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Results On Sequential Conformable Fractional Derivatives With Applications

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Abstract

This paper investigates and states some properties of sequential conformable fractional derivative introduced by R. Khalil et. al. in [1]. Further some theorems of the classical power series are generalized for the fractional power series(CFPS), where the CFPS technique is used to find the solutions of conformable fractional deferential equation with variable coefficients.

1 Introduction

The correspondence between L'Höpital and Leibniz, in 1695, about what might be a derivative of order 1/2, led to the introduction of a generalization of integral and derivative operators, known as Fractional Calculus. Since then, related to the definition of fractional derivatives have been many definitions. The most popular ones of these definitions are Riemann-Liouville and Caputo definitions see [6],[7].

Recently, R. Khalil et al. [1] give a new definition of fractional derivative and fractional integral. In their work they proved the product rule, the fractional Rolle's theorem and Mean Value Theorem utilizing the conformable fractional derivative definition. New construction of the generalized Taylor's power series is obtained by Abdeljawad in[2]. In recent years, many researchers have focused on the approximate analytical solutions of the system of fractional differential equations and some methods have been developed such as fractional power series method(FPS) in [5] . FPS method is a simple technique to find out the recurrence relation that determines the coefficients

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of the fractional power series, where this method is one of the most useful techniques to solve linear system and non-linear system of fractional differential equations with a fast convergence rate and small calculation error.

In this work, we state some properties of sequential conformable fractional derivative, then use the FPS technique to solve conformable fractional differential equation of order two.

2 Conformable fractional derivative

In this section, we present some definitions and some important properties of the conformable fractional derivative. The definition of the conformable fractional derivative is defined as follows;

Definition 1 [1] *Given a function* $f : [0, \infty) \longrightarrow \mathbb{R}$. *For* $\alpha \in (0, 1)$, *the conformable fractional derivative (CFD) of* f *of order* α *is defined by*

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$. *If* f *is* α *differentiable in some* $(0, a)$, $a > 0$, *and* $\lim_{t \to 0^+} f^{(\alpha)}(t)$ *exists, then define*

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

A function f is called α-differentiable at $t \geqslant 0$ if the above limits exists. For simplicity we sometimes use the notation $f^{(\alpha)}(t)$ instead of $T_{\alpha}(f)(t)$.

Consider the limit $\alpha \to 1^-$. In this case , for $t > 0$, we obtain the classical definition for the first derivative of a function $f^{(\alpha)}(t) = f'(t)$.

Theorem 2 [1]*Let* $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a differentiable function in the classical sense. Then* f *is* α *differentiable at* t , $\alpha \in (0, 1)$ *and*

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t), \quad t > 0.$$

Also, if f *is continuously differentiable at* 0, *then* $f^{(\alpha)}(0) = 0$.

Note that the function could be α-differentiable at a point t_0 but not differentiable at that point, as in the following example.

Example 3 *For some fixed* α, *with* $\alpha \in (0, 1)$, *let* $f(t) = \frac{t^{\alpha}}{\alpha}$, $t > 0$. *Note that* $f'(0)$ *does not exists but* $T_{\alpha}(f)(t) = 1$ *for* $t > 0$, *therefore* $f^{(\alpha)}(0) = \lim_{t \to 0^+} T_{\alpha}(f)(t) = 1$.

Theorem 4 [1]*If a function* $f : [0, \infty) \longrightarrow \mathbb{R}$ *is* α*-differentiable at* $a > 0$, $\alpha \in (0,1]$, *then* f *is continuous at* a.

We list some Important properties of the operator T_α as follows.

