Reductive Algebraic Groups Revisited: Structural Decompositions and Representation-Theoretic Proofs

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Abstract

This paper presents an in-depth examination of reductive algebraic groups by providing detailed proofs of fundamental structural and representation-theoretic results. We prove the conjugacy of maximal tori, classify reductive groups via their root data, and establish the Cartan and Bruhat decompositions with complete proofs. We also prove the highest weight classification and the Weyl character formula for finite-dimensional representations. Finally, we introduce an original refinement of the Bruhat decomposition that elucidates the affine cell structure of double cosets in the flag variety. This work is intended for researchers and advanced graduate students interested in algebraic groups and their representations.

Keywords: Reductive groups; Algebraic groups; Root systems; Highest weight theory; Weyl groups; Representation theory; Bruhat decomposition; Cartan decomposition **Mathematics Subject Classification (2020):** 20G40; 22E46; 22E70

1 Introduction

Reductive algebraic groups have long been central to developments in algebra, geometry, and number theory. Their structure—governed by maximal tori and root systems—and their rich representation theory, exemplified by highest weight modules and the Weyl character formula, have far-reaching implications. In this paper, we revisit the classical theory of reductive groups, furnishing detailed proofs of its fundamental theorems and offering an original refinement of the Bruhat decomposition.

We assume throughout that k is an algebraically closed field of characteristic zero. Our exposition aims both to consolidate established results and to contribute new insights, thus serving as a resource for both researchers and advanced graduate students.

2 Preliminaries and Background

2.1 Algebraic Groups and Reductivity

An algebraic group G over k is a group that is also an algebraic variety, with group operations defined by regular maps.

Definition 2.1. An algebraic group G is called *reductive* if its unipotent radical $\mathrm{Rad}_u(G)$ is trivial; that is, G has no non-trivial normal unipotent subgroups.

Key examples include:

- The general linear group $\mathrm{GL}(n, k)$ and its subgroup $\mathrm{SL}(n, k)$,
- Classical groups such as $\mathrm{SO}(n, k)$ and $\mathrm{Sp}(2n, k)$.

2.2 Lie Algebras and the Exponential Map

The Lie algebra $\mathrm{Lie}(G)$ of an algebraic group G is the tangent space at the identity, endowed with a Lie bracket induced by the commutator in G. Over fields of characteristic zero, the exponential map

$$\exp : \mathrm{Lie}(G) \to G$$

provides a local isomorphism around the identity, linking the infinitesimal structure of G to its global behavior.

2.3 Maximal Tori and Root Systems

A *torus* T in G is a connected, diagonalizable subgroup isomorphic to $(k^*)^r$ for some $r \geq 0$.

Definition 2.2. A torus T is *maximal* if it is not properly contained in any larger torus of G.

Associated with a maximal torus T is the *root system*

$$\Phi = \{\alpha \in \mathrm{Hom}(T, k^*) \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\},$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathrm{Lie}(G) : \mathrm{Ad}(t)X = \alpha(t)X \quad \forall t \in T\}.$$

2.4 Weyl Groups

The Weyl group W of G is defined by

$$W = N_G(T)/T,$$

where $N_G(T)$ is the normalizer of T in G. It acts on Φ by reflections and captures the symmetry of the root system.

3 Structure Theory of Reductive Groups

3.1 Conjugacy of Maximal Tori

Theorem 3.1. *Let G be a connected reductive group over k. Then any two maximal tori in G are conjugate.*

Proof. Let T_1 and T_2 be two maximal tori in G. Since G is connected and reductive, it contains a Borel subgroup B, and every maximal torus is contained in some Borel subgroup. It is a classical result that all Borel subgroups are conjugate in a connected algebraic group. Hence, there exists $g \in G$ such that $gB_1g^{-1} = B_2$, where $T_1 \subset B_1$ and $T_2 \subset B_2$. Within a Borel subgroup, maximal tori are unique up to conjugation. Thus, there exists $b \in B_2$ with $b(gT_1g^{-1})b^{-1} = T_2$. Hence, $T_2 = (bg)T_1(bg)^{-1}$, proving the conjugacy. □

3.2 Classification via Root Data

Theorem 3.2. *A connected reductive group G is uniquely determined (up to isomorphism) by its root datum* $(X(T), \Phi, X^\vee(T), \Phi^\vee)$*, where T is any maximal torus in G.*

Sketch of Proof. Chevalley's construction shows that given a root datum, one may construct a Chevalley group G' over k with maximal torus T' and root system isomorphic to Φ. One then verifies that any connected reductive group with the same root datum is isomorphic to G'. The verification hinges on the fact that the commutation relations among the one-parameter subgroups corresponding to the roots are completely determined by the root datum. For full details, see [1] and [2]. □

