \left(\prod_{i=1}^{k} [a_i, b_i], \mathbb{R}_+ \right)$. *Consider* $g_t(r) := f(x + r(t - x))$, $r \geq 0$. *Note that* $g_t(0) = f(x)$, $g_t(1) = f(t)$. *Thus*

$$f(t) - f(x) = g_t(1) - g_t(0) = g_t'(\xi)(1 - 0) = g_t'(\xi), \tag{12}$$

where $\xi \in (0, 1)$.

I.e.

$$f(t) - f(x) = \sum_{i=1}^{k} (t_i - x_i) \frac{\partial f}{\partial t_i} (x + \xi(t - x)). \tag{13}$$

Hence

$$|f(t) - f(x)| \leq \sum_{i=1}^{k} |t_i - x_i| \left| \frac{\partial f}{\partial t_i} (x + \xi(t - x)) \right|$$

$$\leq \sum_{i=1}^{k} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_{\infty}. \tag{14}$$

By (11) we get

$$\left| (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} f(t) \, d\mu(t) - \mu\left(\prod_{i=1}^{k} [a_i, b_i] \right) \wedge f(x) \right| \leq$$

$$(S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |f(t) - f(x)| \, d\mu(t) \overset{(14)}{\leq}$$

$$(S) \int_{\prod_{i=1}^{k} [a_i, b_i]} \left(\sum_{i=1}^{k} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_{\infty} \right) d\mu(t) \overset{(2)}{\leq}$$

$$\sum_{i=1}^{k} (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_{\infty} d\mu(t) \overset{(5)}{\leq}$$

$$\sum_{i=1}^{k} \left(\left\| \frac{\partial f}{\partial t_i} \right\|_{\infty} + 1 \right) \left((S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |t_i - x_i| \, d\mu(t) \right). \tag{15}$$

Here μ *is a fuzzy subadditive measure with* $\mu\left(\prod_{i=1}^{k} [a_i, b_i] \right) > 0$.

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Therefore we get

$$\left| \frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right| \overset{(15)}{\leq} \tag{16}$$

$$\sum_{i=1}^{k}\left(\frac{\left\|\frac{\partial f}{\partial t_i}\right\|_{\infty}+1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\left((S)\int_{\prod_{i=1}^{k}[a_i,b_i]}|t_i-x_i|\,d\mu(t)\right).$$

Notice here $\left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right) \leq 1$, *and*

$\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) \overset{(by\ Thm.\ 3\ (1))}{\leq} 1$,
where $(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) \geq 0$.

If $f : \prod_{i=1}^{k}[a_i,b_i] \to \mathbb{R}_+$ *is a Lipschitz function of order* $0 < \alpha \leq 1$, *i.e.* $|f(x)-f(y)| \leq K\|x-y\|_{l_1}^{\alpha}$, $\forall\ x,y \in \prod_{i=1}^{k}[a_i,b_i]$, $K > 0$, *where* $\|x-y\|_{l_1} := \sum_{i=1}^{k}|x_i-y_i|$, *denoted by* $f \in Lip_{\alpha,K}\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, *then by (11) we get*

$$\left|(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \mu\left(\prod_{i=1}^{k}[a_i,b_i]\right) \wedge f(x)\right| \leq \tag{17}$$

$$(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} |f(t)-f(x)|\,d\mu(t) \leq$$

$$(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} K\|t-x\|_{l_1}^{\alpha}\,d\mu(t) \overset{(5)}{\leq}$$

$$(K+1)(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} \|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We have proved

$$\left| \frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right| \leq \tag{18}$$

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$$\frac{(K+1)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}\|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We have established the following multivariate Ostrowski-Sugeno inequalities.

Theorem 13 *Here μ is a fuzzy subadditive measure with $\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)>0$, $x\in\prod_{i=1}^{k}[a_i,b_i]$.*

1) Let $f\in C^1\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, then

$$\left|\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right|\leq \tag{19}$$

$$\sum_{i=1}^{k}\left(\frac{\left\|\frac{\partial f}{\partial t_i}\right\|_{\infty}+1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\left((S)\int_{\prod_{i=1}^{k}[a_i,b_i]}|t_i-x_i|\,d\mu(t)\right).$$

2) Let $f\in Lip_{\alpha,K}\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, $0<\alpha\leq 1$, then

$$\left|\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right|\leq \tag{20}$$

$$\frac{(K+1)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}\|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We make

Remark 14 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k\geq 1$. Let $f\in\left(C(Q,\mathbb{R}_+)\cap C^{n+1}(Q)\right)$, $n\in\mathbb{N}$ and $x\in Q$ is fixed such that all partial derivatives $f_\alpha:=\frac{\partial^\alpha f}{\partial t^\alpha}$, where $\alpha=(\alpha_1,...,\alpha_k)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,k$, $|\alpha|=\sum_{i=1}^{\kappa}\alpha_i=j$, $j=1,...,n$ fulfill $f_\alpha(x)=0$.*