Theorem 5 [1]*Let* $\alpha \in (0,1]$ *and* f, g *be* α*differentiable at a point* $t > 0$. *Then*

1. $T_\alpha(\lambda f + g) = \lambda T_\alpha(f) + T_\alpha(g)$, *for all* $\lambda \in \mathbb{R}$.
2. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
3. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

Important examples of CFD are listed as follows:

Example 6 [1]

1. $T_\alpha(t^p) = pt^{p-\alpha}$
2. $T_\alpha(e^{at}) = at^{1-\alpha}e^{at}, \quad a \in \mathbb{R}$.
3. $T_\alpha(\sin(at)) = at^{1-\alpha}\cos(at), \quad a \in \mathbb{R}$.
4. $T_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at), \quad a \in \mathbb{R}$.
5. $T_\alpha(e^{\lambda(\frac{1}{\alpha}t^\alpha)}) = \lambda e^{\frac{1}{\alpha}t^\alpha}$.
6. $T_\alpha(\sin(\frac{1}{\alpha}t^\alpha)) = \cos(\dfrac{1}{\alpha}t^\alpha)$.
7. $T_\alpha(\cos(\frac{1}{\alpha}t^\alpha)) = -\sin(\dfrac{1}{\alpha}t^\alpha)$.
8. $T_\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

There are some properties that are not satisfied by operator T_α as follows:

Theorem 7 [2]*For* $\alpha, \beta \in (0,1]$ *and* $\alpha + \beta \in (1,2]$, *and the function* $f : (0, \infty) \longrightarrow \mathbb{R}$ *is twice differentiable on* $(0, \infty)$.T_α *does not satisfy the Index Law;* $T_\alpha T_\beta = T_\beta T_\alpha$,*where*

$$T_\alpha T_\beta\left(f\right)\left(t\right) = t^{1-(\alpha+\beta)}\left(\left(1-\beta\right)f'\left(t\right) + t\ f''\left(t\right)\right),$$

1. *while,*

$$T_\beta T_\alpha\left(f\right)\left(t\right) = t^{1-(\alpha+\beta)}\left(\left(1-\alpha\right)f'\left(t\right) + t\ f''\left(t\right)\right).$$

Proof. Calculating $T_\alpha T_\beta, T_\beta T_\alpha$ gives that

$$T_\alpha T_\beta\left(f\right)\left(t\right) \neq T_\beta T_\alpha\left(f\right)\left(t\right),$$

for $\alpha \neq \beta$. ■

3 Sequential conformable fractional α-derivatives

In this section the higher order of conformable fractional derivative will be defined and the relation between CFD and polynomials will be given.

Definition 8 *The second order CFD operator T_α, will be denoted by*

$$T_\alpha^2 = T_\alpha \left(T_\alpha\right).$$

In general the n^{th} order CFD operator T_α is defined as

$$T_\alpha^n = T_\alpha \left(T_\alpha^{n-1}\right).$$

Note that the operators commute for each positive integers n, m,

$$T_\alpha^n T_\alpha^m = T_\alpha^m T_\alpha^n.$$

The calculation of the second and the third sequential orders of CF operator T_α can be found in [2].

Theorem 9 *Given a function $f : (0, \infty) \longrightarrow \mathbb{R}$ .Then*

$$T_\alpha^2 (f)(t) = T_\alpha T_\alpha (f)(t) = t^{1-2\alpha} \Big((1-\alpha) f'(t) + t\ f''(t) \Big),$$

where f is twice differentiable, and twice α-differentiable at t . Also

$$T_\alpha^3 (f)(t) = t^{1-3\alpha} \Big((1-\alpha)(1-2\alpha) f'(t) + (3-3\alpha)\, t\ f''(t) + t^2\ f'''(t) \Big),$$

where f is three times differentiable at t and three times α-differentiable at t .

Lemma 10 *If m and n are any positive integers and p a real number, then*

$$T_\alpha^m \left(t^p\right) = \Pi_{i=0}^{m-1} \left(p - i\ \alpha\right) t^{p-m\alpha}.$$

Proof.

$$\begin{aligned} T_\alpha^2 \left(t^p\right) &= T_\alpha \left(T_\alpha t^p\right) \\ &= T_\alpha \left(p t^{p-\alpha}\right) \\ &= p \left(p-\alpha\right) t^{p-2\alpha}. \end{aligned}$$