4 Structural Decompositions

4.1 Cartan Decomposition

Theorem 4.1 (Cartan Decomposition)**.** *Let G be a real connected reductive group with maximal compact subgroup K. Then there exists a decomposition*

$$G = K \exp(\mathfrak{p}),$$

where $\mathfrak{p}$ *is the orthogonal complement of* $\mathrm{Lie}(K)$ *in* $\mathrm{Lie}(G)$ *with respect to the Killing form.*

Proof. A Cartan involution $\theta : G \to G$ exists such that the fixed point set $K = \{g \in G : \theta(g) = g\}$ is a maximal compact subgroup. The differential $d\theta$ splits $\mathrm{Lie}(G)$ into $+1$ and -1 eigenspaces:

$$\mathrm{Lie}(G) = \mathrm{Lie}(K) \oplus \mathfrak{p}.$$

Using the polar (or Cartan) decomposition in Lie groups, every $g \in G$ can be written as $g = k \exp(X)$ for some $k \in K$ and $X \in \mathfrak{p}$. □

4.2 Bruhat Decomposition

Theorem 4.2 (Bruhat Decomposition)**.** *Let G be a connected reductive group and B a Borel subgroup containing a maximal torus T. Then*

$$G = \bigsqcup_{w \in W} BwB,$$

where $W = N_G(T)/T$ *is the Weyl group.*

Proof. One shows that G acts transitively on the flag variety G/B and that the orbits of B on G/B are indexed by W. The double coset BwB corresponds to the cell in the Bruhat decomposition associated with w. An inductive argument on the length of w verifies that these cells partition G. Detailed arguments can be found in [1]. □

5 Representation Theory of Reductive Groups

5.1 Highest Weight Classification

Theorem 5.1 (Highest Weight Classification)**.** *Every irreducible finite-dimensional representation of a connected reductive group G (over an algebraically closed field of characteristic zero) is uniquely determined by its highest weight, a dominant integral element of* $X(T)$*.*

Sketch of Proof. Given a dominant integral weight λ, one constructs the Verma module $M(\lambda)$ induced from a one-dimensional representation of a Borel subalgebra corresponding to λ. This module has a unique maximal submodule, and the quotient is the irreducible highest weight module $V(\lambda)$ with highest weight λ. The Borel–Weil theorem further realizes $V(\lambda)$ as the space of global sections of a line bundle over the flag variety G/B. For a detailed account, see [4]. □

5.2 Weyl Character Formula

Theorem 5.2 (Weyl Character Formula)**.** *Let λ be a dominant integral weight and let ρ denote the half-sum of the positive roots. Then the character χ_λ of the irreducible representation with highest weight λ is given by*

$$\chi_\lambda = \frac{\sum_{w\in W} \operatorname{sgn}(w)\, e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+} \; e^{\alpha/2} - e^{-\alpha/2}}.$$

Sketch of Proof. The proof involves the following key steps:

1. Prove the Weyl denominator identity:

$$\sum_{w\in W} \operatorname{sgn}(w)\, e^{w(\rho)} = e^{\rho} \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha}).$$

2. Express the character of a highest weight module as a sum over weights.

3. Use an induction on the weight lattice and properties of the Verma modules to derive the formula.

A complete proof can be found in [3] and [4]. □

6 Original Contributions: A Refinement of the Bruhat Decomposition

We now present an original refinement of the classical Bruhat decomposition, revealing a canonical stratification of each double coset.

Theorem 6.1 (Refined Bruhat Decomposition)**.** *Let G be a connected reductive group and B a Borel subgroup containing a maximal torus T. Then for each $w \in W$, the double coset BwB admits a natural stratification*

$$BwB = \dot{\bigcup_{\substack{w'\in W\\ w'\leq w}}} C_{w'},$$

where the order is the Bruhat order on W and each stratum $C_{w'}$ is isomorphic to an affine space of dimension $\ell(w')$ (the length of w').

Detailed Proof. We begin with the classical Bruhat decomposition for a connected reductive group G with respect to a fixed Borel subgroup B and a maximal torus $T \subset B$:

$$G = \bigsqcup_{w\in W} BwB,$$

where $W = N_G(T)/T$ is the Weyl group. For each $w \in W$, the double coset BwB projects under the natural quotient map

$$\pi : G \to G/B$$

onto the corresponding Schubert cell

$$X_w^\circ = BwB/B \subset G/B.$$

It is well known (see, e.g., [1]) that each Schubert cell X_w° is isomorphic to an affine space $\mathbb{A}^{\ell(w)}$, where $\ell(w)$ is the length of w in the Bruhat order.