By [2], p. 513, we get that

$$|f(t)-f(x)|\leq\frac{\left[\left(\sum_{i=1}^{k}|t_i-x_i|\left\|\frac{\partial}{\partial t_i}\right\|_{\infty}\right)^{n+1}f\right]}{(n+1)!},\quad\forall\ t\in Q. \tag{21}$$

Call

$$D_{n+1}(f) := \max_{\alpha:|\alpha|=n+1} \|f_\alpha\|_\infty . \tag{22}$$

For example, when $k=2$ and $n=1$, we get that

$$\left[\left(\sum_{i=1}^{2} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^2 f\right] =$$

$$(t_1 - x_1)^2 \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty + 2|t_1 - x_1||t_2 - x_2| \left\|\frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_\infty + (t_2 - x_2)^2 \left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty , \tag{23}$$

and

$$D_2(f) = \max_{\alpha:|\alpha|=2} \|f_\alpha\|_\infty . \tag{24}$$

Clearly, it holds

$$\left[\left(\sum_{i=1}^{2} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^2 f\right] \leq D_2(f)\left(|t_1 - x_1| + |t_2 - x_2|\right)^2 . \tag{25}$$

Consequently, we derive that

$$\left[\left(\sum_{i=1}^{k} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^{n+1} f\right] \leq D_{n+1}(f) \|t - x\|_{l_1}^{n+1}, \quad \forall\ t \in Q. \tag{26}$$

By (11) we get

$$\left|(S)\int_Q f(t)\,d\mu(t) - \mu(Q) \wedge f(x)\right| \leq$$

$$(S)\int_Q |f(t) - f(x)|\,d\mu(t) \overset{(21)}{\leq} \tag{27}$$

$$(S)\int_Q \frac{\left[\left(\sum_{i=1}^{k} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^{n+1} f\right]}{(n+1)!} d\mu(t) \overset{(26)}{\leq}$$

$$(S)\int_Q \frac{D_{n+1}(f) \|t - x\|_{l_1}^{n+1}}{(n+1)!} d\mu(t) \overset{(5)}{\leq} \tag{28}$$

$$\left(\frac{D_{n+1}(f)}{(n+1)!} + 1\right)(S)\int_Q \|t - x\|_{l_1}^{n+1} d\mu(t) .$$

Here μ is a fuzzy subadditive measure with $\mu(Q) > 0$.

By (27) and (28) we obtain

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!}+1\right)}{\mu(Q)}(S)\int_Q \|t-x\|_{l_1}^{n+1} d\mu(t). \tag{29}$$

We have established the following multivariate Ostrowski-Sugeno general inequality:

Theorem 15 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 1$. Let $f \in \left(C(Q,\mathbb{R}_+) \cap C^{n+1}(Q)\right)$, $n \in \mathbb{N}$, $x \in Q$ be fixed: $f_\alpha(x) = 0$, all $\alpha : |\alpha| = j$, $j = 1, ..., n$. Here μ is a fuzzy subadditive measure with $\mu(Q) > 0$. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!}+1\right)}{\mu(Q)}(S)\int_Q \|t-x\|_{l_1}^{n+1} d\mu(t). \tag{30}$$

Corollary 16 *All as in Theorem 15. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(1+\frac{1}{(n+1)!}\right)}{\mu(Q)}(S)\int_Q \left[\left(\sum_{i=1}^k |t_i - x_i| \left\|\frac{\partial}{\partial x_i}\right\|_\infty\right)^{n+1} f\right] d\mu(t). \tag{31}$$

Next we take again $Q := \prod_{i=1}^k [a_i, b_i]$, we set $a := (a_1, ..., a_k)$, $b := (b_1, ..., b_k)$, and $\frac{a+b}{2} = \left(\frac{a_1+b_1}{2}, ..., \frac{a_k+b_k}{2}\right) \in \prod_{i=1}^k [a_i, b_i]$.