Also,

$$\begin{aligned} T_{\alpha}^{3}\left(t^{p}\right) &= T_{\alpha}\left(p\left(p-\alpha\right)t^{p-2\alpha}\right) \\ &= p\left(p-\alpha\right)\left(p-2\alpha\right)t^{p-3\alpha}. \end{aligned}$$

Then

$$\begin{aligned} T_{\alpha}^{m}\left(t^{p}\right) &= p\left(p-\alpha\right)\left(p-2\alpha\right)\ldots\left(p-\left(m-1\right)\alpha\right)t^{p-m\alpha} \\ &= \Pi_{i=0}^{m-1}\left(p-i\ \alpha\right)t^{p-m\alpha}. \end{aligned}$$

■

Corollary 11 *If n is positive integer , then*

$$T_{\alpha}^{n}\left(t^{n}\right) = \Pi_{i=0}^{n-1}\left(n-i\ \alpha\right)t^{n(1-\alpha)}.$$

Corollary 12 *If m and n are any positive integers and $0<\alpha<1$,then*

$$T_{\alpha}^{m}\left(t^{n\alpha}\right) = \Pi_{i=0}^{m-1}\alpha^{m}\left(n-i\ \right)t^{\alpha(n-m)}.$$

Applying Lemma10 and using the linearity property of Theorem 5, we get

Lemma 13 *If $P_n(t) = a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_n$ is a polynomial in t of degree n, then*

$$T_{\alpha}^{m}P_{n}\left(t\right) = \sum_{j=0}^{n} a_{j}\ \Pi_{i=0}^{m-1}\left(n-j-i\ \alpha\right)t^{n-j-m\alpha}.$$

Changing variable from $t \to t^{\alpha}$, we get the following result.

Corollary 14 *If $f(t) = a_0t^{\alpha n} + a_1t^{\alpha(n-1)} + a_2t^{\alpha(n-2)} + \cdots + a_n$, then*

$$T_{\alpha}^{m}f\left(t\right) = \sum_{j=0}^{n} a_{j}\ \Pi_{i=0}^{m-1}\left(\alpha\left(n-i\ \right)-j\right)t^{\alpha(n-m)-j}.$$

The following theorem presents the n^{th} sequential CF α-derivative utilizing the limit definition as follows:

Theorem 15 *Given a function $f:[0,\infty)\longrightarrow\mathbb{R}$. Then for $0<\alpha<1$ and n a positive integer, the n th order of the $\alpha-$conformable fractional derivative of f of is as follows*

$$T_{\alpha}^{n}f\left(t\right) = \lim_{\varepsilon\to 0}\frac{\sum_{j=0}^{n}\binom{n}{j}\left(-1\right)^{n-j}f\left(\left(1+\varepsilon t^{-\alpha}\right)^{j}t\right)}{\varepsilon^{n}},$$

where $t>0$.

Proof. Since

$$\begin{aligned} T_\alpha f(t) &= \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{f(t(1+\varepsilon t^{-\alpha})) - f(t)}{\varepsilon}. \end{aligned}$$

Let $\delta = 1 + \varepsilon t^{-\alpha}$, we get

$$T_\alpha f(t) = \lim_{\delta\to 1} \frac{f(\delta t) - f(t)}{(\delta - 1)\, t^\alpha}.$$

Thus

$$\begin{aligned} T_\alpha^2 f(t) &= \lim_{\delta\to 1} \frac{T_\alpha f(\delta t) - T_\alpha f(t)}{(\delta - 1)\, t^\alpha} \\ &= \lim_{\delta\to 1} \frac{f(\delta^2 t) - 2f(\delta t) + f(t)}{(\delta - 1)^2\, t^{2\alpha}}. \end{aligned}$$