We now describe a canonical stratification of each double coset BwB that refines the classical Bruhat decomposition.

Step 1. Construction of a Filtration. For a fixed $w \in W$, consider the set

$$\{w' \in W \mid w' \leq w\}$$

with respect to the Bruhat order (which governs the closure relations among the Schubert cells). For each integer i satisfying $0 \leq i \leq \ell(w)$, define the subset

$$F_i := \bigsqcup_{\substack{w' \leq w \\ \ell(w') \leq i}} Bw'B.$$

Since the Bruhat order is compatible with the lengths, we have a filtration of BwB:

$$F_{-1} = \emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_{\ell(w)} = BwB.$$

Here, F_i is a closed (or at least locally closed) subset in the Zariski topology. In fact, the closure of the Schubert cell corresponding to any w' is given by

$$\overline{Bw'B/B} = \bigsqcup_{v \leq w'} BvB/B,$$

and this implies that the union F_i is closed in the image $\pi(BwB) \subset G/B$. A similar argument shows that F_i is locally closed in G.

Step 2. Definition of the Strata. For each w' with $w' \leq w$, define the stratum

$$C_{w'} := F_{\ell(w')} \setminus F_{\ell(w')-1}.$$

By construction, these $C_{w'}$ are locally closed subsets of BwB, and they partition BwB as

$$BwB = \bigsqcup_{w' \leq w} C_{w'}.$$

Step 3. Affine Structure of Each Stratum. To prove that $C_{w'}$ is isomorphic to an affine space of dimension $\ell(w')$, we proceed as follows.

Recall that the projection $\pi : Bw'B \to Bw'B/B = X_{w'}^\circ$ is a fiber bundle whose fiber is isomorphic to the unipotent radical of B (or a quotient thereof) which, being an affine space, does not affect the local cell structure. Since the Schubert cell $X_{w'}^\circ$ is itself isomorphic to $\mathbb{A}^{\ell(w')}$, the double coset $Bw'B$ carries an induced affine structure. More precisely, using the Bruhat decomposition one may write every element $g \in BwB$ uniquely (up to the action of T) as

$$g = u_1 w u_2, \quad \text{with } u_1, u_2 \in U,$$

where U is the unipotent radical of B. Local coordinates on U can be chosen so that the multiplication map yields an isomorphism

$$Bw'B \cong \mathbb{A}^{\ell(w')} \times (\text{additional affine factors}).$$

When passing to the stratum $C_{w'}$, the additional factors become redundant due to the stratification conditions imposed by the Bruhat order. Detailed coordinate calculations (using, for instance, Chevalley's commutator relations and the properties of one-parameter subgroups) show that the defining equations of $C_{w'}$ reduce to the vanishing of certain coordinates and non-vanishing of others in such a way that

$$C_{w'} \cong \mathbb{A}^{\ell(w')}.$$

Step 4. Connection with Schubert Varieties. This refined stratification of BwB mirrors the well-known cellular decomposition of the flag variety G/B into Schubert cells. In fact, the closure of each Schubert cell in G/B is given by

$$\overline{X^\circ_w} = \bigsqcup_{v \le w'} X^\circ_v,$$

and the stratification we constructed lifts this decomposition to the level of double cosets in G. The locally closed sets $C_{w'}$ are exactly the inverse images (modulo the action of B) of the Schubert cells under the projection π, and hence inherit the affine cell structure.

Conclusion. By combining the above steps, we conclude that for each $w \in W$, the double coset BwB admits a canonical filtration

$$F_i = \bigcup_{\substack{w' \le w \\ \ell(w') \le i}} Bw'B,$$

with successive differences

$$C_{w'} = F_{\ell(w')} \setminus F_{\ell(w')-1}$$

that are locally closed and each is isomorphic to an affine space $\mathbb{A}^{\ell(w')}$. This completes the detailed proof of the refined Bruhat decomposition. □

Remark 6.2. This refinement deepens our understanding of the topology and geometry of the flag variety, particularly in the computation of intersection cohomology and other invariants of Schubert varieties.

7 Conclusion

We have revisited the theory of reductive algebraic groups by furnishing detailed proofs of core results, including the conjugacy of maximal tori, classification via root data, the Cartan and Bruhat decompositions, the highest weight classification, and the Weyl character formula. Moreover, we have introduced an original refinement of the Bruhat decomposition that elucidates the affine cell structure of double cosets in the flag variety.

These contributions not only consolidate classical theory but also offer new perspectives that may facilitate further research in geometric representation theory and the computation of topological invariants. We trust that the detailed proofs and original results presented here will serve as a valuable resource for researchers and advanced graduate students in the field.

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