Corollary 17 *Let $f \in \left(C\left(\prod_{i=1}^k [a_i, b_i], \mathbb{R}_+\right) \cap C^{n+1}\left(\prod_{i=1}^k [a_i, b_i]\right)\right)$, $n \in \mathbb{N}$, such that $f_\alpha\left(\frac{a+b}{2}\right) = 0$, all $\alpha : |\alpha| = j$, $j = 1, ..., n$. Here μ is a fuzzy subadditive measure with $\mu\left(\prod_{i=1}^k [a_i, b_i]\right) > 0$. Then*

$$\left|\frac{1}{\mu\left(\prod_{i=1}^k [a_i, b_i]\right)}(S)\int_{\prod_{i=1}^k [a_i, b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f\left(\frac{a+b}{2}\right)}{\mu\left(\prod_{i=1}^k [a_i, b_i]\right)}\right)\right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!}+1\right)}{\mu\left(\prod_{i=1}^k [a_i, b_i]\right)}(S)\int_{\prod_{i=1}^k [a_i, b_i]} \left\|t - \frac{a+b}{2}\right\|_{l_1}^{n+1} d\mu(t). \tag{32}$$

Proof. By Theorem 15. ■

We make

Remark 18 *By multinomial theorem we have that*

$$\|t-x\|_{l_1}^{n+1} = \left(\sum_{i=1}^{k} |t_i - x_i|\right)^{n+1} =$$

$$\sum_{r_1+r_2+...+r_k=n+1} \binom{n+1}{r_1, r_2, ..., r_k} |t_1 - x_1|^{r_1} |t_2 - x_2|^{r_2} ... |t_k - x_k|^{r_k}, \qquad (33)$$

where

$$\binom{n+1}{r_1, r_2, ..., r_k} = \frac{(n+1)!}{r_1! r_2! ... r_k!}. \qquad (34)$$

By (27), (28) we get

$$\left|(S)\int_Q f(t)\, d\mu(t) - \mu(Q) \wedge f(x)\right| \leq$$

$$(S)\int_Q \frac{D_{n+1}(f)}{(n+1)!} \|t-x\|_{l_1}^{n+1} d\mu(t) \overset{(by\ (33),\ (34))}{=}$$

$$(S)\int_Q \left[\sum_{r_1+r_2+...+r_k=n+1} \left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!}\right) \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right)\right] d\mu(t) \overset{(2)}{\leq}$$

$$\sum_{r_1+r_2+...+r_k=n+1} (S)\int_Q \left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!}\right) \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right) d\mu(t) \overset{(5)}{\leq}$$

$$\sum_{r_1+r_2+...+r_k=n+1} \left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} + 1\right) (S)\int_Q \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right) d\mu(t). \qquad (35)$$

We have proved the following multivariate Ostrowski-Sugeno general inequality:

Theorem 19 *Here all as in Theorem 15. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\sum_{r_1+r_2+...+r_k=n+1} \left(\frac{\left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} + 1\right)}{\mu(Q)}\right) (S)\int_Q \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right) d\mu(t). \qquad (36)$$

We make

Remark 20 *In case $k=2$, $n=1$, by (27), (28) we get*

$$\left|(S)\int_Q f(t)\,d\mu(t)-\mu(Q)\wedge f(x)\right|\leq$$

$$(S)\int_Q \frac{D_2(f)}{2}\|t-x\|_{l_1}^2\,d\mu(t)=$$

$$(S)\int_Q \frac{D_2(f)}{2}\left[(t_1-x_1)^2+2|t_1-x_1||t_2-x_2|+(t_2-x_2)^2\right]d\mu(t)\leq \quad (37)$$

$$(S)\int_Q \frac{D_2(f)}{2}(t_1-x_1)^2\,d\mu(t)+(S)\int_Q D_2(f)|t_1-x_1||t_2-x_2|\,d\mu(t)$$

$$+(S)\int_Q \frac{D_2(f)}{2}(t_2-x_2)^2\,d\mu(t)\leq$$

$$\left(1+\frac{D_2(f)}{2}\right)(S)\int_Q (t_1-x_1)^2\,d\mu(t)+(1+D_2(f))(S)\int_Q |t_1-x_1||t_2-x_2|\,d\mu(t)$$

$$+\left(1+\frac{D_2(f)}{2}\right)(S)\int_Q (t_2-x_2)^2\,d\mu(t).$$

We have proved

Corollary 21 *Let Q be a compact and convex subset of $\mathbb{R}^2$. Let $f\in(C(Q,\mathbb{R}_+)\cap C^2(Q))$, $x=(x_1,x_2)\in Q$ be fixed: $\frac{\partial f}{\partial t_1}(x_1,x_2)=\frac{\partial f}{\partial t_2}(x_1,x_2)=0$. Here μ is a fuzzy subadditive measure with $\mu(Q)>0$. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu(Q)}\right)\right|\leq$$

$$\frac{\left(1+\frac{D_2(f)}{2}\right)}{\mu(Q)}(S)\int_Q (t_1-x_1)^2\,d\mu(t)+\frac{(1+D_2(f))}{\mu(Q)}(S)\int_Q |t_1-x_1||t_2-x_2|\,d\mu(t)$$

$$+\frac{\left(1+\frac{D_2(f)}{2}\right)}{\mu(Q)}(S)\int_Q (t_2-x_2)^2\,d\mu(t). \quad (38)$$

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