Again calculating the 3^{rd} order of the $\alpha-$conformable fractional derivative, we get

$$\begin{aligned} T_\alpha^3 f(t) = T_\alpha T_\alpha^2 f(t) &= \lim_{\delta\to 1} \frac{T_\alpha f(\delta^2 t) - 2T_\alpha f(\delta t) + T_\alpha f(t)}{(\delta - 1)^2\, t^{2\alpha}} \\ &= \lim_{\delta\to 1} \frac{f(\delta^3 t) - 3f(\delta^2 t) + 3f(\delta t) + f(t)}{(\delta - 1)^3\, t^{3\alpha}} \\ &= \lim_{\delta\to 1} \frac{\sum_{j=0}^{3} \binom{3}{j} (-1)^{3-j} f\left(\delta^j t\right)}{(\delta - 1)^3\, t^{3\alpha}}. \end{aligned}$$

Repeating this process n times, we get

$$T_\alpha^n f(t) = \lim_{\delta\to 1} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f\left(\delta^j t\right)}{(\delta - 1)^n\, t^{n\alpha}}.$$

Substituting $\varepsilon = (\delta - 1)\, t^\alpha$, then

$$T_\alpha^n f(t) = \lim_{\varepsilon\to 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f\left((1+\varepsilon t^{-\alpha})^j t\right)}{\varepsilon^n}.$$

■

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4 Conformable Fractional Power Series Representation

Power series is an important tool in the study of elementary functions. Using this power expansion gives us the ability to make an approximate study of many differential equations . In this section, we will recall some important definitions and theorems of fractional power series theory .

Definition 16 *For $\alpha \in (0,1)$ a conformable fractional power series of the form*

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots\cdots ,$$

where $t > t_0 \geq 0$ tis called the conformable fractional power series (CFPS) about t_0,where c_n denote the coefficients of the series, where $n \in \mathbb{N}$.

Note that for CFPS, we have the value c_0 for$n = 0$,at $t = t_0$, while $c_n = 0$ for $n \geq 1$at $t = t_0$ .Also note that for $t_0 = 0$, then the CFPS becomes

$$\sum_{n=0}^{\infty} c_n t^{n\alpha} = c_0 + c_1 t^{\alpha} + c_2 t^{2\alpha} + \cdots\cdots$$

Proposition 17 [4]*If $\sum_{n=0}^{\infty} c_n t^{n\alpha}$converges absolutely for $t = t_0 > 0$, then we it converges absolutely for $t \in (0, t_0)$.*

Theorem 18 *The series$\sum_{n=0}^{\infty} c_n t^{n\alpha}$converges,$-\infty < t < \infty$ has radius of convergence R, if and only if the series $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$ has radius of convergence $R^{\frac{1}{\alpha}}$, $R > 0$.*

Theorem 19 [2]*Assume f is an infinitely α-differentiable function, for some $\alpha \in (0,1]$ at a neighborhood of a point t_0. Then f has the Taylor CFPS series expansion as follows*

$$f(t) = \sum_{k=0}^{\infty} \frac{T_{\alpha}^{k} f(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}, \ t \in \left(t_0, t_0 + R^{\frac{1}{\alpha}}\right) \ , \ R > 0.$$

The next following examples doesn't have the Taylor PS expansion about $t_0 \geq 0$ since there are not differentiable there. But they have Taylor CFPS expansion at t_0.

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Example 20 [2]

1. $e^{\frac{1}{\alpha}(t-t_0)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\alpha^k k!}$, *for* $t \in [t_0, \infty)$.
2. $\sin(\frac{1}{\alpha}(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-t_0)^{(2k+1)\alpha}}{\alpha^{2k+1} 2k+1!}$, *for* $t \in [t_0, \infty)$.
3. $\cos(\frac{1}{\alpha}(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-t_0)^{(2k)\alpha}}{\alpha^{2k} 2k!}$, *for* $t \in [t_0, \infty)$.
4. $\frac{1}{1-\frac{(t-t_0)^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} (t-t_0)^{k\alpha}$, *for* $t \in [t_0, t_0+1)$.

5 Solving CFD equation's using CFPS

Example 21 *Consider the following conformable fractional differential equation*

$$T_{\alpha}^{2} y(t) - t^{\alpha} y(t) = 0, \tag{2}$$

with initial conditions,

$$y(0) = 0, \; T_{\alpha} y(0) = y_0, \tag{3}$$

where y_0 is a real constant.
Now using CFPS technique, let

$$y(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}.$$

Then

$$\begin{aligned} T_{\alpha}^{2} y(t) &= \sum_{n=2}^{\infty\alpha} \alpha^2 (n)(n-1) c_n t^{(n-2)\alpha} \\ &= \sum_{n=0}^{\infty} \alpha^2 (n+1)(n+2) c_{n+2} t^{n\alpha} \end{aligned}$$

and

$$\begin{aligned} t^{\alpha} y(t) &= \sum_{n=0}^{\infty} c_n t^{(n+1)\alpha} \\ &= \sum_{n=1}^{\infty} c_{n-1} t^{n\alpha} \end{aligned} \tag{4}$$

Substituting $T_\alpha^2 y(t)$ and $t^\alpha y(t)$ in CFDE, we get

$$\sum_{n=0}^{\infty} \alpha^2 (n+1)(n+2) c_{n+2} t^{n\alpha} - \sum_{n=1}^{\infty} c_{n-1} t^{n\alpha} = 0.$$

Then from Formula (), one can obtain

$$2\alpha^2 c_2 + \sum_{n=1}^{\infty} \left[\alpha^2 (n+1)(n+2) c_{n+2} - c_{n-1}\right] t^{n\alpha} = 0.$$

Equating the coefficients of $t^{n\alpha}$ to zero in both sides gives the following;

$$c_2 = 0, \text{ and } c_{n+2} = \frac{c_{n-1}}{\alpha^2 (n+1)(n+2)}, \; n = 1, 2, 3, \ldots. \tag{5}$$

Considering the initial conditions of the CFDE,

$$c_0 = 0, \text{ and } c_1 = \frac{1}{\alpha} y_0.$$

Based on Equation 5,the coefficients of $t^{n\alpha}$ can be divided into two categories: zero terms

$$c_2 = c_3 = c_5 = c_6 = c_8 = c_9 = c_{11} = c_{14} = \cdots = 0,$$

in general

$$c_{3n+2} = c_{3n+3}, \text{ for } n = 0, 1, 2, 3, \ldots$$

and non zero terms

$$c_{3n+1} \neq 0, \text{for } n = 0, 1, 2, 3, \ldots,$$

that is $c_1, c_4, c_7, c_{10}, c_{13}, c_{16} \cdots$, where

$$c_4 = \frac{c_1}{\alpha^2 (3.4)} = \frac{y_0}{\alpha^3 (3.4)},$$

$$c_7 = \frac{c_4}{\alpha^2 (6.7)} = \frac{y_0}{\alpha^5 (3.4.6.7)},$$
$$c_{10} = \frac{c_7}{\alpha^2 (9.10)} = \frac{y_0}{\alpha^7 (3.4.6.7.9.10)},$$
$$c_{10} = \frac{c_{10}}{\alpha^2 (9.10)} = \frac{y_0}{\alpha^9 (3.4.6.7.9.10)},$$
$$\ldots\ldots\ldots$$

Then, one can obtain the following CFPS as follows

$$\begin{aligned} y(t) &= c_1 t^{\alpha} + c_4 t^{4\alpha} + c_7 t^{7\alpha} + c_{10} t^{10\alpha} + \cdots\cdots\cdots \\ &= \frac{1}{\alpha} y_0 t^{\alpha} + \frac{y_0}{\alpha^3 (3.4)} t^{4\alpha} + \frac{y_0}{\alpha^5 (3.4.6.7)} t^{7\alpha} + \frac{y_0}{\alpha^9 (3.4.6.7.9.10)} t^{10\alpha} + \cdots\cdots \\ &= \frac{1}{\alpha} y_0 \left(1 + \sum_{k=1}^{\infty} \frac{1}{\alpha^{2k} \left(\prod_{i=0}^{k} (3i)(3i+1) \right)} t^{(3k+1)\alpha} \right). \end{aligned}$$

Conclusion 22 *The main aim of this work is to provide a reliable algorithm for the solutions to the systems of fractional differential equations by using the CFPS.*

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